Open and Closed Mirror Symmetry

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Resumo: A simetria de espelho conjectura uma correspondência profunda entre a geometria simpléctica de um espaço e a geometria algébrica do seu "espelho". Existem várias versões desta correspondência, desde a igualdade de alguns invariantes numéricos, inicialmente conjecturada por físicos, a versões categóricas propostas por Kontsevich.

Este artigo revê algumas destas versões e ilustra-las num exemplo relativamente simples: uma esfera com três orbi-pontos (no lado simpléctico). Explicamos como construir o espaço "espelho", enunciamos as conjecturas de espelho e descrevemos uma abordagem à sua prova.

Abstract Mirror symmetry predicts a deep correspondence between the symplectic geometry of a space and the algebraic geometry of its "mirror". There are different versions of this correspondence, from the equality of some numerical invariants, first predicted by physicists, to categorical versions proposed by Kontsevich.

This paper reviews some of these versions and illustrates them on a relatively simple example: a sphere with three orbifold points (on the symplectic side). We explain how to construct the "mirror" space, state the mirror predictions and describe an approach to prove them.

palavras-chave: Simetria de espelho; categoria de Fukaya; orbi-variedade.

keywords: Mirror symmetry; Fukaya category; orbifold.

1 Introduction

1.1 A brief history

Mirror symmetry is a set of predictions from string theory relating the symplectic and complex geometry of certain pairs of Calabi-Yau manifolds. Superstring theory proposes that the space-time is (locally) of the form $\mathbb{R}^{1,3} \times X$, where $\mathbb{R}^{1,3}$ is the usual Minkowski space (that we see around us) and X is a very (very) small Calabi-Yau three-fold. Meaning X is a

Kähler manifold (therefore both complex and symplectic) of complex dimension three with a Ricci-flat metric. While looking for the X that would help describe our universe, string theorists produced large lists of Calabi-Yau manifolds and found a surprising symmetry. There are many pairs of Calabi-Yau manifolds X and $\check{X}$ which exchange Hodge numbers, that is $h^{1,1}(X) = h^{1,2}(\check{X})$ and $h^{1,2}(X) = h^{1,1}(\check{X})$.

The most famous example of this is the quintic threefold and its mirror. Let X_a be the solution set in $\mathbb{CP}^4$ of the equation

$$X_a = \{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - a z_0 z_1 z_2 z_3 z_4 = 0\} \subset \mathbb{CP}^4,$$

for some $a \in \mathbb{C}$. For most values of a this is a smooth Calabi-Yau manifold. The group $(\mathbb{Z}/5)^3$ acts on X_a and the quotient X_a/G is singular but there is a resolution $\check{X}_a \to X_a/G$, which is smooth and the mirror partner of X_a.

Mirror symmetry predictions are much deeper than the equality of Hodge numbers. Candelas, de la Ossa, Green and Parkes [6] predicted the number of rational curves of degree d in X_a could be obtained from certain period integrals of the family $\check{X}_a$. This was remarkable since only the cases with $d \leq 3$ were known. This led to the development of Gromov-Witten invariants, a way to define precisely the counting of rational curves. Using Gromov-Witten invariants Givental [15] proved the predictions for the quintic. Both the (genus zero) Gromov-Witten invariants and the period integrals can be organized into *Frobenius* manifolds. Mirror symmetry then predicts an isomorphism between these two Frobenius manifolds. This is known as *closed-string mirror symmetry*.

In [18], Kontsevich proposed a new version of mirror symmetry at a categorical level. To a Calabi-Yau manifold X one can associate two categories: the Fukaya category $Fuk(X)$ and the category of coherent sheaves $Coh(X)$. The Fukaya category is an A_∞-category, which depends only on the symplectic structure of X, and whose objects are, roughly speaking, the Lagrangians submanifolds of X. The category of coherent sheaves $Coh(X)$ is an abelian category (which can be promoted to an A_∞-category) which depends only on the complex structure of X. Kontsevich proposed that mirror symmetry exchanges these categories, that is, the derived categories of $Fuk(X)$ and $Coh(\check{X})$ are equivalent. This is usually called the homological mirror symmetry conjecture, or using physics terminology *open-string mirror symmetry*. This has been verified in several cases, see [23] for example.

Starting with the works of Givental and Batyrev it was suggested that mirror symmetry is not restricted to Calabi-Yau manifolds. It was conjectured that when X is Fano [16] or when X is of general type [17], there is also a mirror partner. In this case, the mirror is not simply a space, it's a non-compact manifold $\check{X}$ together with a holomorphic function $W : \check{X} \to \mathbb{C}$, which is called a Landau-Ginzburg model. For Landau-Ginzburg models one has to modify the mirror symmetry conjectures accordingly, for example replacing $Coh(\check{X})$ with the category of matrix factorizations $MF(\check{X}, W)$.

In 1996, Strominger-Yau-Zaslow [24] proposed a geometric explanation for mirror symmetry. Mirror (Calabi-Yau) pairs X and $\check{X}$ should admit dual, special Lagrangian torus fibrations over the same base B. This is known as the SYZ conjecture. A proof of the SYZ conjecture seems to be out of reach and in fact these fibrations might only exist after deforming X. Nevertheless this conjecture has been very influential and inspired many important insights into mirror symmetry.

