A note on reversibility and Pell equations

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Resumo: Esta nota é sobre automorfismos hiperbólicos do toro que são reversíveis relativamente a uma involução linear conservativa. Por estarmos em dimensão dois, será possível associar à reversibilidade uma equação de Pell generalizada de cujo conjunto de soluções retiraremos informação adicional sobre aquela propriedade. Provaremos ainda que a reversibilidade é uma propriedade rara neste contexto e descreveremos o caso genérico.

Abstract: This note concerns hyperbolic toral automorphisms which are reversible with respect to a linear area-preserving involution. Due to the low dimension, we will be able to associate the reversibility with a generalized Pell equation from whose set of solutions we will infer further information. Additionally, we will show that reversibility is a rare feature and will characterize the generic setting.

palavras-chave: Automorfismos hiperbólicos do toro; reversibilidade; equações de Pell generalizadas.

keywords: Hyperbolic toral automorphisms; reversibility; generalized Pell equations.

1 Introduction

Let M be a compact, connected, smooth, Riemannian two-dimensional manifold without boundary and μ its normalized area. Denote by $\mathrm{Diff}_{\mu}^{1}(M)$ the set of all area-preserving C^1-diffeomorphisms of M endowed with the C^1-topology. A diffeomorphism $f : M \to M$ is said to be Anosov if M is a hyperbolic set for f, that is, if the tangent bundle of M admits a splitting $E^s \oplus E^u$ such that there exist an adapted norm $\| \ \|$ and a constant $\sigma \in (0,1)$ satisfying $\|Df_x(v)\| \leq \sigma$ and $\|Df_x^{-1}(u)\| \leq \sigma$ for every $x \in M$ and any unitary vectors $v \in E^s_x$ and $u \in E^u_x$. It is known, after [6, 8], that among the 2-dimensional manifolds only the torus (we will denote by $\mathbb{T}^2$) may support this type of diffeomorphisms. Moreover, each Anosov diffeomorphism on $\mathbb{T}^2$ is topologically conjugate to a linear model, that is, to a diffeomorphism induced on the torus by an element of the linear group $SL(2,\mathbb{Z})$ of the 2×2 matrices with integer entries, determinant equal to ± 1 and whose eigenvalues do not belong to the unit circle (the so called toral automorphisms). In this note, we consider linear Anosov diffeomorphisms in $\mathrm{Diff}^1_\mu(\mathbb{T}^2)$ which exhibit a certain kind of symmetry. More precisely, let $R : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism such that $R \circ R$ is the identity map of $\mathbb{T}^2$ (R is called an involution); our attention will be turned to the subset $\mathrm{Diff}^1_{\mu,R}(\mathbb{T}^2)$ of maps $f \in \mathrm{Diff}^1_\mu(\mathbb{T}^2)$, said to be R-reversible, such that R conjugates f and f^{-1}, that is,

$$R \circ f = f^{-1} \circ R.$$

Reversibility plays a fundamental role in physics and dynamical systems. In view of the many known applications, the references [7] and [11] present a thorough survey on reversible systems. Regarding this subject, important work was done in [12] and [1]. In the former, the authors derived necessary conditions for local reversibility within mappings of the plane with a symmetric fixed point, expressing them through the quadratic and cubic coefficients of the Taylor expansion about the fixed point. This way, they established an efficient negative criteria to show that a mapping is not reversible. On the other hand, on Section 2.2 of [1] we may read a detailed characterization of the kind of groups that are admissible as groups of reversible symmetries associated with unimodular matrices. Using algebraic techniques, the authors unveiled the possible structure of those groups, showing that it is completely resolvable when the matrices belong to $Gl(2,\mathbb{Z})$ or $PG(2,\mathbb{Z})$, as summarized in [1, Theorem 3].

Our work has followed a different, not purely algebraic, approach and addressed other questions, thus complementing the information disclosed by the two previous references. First, we will present a simple method to construct a (non unique) Anosov diffeomorphism that anti-commutes with a given linear involution (Section 4). Conversely, after concluding that an automorphism that reverses orientation is not reversible, we will settle (cf. Section 5) a connection between the existence of reversible symmetries for a hyperbolic toral automorphism and the set of solutions of a generalized

Pell equation [9]. Moreover, we will make clear in Section 6 why generically, in the C^1 topology, the r-centralizer (definition (1)) of a toral Anosov diffeomorphism is trivial, summoning a similar result obtained in [2] within the conservative context for the centralizer. We believe that an extension of these results may be proved for reversing symmetries that are not involutions, and on higher dimensions. Yet, those are so far open problems.

