

ON SIMPLE BOUNDS FOR EIGENVALUES OF SYMMETRIC TRIDIAGONAL MATRICES¹

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Abstract How much can be said about the location of the eigenvalues of a symmetric tridiagonal matrix just by looking at its diagonal entries? We use classical results on the eigenvalues of symmetric matrices to show that the diagonal entries are bounds for some of the eigenvalues regardless of the size of the off-diagonal entries. Numerical examples are given to illustrate that our arithmetic-free technique delivers useful information on the location of the eigenvalues.

keywords: permutations, separation theorem.

1 Introduction

Gerschgorin theorems (see [1], p. 71) are the best known results on the location of the eigenvalues, in the complex plane, of a general matrix $A = (a_{ij})$, of size n . For real symmetric matrices, the Gerschgorin “disks” are just the n real intervals $[a_{ii} - R_i, a_{ii} + R_i]$, where $R_i = \sum_{j \neq i} |a_{ij}|$. If the union of $m < n$ of these intervals form a connected domain which is isolated from the other intervals, then there are precisely m eigenvalues in such domain.

Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of a real symmetric matrix A , not necessarily tridiagonal. Since $\lambda_1 = \min_{x \neq 0} (x^T A x) / (x^T x)$ and $\lambda_n = \max_{x \neq 0} (x^T A x) / (x^T x)$, by choosing $x = e_i$, the i th column of the identity matrix, for each $i = 1, \dots, n$, it follows that $\lambda_1 \leq \min\{a_{ii}\}$ and $\lambda_n \geq \max\{a_{ii}\}$. These simple bounds for the extremal eigenvalues are well known. For symmetric tridiagonal matrices, we present a result which we

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think to be new. The bounds just mentioned are particular cases of our more general result.

2 A new result

A symmetric tridiagonal is the adjacency matrix of a weighted graph which is simply a chain of nodes where the edges are bidirectional and the nodes are ordered in the “natural way”, i.e., consecutive numbers are assigned to nodes which are connected. A different ordering of the nodes (there are, of course, $n!$ different orderings) will produce a similar matrix PAP^T which differs from the tridiagonal A by a permutation of rows and columns, i.e., P is a permutation matrix. Different permutations will produce different patterns of sparsity and this may be explored in different contexts. For instance, for the matrix

$$A = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & b_2 & a_3 & b_3 \\ & & b_3 & a_4 \end{pmatrix}, \quad (1)$$

with $P = [1, 3, 4, 2]$, we get

$$PAP^T = \begin{pmatrix} a_1 & & & b_1 \\ & a_3 & b_3 & b_2 \\ & b_3 & a_4 & \\ b_1 & b_2 & & a_2 \end{pmatrix}. \quad (2)$$

The leading principal submatrix of order 2 is diagonal and we may use the so-called “separation theorem” (see, for instance, [1], pp.103-104) to show that a_1 and a_3 are bounds to some eigenvalues of A . In general, we have the following result.

Theorem 2.1. *Let*

$$A = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & \ddots & & \\ & \ddots & \ddots & b_{n-1} & \\ & & b_{n-1} & a_n & \end{pmatrix} \quad (3)$$

and $S = \{s_1, s_2, \dots, s_m\}$ be a subset of $\{1, 2, \dots, n\}$ such that

$$|s_i - s_j| \geq 2 \quad (4)$$

for any s_i and s_j ($i \neq j$). Then, independently of the off-diagonal entries b_k , there are at least m eigenvalues of A which are not larger than $\max_{i \in S} \{a_i\}$; also, there are at least m eigenvalues not smaller than $\min_{i \in S} \{a_i\}$.

Proof. First, it must be observed that the condition (4) imposes an upper bound on m which is $n/2$ for n even (in this case $S = \{1, 3, \dots, n-1\}$ or $S = \{2, 4, \dots, n\}$) and is $(n+1)/2$ for n odd, with $S = \{1, 3, \dots, n\}$. To prove the result, consider a similarity transformation of A based on permutations which brings the m diagonal entries a_i for $i \in S$ to the top m rows. From (4) it follows that the leading principal matrix, of order m , say A_m , is diagonal, i.e., no off-diagonal entry is inside this submatrix. It is well known that the eigenvalues of A_m , $\lambda_1^{(m)} \leq \lambda_2^{(m)} \leq \dots \leq \lambda_m^{(m)}$, say, separate those of A_{m+1} , i.e.,

$$\lambda_1^{(m+1)} \leq \lambda_1^{(m)} \leq \lambda_2^{(m+1)} \leq \dots \leq \lambda_m^{(m+1)} \leq \lambda_m^{(m)} \leq \lambda_{m+1}^{(m+1)}.$$