2 Overview

Whereas it is often useful to check if there is a dynamical system which is reversible under a given involution, we may also be interested in ascertaining if there exists an involution under which a given dynamics is reversible, and to what extent this is a common feature. The latter query is hard to answer in general, and that is why we will confine our study to hyperbolic toral automorphisms, benefiting both from the linear structure and the low dimensional setting. In what follows we will address three questions.

> Q_1. *Given a linear involution $R \in Diff_\mu^{\,1}(\mathbb{T}^2)$, is there an R-reversible linear area-preserving Anosov diffeomorphism f?*

The answer is obvious (and no) if $R = \pm Id$, the so called trivial case. Concerning the other possible involutions, we will look for a linear Anosov diffeomorphism f whose derivative at any point of $\mathbb{T}^2$ is a fixed linear map with matrix $L \in SL(2,\mathbb{Z})$. Observe that, as R is induced by a matrix $A \in SL(2,\mathbb{Z})$ as well, if we lift the equality $R \circ f = f^{-1} \circ R$ by differentiating it at any point of $\mathbb{T}^2$, we obtain $A \circ L = L^{-1} \circ A$; as we will verify in Remark 3.2, this equality implies that $det\, L = 1$. Analyzing the entries of the matrices involved we will answer positively to question Q_1.

Proposition 2.1 *Let $\mathcal{L}$ be the set of linear Anosov diffeomorphisms on $\mathbb{T}^2$. If $R \in Diff_\mu^{\,1}(\mathbb{T}^2) \setminus \{\pm Id\}$ is a linear involution, then $Diff_{\mu,R}^{\,1}(\mathbb{T}^2) \cap \mathcal{L} \neq \emptyset$.*

The proof of this result will be presented on Section 4. The argument is straightforward once the list of involutions in $SL(2,Z)$ is determined. Afterwards we will deal with the converse query.

> Q_2. *Given a linear Anosov diffeomorphism $f \in Diff_\mu^{\,1}(\mathbb{T}^2)$, is there a linear involution $R \in Diff_\mu^{\,1}(\mathbb{T}^2)$ such that f is R-reversible?*

If such an f reverses orientation (meaning that f is induced by a linear transformation $L \in SL(2,\mathbb{Z})$ such that $det\,(L) = -1$), then f is never R-reversible for every linear involution $R \in \mathrm{Diff}^1_\mu(\mathbb{T}^2)$. We will show (cf. Section 5) that, if f preserves orientation, then the reversibility of f with respect to a linear involution R determines a generalized Pell equation (see [4, 10] for this concept) whose set of solutions displays the entries of the R's matrix.

Theorem A *Let $f \in \mathit{Diff}^1_\mu(\mathbb{T}^2)$ be a linear Anosov diffeomorphism induced by a transformation*

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

such that $det\,(L) = 1$. Then:

(i) b divides $a-d$ if and only if f is R-reversible, where R is the projection on $\mathbb{T}^2$ of either

$$A = \begin{pmatrix} 1 & 0 \\ \frac{d-a}{b} & -1 \end{pmatrix} \quad or \quad A = \begin{pmatrix} -1 & 0 \\ \frac{a-d}{b} & 1 \end{pmatrix}.$$

(ii) c divides $a-d$ if and only if f is R-reversible, where R is the projection on $\mathbb{T}^2$ of either

$$A = \begin{pmatrix} 1 & \frac{d-a}{c} \\ 0 & -1 \end{pmatrix} \quad or \quad A = \begin{pmatrix} -1 & \frac{a-d}{c} \\ 0 & 1 \end{pmatrix}.$$

(iii) Given $\alpha, \beta \in \mathbb{Z}\backslash\{0\}$ such that $1-\alpha^2 \neq 0$ and β divides $1-\alpha^2$, consider the involution R obtained by projecting on $\mathbb{T}^2$ the matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Then f is R-reversible if and only if the generalized Pell equation $x^2 - Dy^2 = N$, where $D = (a+d)^2-4$ and $N = 4b^2$, has among its solutions

$$x = 2b\alpha + (d-a)\beta \quad and \quad y = \beta.$$

Observe that, if $f, g \in \mathrm{Diff}^1_{\mu,R}(\mathbb{T}^2)$, then $R \circ f^{-1} = f \circ R$ and $R \circ g^{-1} = g \circ R$ but $R \circ (f \circ g) = (f^{-1} \circ R) \circ g = (f^{-1} \circ g^{-1}) \circ R = (g \circ f)^{-1} \circ R$; this means that the set $\text{Diff}_{\mu,R}^{\,1}(\mathbb{T}^2)$, endowed with the composition of maps is, in general, not a group. Yet, $\text{Diff}^{\,1}(\mathbb{T}^2)$, $\text{Diff}_{\mu}^{\,1}(\mathbb{T}^2)$ and $\text{Diff}_{\mu,R}^{\,1}(\mathbb{T}^2)$ are Baire spaces (cf. [5] and Subsection 3.2), so it is natural to adopt another perspective towards reversibility.

Q_3. *Is reversibility generic?*

Given a diffeomorphim $f : \mathbb{T}^2 \to \mathbb{T}^2$, the r-centralizer of f is the set

$$\mathcal{Z}_r(f) = \{R \in \text{Diff}^{\,1}(\mathbb{T}^2) : R \circ f = f^{-1} \circ R\}. \tag{1}$$

$\mathcal{Z}_r(f)$ is said to be trivial if it is either empty or reduces to a set $\{R \circ f^n : n \in \mathbb{Z}\}$ for some $R \in \text{Diff}^{\,1}(\mathbb{T}^2)$. We will verify on Section 6 that, for any diffeomorphism f in a residual subset of $\text{Diff}_{\mu}^{\,1}(\mathbb{T}^2)$, the equation $R \circ f = f^{-1} \circ R$ has only trivial solutions.

Theorem B *C^1-generically in $\text{Diff}_{\mu}^{\,1}(M)$, the r-centralizer is trivial.*

Notice that, in spite of the fact that, on the torus $\mathbb{T}^2$, each Anosov diffeomorphism is conjugate to a hyperbolic toral automorphism, the conclusions we have drawn cannot be extended to all Anosov diffeomorphisms because the R-reversibility, for a fixed R, is not preserved by conjugacy. In fact, if $f \in \text{Diff}_{\mu,R}^{\,1}(\mathbb{T}^2)$ is conjugate to $g \in \text{Diff}_{\mu}^{\,1}(\mathbb{T}^2)$ through a homeomorphism h, then, although we have $(R \circ h) \circ g = f^{-1}(R \circ h)$, the map g may be not R-reversible. Nevertheless, the last equation indicates that $(h^{-1} \circ R \circ h) \circ g = g^{-1} \circ (h^{-1} \circ R \circ h)$, so a diffeomorphism conjugate to an R-reversible linear Anosov diffeomorphism is reversible as well, although with respect to another involution, which is conjugate to R but may be neither linear nor even differentiable.

3 Linear involutions on $\mathbb{T}^2$

We start characterizing the linear involutions $R : \mathbb{T}^2 \to \mathbb{T}^2$ of the torus, induced by matrices A in $SL(2, \mathbb{Z})$. Given this, we will describe the set of fixed points of such involutions.

3.1 Classification

After differentiating the equality $R^2 = Id_{\mathbb{T}^2}$ at any point of $\mathbb{T}^2$, we obtain $A^2 = Id_{\mathbb{R}^2}$. Comparing the entries of the matrices in this equality, we conclude that:

Lemma 3.1 *A matrix* $A \in SL(2,\mathbb{Z}) \setminus \{\pm Id\}$ *is an involution if and only if it belongs to the following list:*

- $A = \begin{pmatrix} \pm 1 & 0 \\ \gamma & \mp 1 \end{pmatrix}$ *or its transpose, for some* $\gamma \in \mathbb{Z}$.
- $A = \begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix}$ *for* $\alpha \in \mathbb{Z}\backslash\{\pm 1\}$ *and* $\beta \in \mathbb{Z}\backslash\{0\}$ *such that* β *divides* $1-\alpha^2$.