As a consequence of this, for the eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ of $A = A_n$, we have

$$\lambda_m \leq \lambda_m^{(n-1)} \leq \dots \leq \lambda_m^{(m+1)} \leq \lambda_m^{(m)} = \max_{i \in S} \{a_i\} \quad (5)$$

and

$$\min_{i \in S} \{a_i\} = \lambda_1^{(m)} \leq \lambda_2^{(m+1)} \leq \dots \leq \lambda_{n-m}^{(n-1)} \leq \lambda_{n-m+1}. \quad (6)$$

□

When n is odd and $m = (n+1)/2$, we find an inclusion interval for the middle eigenvalue λ_m

$$\min_{i \in S} \{a_i\} \leq \lambda_m \leq \max_{i \in S} \{a_i\}. \quad (7)$$

We now apply our results to symmetric tridiagonal matrices with null main diagonal entries. For even n and $m = n/2$, we conclude that zero halves the spectrum, i.e, the number of positive and the number of negative eigenvalues are equal. This is not new, the so-called Golub-Kahan matrices are similar (again, the similarity transformation is just a permutation of rows and columns) to

$$\begin{pmatrix} O & B \\ B^T & 0 \end{pmatrix}$$

(see Lemma 5.5 in [2]) whose eigenvalues are $\pm\sigma_i, i = 1, \dots, n/2$, where σ_i is a singular value of a bidiagonal matrix B . For odd n , the matrix is singular and that zero is in fact an eigenvalue may be concluded immediately from (7).

In the case of the Wilkinson matrix of order 5

$$A = \begin{pmatrix} 2 & 1 & & & \\ 1 & 1 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 1 & 1 \\ & & & 1 & 2 \end{pmatrix},$$

whose eigenvalues, with 4 decimal places, are -1.1149, 0.3820, 1.2541, 2.6180, 2.8608, with $m = 3$ and $S = \{1, 3, 5\}$, from (7) we conclude that

$$0 \leq \lambda_3 \leq 2$$

and from (6) we also have, with $S = \{1, 5\}$, that

$$2 \leq \lambda_4 \leq \lambda_5.$$

Furthermore, (5) with $S = \{3\}$ and $S = \{2, 4\}$, also gives

$$\lambda_1 \leq 0, \lambda_2 \leq 1.$$

The key aspect we want to stress out is that all these bounds are totally independent of the off-diagonal entries. This may be used in a different direction which is that of investigating the sensitivity of small eigenvalues of symmetric tridiagonals, in some special cases. For instance, if A has odd order and constant main diagonal entries $a_i = a \neq 0$, then it follows from (7) that the middle eigenvalue is $\lambda_m = a$. In fact, for this to hold we just need to have $a_i = a$ for $i \in S = \{1, 3, \dots, n\}$. Now, if we introduce small relative perturbations in such diagonal entries of A , i.e.,

$$\tilde{a}_i = a(1 + \delta_i),$$

with $|\delta_i| \leq \varepsilon$, we get

$$\left| \frac{\tilde{\lambda}_m - a}{a} \right| \leq \varepsilon$$

(it is $\tilde{\lambda}_m \in [a(1 - \varepsilon), a(1 + \varepsilon)]$ for $a > 0$ and $\tilde{\lambda}_m \in [a(1 + \varepsilon), a(1 - \varepsilon)]$ for $a < 0$).

In [3] we presented a general result for the relative perturbations of the eigenvalues of symmetric tridiagonal matrices. For matrices with off-diagonal entries much larger (in absolute value) than diagonal entries, Theorem 3.1 in [3] allows us to get much more accurate bounds than those produced by classical perturbation theory (where the perturbation in each

eigenvalue is bounded by the norm of the perturbation matrix). However, it is required that entry-wise relative perturbations are all bounded by some small constant. In the present case we have a much stronger result for the perturbation of the middle eigenvalue λ_m because it is determined solely by a_i , $i \in S = \{1, 3, \dots, n\}$, and remains unaltered by arbitrarily large perturbations of the remaining entries.

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