Let A be a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z})$ such that $A^2 = Id$. These properties imply that $\alpha\delta - \beta\gamma = \pm 1$ and

$$\begin{cases} \alpha^2 + \beta\gamma = 1 \\ \beta(\alpha+\delta) = 0 \\ \gamma(\alpha+\delta) = 0 \\ \gamma\beta + \delta^2 = 1 \end{cases}$$

so

$$\begin{cases} \beta = 0 \vee \alpha = -\delta \\ \gamma = 0 \vee \alpha = -\delta. \end{cases}$$

1st case: $\beta = 0$. One must have $\alpha = \pm 1$ and $\delta = \pm 1$. If $\alpha = \delta = 1$ or $\alpha = \delta = -1$, we conclude that $\gamma = 0$ and so $A = \pm Id$. Therefore, $-\alpha = \delta = 1$ or $\alpha = -\delta = 1$, and there are no restrictions on the value of γ. Hence $A = \begin{pmatrix} \pm 1 & 0 \\ \gamma & \mp 1 \end{pmatrix}$ for $\gamma \in \mathbb{Z}$.

2nd case: $\gamma = 0$. Again $\alpha = \pm 1$ and $\delta = \pm 1$, and so either $-\alpha = \delta = 1$ or $\alpha = -\delta = 1$. Therefore $A = \begin{pmatrix} \pm 1 & \beta \\ 0 & \mp 1 \end{pmatrix}$ for $\beta \in \mathbb{Z}$.

3rd case: $\beta \neq 0$ and $\gamma \neq 0$. We must have $\alpha = -\delta$, hence $\alpha^2 + \beta\gamma = 1$ and so $\gamma = \frac{1-\alpha^2}{\beta}$. Moreover, for A to belong to $SL(2,\mathbb{Z})$, the entry β must divide $1-\alpha^2$. Thus $A = \begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix}$. To prevent repetitions with the preceding case, α has to be different from ± 1.

Remark 3.2 *From the previous argument we also conclude that:*

1. *A linear map $A \in SL(2,\mathbb{Z}) \setminus \{\pm Id\}$ is an involution if and only if* $\det A = -1$ *and trace* $A = 0$.[1] *Therefore, the eigenvalues of both A and $-A$ are precisely -1, 1.*

2. *Given a linear involution $A \in SL(2,\mathbb{Z}) \setminus \{\pm Id\}$ and a matrix $L \in SL(2,\mathbb{Z}) \setminus \{\pm Id\}$, then L is A-reversible (that is, $A \circ L = L^{-1} \circ A$) if and only if $(LA)^2 = Id$. If the spectrum of the matrix L does not contain -1, 1, then the involution LA cannot be $\pm Id$ (otherwise, $L = A$ or $L = -A$ and so -1, 1 would be eigenvalues of L). Therefore, there is a linear involution $B \in SL(2,\mathbb{Z}) \setminus \{\pm Id\}$ such that $L = BA$. In particular, we must have* $\det L = 1$.

3.2 Fixed points

From the previous description of the matrices A and by solving the equation $A(x,y) = (x,y)$ in $\mathbb{R}^2$, we deduce that the fixed point set of a linear non-trivial involution R of the torus is a finite union of smooth closed curves obtained by projecting subspaces of $\mathbb{R}^2$ with dimension one. Firstly, we know that such an R is induced by a matrix $A \in SL(2,\mathbb{Z})$ as specified by Lemma 3.1, so we get

A	***Fixed points subspace of*** A
$\begin{pmatrix} 1 & 0 \\ \gamma & -1 \end{pmatrix}$	$y = \frac{\gamma}{2}x$
$\begin{pmatrix} 1 & \gamma \\ 0 & -1 \end{pmatrix}$	$y = 0$
$\begin{pmatrix} -1 & 0 \\ \gamma & 1 \end{pmatrix}$	$x = 0$
$\begin{pmatrix} -1 & \gamma \\ 0 & 1 \end{pmatrix}$	$x = \frac{\gamma}{2}y$
$\begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix}$, $\beta \neq 0$	$y = \frac{1-\alpha}{\beta}x$

Moreover, if $\pi : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denotes the usual projection and $A(x,y) = (x,y)$, then $R(\pi(x,y)) = \pi(x,y)$, so the projections on the torus of the subspaces of the previous table are sets of fixed points by R. As the slopes of these lines are rational numbers, these projections are closed curves on $\mathbb{T}^2$.

Conversely, observe that $R(\pi(x,y)) = \pi(x,y)$ if and only if $\pi(A(x,y)) = \pi(x,y)$, and this implies that there exists $(n,m) \in \mathbb{Z}^2$ such that $A(x,y) =$

[1]We thank the referee for calling our attention to this property.

$(x, y) + (n, m)$. So, for instance, if $A = \begin{pmatrix} 1 & 0 \\ \gamma & -1 \end{pmatrix}$, then

$$A(x, y) = (x, y) + (n, m) \quad \Leftrightarrow \quad x \in \mathbb{R},\ n = 0,\ y = \frac{\gamma}{2}\, x - \frac{m}{2}$$

and therefore, for the involution R induced by such a matrix A, we have

$$Fix(R) = \pi\left(\left\{(x, \frac{\gamma}{2}\, x) : x \in \mathbb{R}\right\}\right) \cup \pi\left(\left\{(x, \frac{\gamma}{2}\, x - \frac{1}{2}) : x \in \mathbb{R}\right\}\right).$$

This is the union of two closed curves on $\mathbb{T}^2$. The other cases are analogous.

4 Answer to question Q_1

Recall that, if a linear Anosov diffeomorphism f is induced by a matrix $L(x, y) = (ax + by, cx + dy)$ of $SL(2, \mathbb{Z})$, then the entries of L must satisfy the conditions:

(IL) (Integer lattice invariance) $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$.

(H1) (Hyperbolicity) If $ad - bc = 1$, then $(a + d)^2 - 4 > 0$.

(H2) (Hyperbolicity) If $ad - bc = -1$, then $(a + d)^2 + 4$ is not a perfect square.

These requirements explain why a linear Anosov diffeomorphism is never $\pm Id$–reversible. Indeed, if $R = Id$ and f is R-reversible, then $f^2 = Id$; however, this equality does not hold among Anosov diffeomorphisms, whose periodic points are hyperbolic and so isolated. If $R = -Id$ and f is induced by a matrix $L \in SL(2, \mathbb{Z})$ and is R-reversible, then the equality $(-Id) \circ L = L^{-1} \circ (-Id)$ yields

$$\begin{cases} a = d,\ b = c = 0, & \text{if } ad - bc = 1 \\ a = -d, & \text{if } ad - bc = -1 \end{cases}$$

which contradicts one of the properties (H1) or (H2).

Going through the available matrices A, given by Lemma 3.1, we will determine, for each R, an orientation-preserving R-reversible and linear Anosov diffeomorphism f.

(a) $A = \begin{pmatrix} 1 & 0 \\ \gamma & -1 \end{pmatrix}$. We start noticing that the equality $A \circ L = L^{-1} \circ A$, with $det\,(L) = 1$, is equivalent to $b\gamma = d - a$. If $\gamma = 0$, we may take $L = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$ with integer entries such that $a^2 - bc = 1$ and $4a^2 > 4$ (so $b \neq 0$ and $c \neq 0$). For instance, $a = d = 3$, $b = 4$ and $c = 2$. If $\gamma \neq 0$, it must divide $d - a$ and L has to be $\begin{pmatrix} a & \frac{d-a}{\gamma} \\ c & d \end{pmatrix}$, with integer entries such that $ad - bc = 1$, $(a+d)^2 > 4$, $d - a \neq 0$ and $c \neq 0$. For example, $a = \gamma \in \mathbb{Z}\backslash\{0\}$, $b = 1$, $c = 2\gamma^2 - 1$ and $d = 2\gamma$. As the reversibility condition $A \circ L = L^{-1} \circ A$, with $det\,(L) = 1$, is equivalent to $A^T \circ L^T = (L^T)^{-1} \circ A^T$, with $det\,(L^T) = 1$, the case of the transpose matrix is equally solved.

(b) $A = \begin{pmatrix} -1 & 0 \\ \gamma & 1 \end{pmatrix}$. As in the previous case, the equality $A \circ L = L^{-1} \circ A$, with $det\,(L) = 1$, is equivalent to $b\gamma = a - d$. So, if $\gamma = 0$, we may take $L = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$. If $\gamma \neq 0$, we may choose, for instance, $a = \gamma \in \mathbb{Z}\backslash\{0\}$, $b = -1$, $c = 1 - 2\gamma^2$ and $d = 2\gamma$. Again, for $A^T = \begin{pmatrix} -1 & \beta \\ 0 & 1 \end{pmatrix}$, we may just pick the Anosov diffeomorphism induced by L^T.

(c) $A = \begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix}$, with $\alpha, \beta, 1 - \alpha^2 \neq 0$ and β a divisor of $1 - \alpha^2$. The equality

$$\begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix}$$

is equivalent to the equation

$$\alpha\beta a + (1 - \alpha^2)b + \beta^2 c - \alpha\beta d = 0. \tag{2}$$

To ease our task, we may try to find a matrix satisfying $a = d$. Under this assumption, equation (2) becomes $(1-\alpha^2)b + \beta^2 c = 0$. As c must also comply with the equality $a^2 - bc = 1$ and b cannot be zero, we must have $c = \frac{a^2-1}{b}$. In addition, we know that β divides $1 - \alpha^2$ and that $\alpha^2 \neq 1$, so $4\alpha^2 - 4 > 0$. Therefore, a convenient choice is $a = d = \alpha$, $b = \pm\beta$ and $c = \frac{\alpha^2-1}{\pm\beta}$. Namely,

$$L = \begin{pmatrix} \alpha & \beta \\ \frac{\alpha^2-1}{\beta} & \alpha \end{pmatrix} \quad \text{or} \quad L = \begin{pmatrix} \alpha & -\beta \\ \frac{1-\alpha^2}{\beta} & \alpha \end{pmatrix}.$$

5 Answer to question Q_2

Let f be a linear Anosov diffeomorphism, induced by a matrix $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2, \mathbb{Z})$. As seen in Remark 3.2, we must have $ad - bc = 1$. Take a linear involution R given by the projection on the torus of a matrix A as described by Lemma 3.1.

Case 1: $A = \begin{pmatrix} 1 & 0 \\ \gamma & -1 \end{pmatrix}$ or $A = \begin{pmatrix} -1 & 0 \\ \gamma & 1 \end{pmatrix}$. The reversibility equality is equivalent to $b\gamma = d - a$ or $b\gamma = a - d$. So there is such an involution A if and only if b divides $d - a$, in which case only one valid γ exists, namely, $\gamma = \frac{d-a}{b}$ or $\gamma = \frac{a-d}{b}$, respectively.

Case 2: $A = \begin{pmatrix} 1 & \gamma \\ 0 & -1 \end{pmatrix}$ or $A = \begin{pmatrix} -1 & \gamma \\ 0 & 1 \end{pmatrix}$. Dually, the reversibility condition is equivalent to $c\gamma = d - a$ or $c\gamma = a - d$. So there is such an involution A if and only if c divides $d - a$, and then we get a unique value for γ.

Case 3: $A = \begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix}$, where $\alpha, \beta, 1 - \alpha^2 \neq 0$ and β divides $1 - \alpha^2$. The pairs $(\alpha, \beta) \in \mathbb{Z}^2$ for which f is R-reversible are the integer solutions of the equation, in the variables α and β, given by

$$b\alpha^2 + \alpha\beta(d - a) - \beta^2 c = b \tag{3}$$

which satisfy the mentioned constraints. This quadratic form defines a conic whose kind depends uniquely on the sign of

$$\Delta = (d - a)^2 + 4bc = (a + d)^2 - 4$$

which we know to be always positive within our setting. So the conic is a hyperbola. After the change of variables

$$x = 2b\alpha + (d - a)\beta \quad \text{and} \quad y = \beta$$

the equation of the conic becomes

$$x^2 - Dy^2 = N \tag{4}$$

where $D = \Delta = (a + d)^2 - 4 > 0$ and $N = 4b^2$. Thus the problem of finding the intersections of the conic (3) with the integer lattice is precisely the one of existence of solutions of the generalized Pell equation (4). According to [3, 4, 10], this sort of Pell equation either has no integer solutions or has infinitely many, and there are several efficient algorithms to determine which one holds in each particular case. However, if they exist, the solutions we are interested in have also to fulfill the other requirements, namely $\alpha, \beta, 1-\alpha^2 \neq 0$ and β divides $1-\alpha^2$.

5.1 Examples

We may test the previous information in a few examples.

Anosov			***Involutions***		
	$\begin{pmatrix} 1 & 0 \\ \gamma & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \gamma \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ \gamma & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & \gamma \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix}$
$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$	$\gamma = 0$	$\gamma = 0$	$\gamma = 0$	$\gamma = 0$	*e.g.* $\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$
$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$	$\gamma = -1$	$\gamma = -1$	$\gamma = 1$	$\gamma = 1$	*e.g.* $\begin{pmatrix} 5 & 3 \\ -8 & -5 \end{pmatrix}$
$\begin{pmatrix} 4 & 9 \\ 7 & 16 \end{pmatrix}$	$-$	$-$	$-$	$-$	$-$

When $L = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, the generalized Pell equation is $x^2 - 12y^2 = 4$ and there are infinitely many matrices A of the third kind which correspond to linear involutions R such that f is R-reversible. Similarly, for $L = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ the generalized Pell equation is $x^2 - 5y^2 = 4$ and there are infinitely many involutions of type 3. The third example in this table has no linear involutions, although its Pell equation $x^2 - 396y^2 = 324$ has infinitely many solutions.

Remark 5.1 *Notice that, if R is an involution such that $R \circ f = f^{-1} \circ R$, then, for each $n \in \mathbb{Z}$, the diffeomorphism $R \circ f^n$ is also an involution, since $(R \circ f^n)^2 = (R \circ f^n) \circ (f^{-n} \circ R) = Id$; and f is $R \circ f^n$-reversible due to the equality*

$$(R \circ f^n) \circ f = (R \circ f) \circ f^n = (f^{-1} \circ R) \circ f^n = f^{-1} \circ (R \circ f^n).$$

Therefore, once such an involution R is found for an Anosov diffeomorphism f, then we have infinitely many involutions with respect to which f

is reversible. This is so because, as no non-trivial power of an Anosov diffeomorphism is equal to the Identity, we have $R \circ f^k \neq R \circ f^m$ for every $k \neq m \in \mathbb{Z}$.

The next table shows examples of a similar construction of generalized Pell equations and conics in the orientation-reversing setting (although no involution exists). If $A = \begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix}$, where $\alpha, \beta, 1-\alpha^2 \neq 0$ and β divides $1-\alpha^2$, then the pairs $(\alpha, \beta) \in \mathbb{Z}^2$ satisfying the R-reversibility condition for f should belong to the set of integer solutions, in the variables α and β, of the equations $\alpha b + \beta d = 0$, $\alpha c - \frac{a}{\beta}(1-\alpha^2) = 0$ and $\alpha a + \beta c = -\alpha d + \frac{b}{\beta}(1-\alpha^2)$. The last equality describes a (possibly degenerate) conic

$$b\alpha^2 + \alpha\beta(a+d) + \beta^2 c = b$$

whose type is determined by the sign of

$$\Delta = (a+d)^2 - 4bc = (a-d)^2 - 4.$$

Once again, the problem of finding the points of this conic in the integer lattice is linked to the existence of solutions of the generalized Pell equation $x^2 - Dy^2 = N$, where $x = 2b\alpha + (a+d)\beta$, $y = \beta$, $D = \Delta = (a-d)^2 - 4$ and $N = 4b^2$.

Anosov	Δ	***Pell eq.***	***$\sharp$ of solutions***	***Conic***	***Involutions***
$\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$	-3	$x^2 + 3y^2 = 36$	6	Ellipse	–
$\begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$	0	$x^2 = 64$	∞	Two lines	–
$\begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}$	5	$x^2 - 5y^2 = 100$	∞	Hyperbola	–

6 Answer to question Q_3

Given an area-preserving diffeomorphim f, if $f^2 = Id$, then f^n belongs to $\mathcal{Z}_r(f)$ for all $n \in \mathbb{Z}$; and conversely. However, a generic $f \in \mathrm{Diff}^1_\mu(M)$ does not satisfy the equality $f^n = Id$, for any integer $n \neq 0$. This is due to the fact that, by Kupka-Smale Theorem for area-preserving diffeomorphisms [5], given $k \in \mathbb{N}$, generically in the C^1 topology the periodic orbits of period less or equal to k are isolated.

The existence of hyperbolic toral diffeomorphisms (which are structurally stable) without reversible symmetries shows that there are non-empty open subsets of $\mathrm{Diff}_{\mu}^{1}(M)$ where reversibility is absent. Moreover, if $R \neq S$ are in $\mathcal{Z}_r(f)$, then $R \circ S$ belongs to the centralizer of f, as shown by the equalities $(R \circ S) \circ f = R \circ (S \circ f) = R \circ (f^{-1} \circ S) = f \circ (R \circ S)$. Now, according to [2], for a C^1-generic $f \in \mathrm{Diff}_{\mu}^{1}(M)$, the centralizer of f is trivial, meaning that it reduces to the powers of f. Therefore, there must exist $n \in \mathbb{Z}$ such that $S = R \circ f^n$. Thus, if $\mathcal{Z}_r(f) \neq \emptyset$, then its elements are obtained from the composition of one of them with the powers of f. And so, C^1-generically, $\mathcal{Z}_r(f)$ is trivial.

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