1.2 Family Floer theory

Mirror symmetry and the SYZ conjecture become more manageable if one takes a less symmetric approach. That is, if we consider the Calabi-Yau manifold X just as a symplectic manifold and construct a variety (or rigid analytic space) $\check{X}$ over the (non-archimedian) Novikov field:

$$\Lambda := \left\{ \sum_{k=0}^{\infty} a_k T^{\lambda_k} | a_k \in \mathbb{C}, \lambda_k \in \mathbb{R}, \lambda_k \to \infty \right\}.$$

Then one can try to prove half of mirror symmetry, that is, to relate the symplectic geometry of X (Gromov-Witten invariants or Fukaya category) to the algebraic/analytic geometry of $\check{X}$.

In this approach, proposed by Fukaya [11] (see also [19]), one starts with a (possibly singular) SYZ fibration or, more generally, some "interesting" family of Lagrangians in X and constructs $\check{X}$ as the moduli space of objects in the Fukaya category supported on this family of Lagrangians. The fact that $Fuk(X)$ is a linear category over Λ is then the reason why $\check{X}$ is not a complex manifold. This approach comes with an additional benefit: using family Floer cohomology (introduced by Fukaya), one has a canonically defined functor from $Fuk(X)$ to the category of coherent sheaves (or matrix factorizations) on $\check{X}$. This construction has been carried out for the case of smooth fibrations by Abouzaid [2].

In this note, we will illustrate these ideas in an example, first studied by Cho-Hong-Lau [9], which is as simple as possible: the family of Lagrangians consists of a single Lagrangian. The mirror will then be a Landau-Ginzburg model $(\check{X}, W)$ where $\check{X}$ is an affine space.

2 Orbifold spheres

2.1 Our example

Let $X := \mathbb{P}^1_{a,b,c}$ be an orbifold sphere with three orbifolds points with isotropy groups $\mathbb{Z}/a$, $\mathbb{Z}/b$, $\mathbb{Z}/c$, where $a, b, c \geq 2$. The orbifold Euler characteristic is given by $\chi\left(\mathbb{P}^1_{a,b,c}\right) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1$. The orbifold $\mathbb{P}^1_{a,b,c}$ can be constructed as a global quotient of a Riemann surface Σ by a finite group. If $\chi\left(\mathbb{P}^1_{a,b,c}\right) > 0$ then Σ is a sphere, if $\chi\left(\mathbb{P}^1_{a,b,c}\right) = 0$ then Σ is an elliptic curve and in the other cases Σ is a surface of genus ≥ 2. For example, $\mathbb{P}^1_{3,3,3} = E/(\mathbb{Z}/3)$, where E is an elliptic curve with a $\mathbb{Z}/3$ symmetry.

We now introduce our Lagrangian: $\mathbb{L}$ which we call the Seidel Lagrangian, since it first appeared in [22]. This is an immersed circle $\mathbb{S}^1 \looparrowright \mathbb{P}^1_{a,b,c}$ with three transversal (double) self-intersections (see Figure 1). The three immersed points lie in the equator, determined by the three orbifold points. Moreover we assume that the image of $\mathbb{L}$ is invariant under reflection on the equator. The image of $\mathbb{L}$ and the equator divide the sphere into eight regions: two triangles and six bigons. We take $\mathbb{L}$ and scale the symplectic form so that each of these regions has area 1.

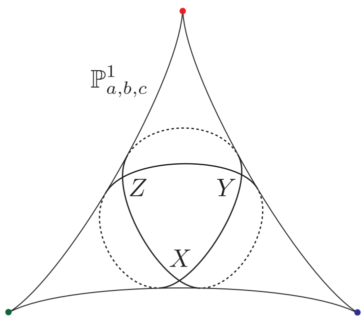

Figura 1: The orbifold sphere

2.2 The Fukaya algebra

We will start with a sketch of the construction of the Fukaya algebra of a Lagrangian submanifold. In fact, there is a family of Fukaya algebras parametrized by $H^*_{orb}(X)$ the orbifold cohomology of X. The orbifold cohomology is the singular cohomology of $\mathcal{I}X$ the inertia orbifold of X. In our example, we have

$$\mathcal{I}X = S^2 \bigcup_{i=1}^{a-1} pt \bigcup_{j=1}^{b-1} pt \bigcup_{k=1}^{c-1} pt,$$

that is a copy of X (which as a topological space is a sphere) and one point for each non-trivial element in the isotropy groups of the three orbifold points in X. We define $H^*_{orb}(X) := H^*(\mathcal{I}X, \Lambda)$. The Novikov field Λ has a real valuation $\nu : \Lambda \to \mathbb{R}$, given by the lowest power of T. Therefore $H^*_{orb}(X)$ inherits a valuation ν. We fix $\tau \in H^*_{orb}(X)$ with $\nu(\tau) > 0$ and define the Fukaya algebra $\mathcal{F}_\tau(\mathbb{L})$.

Consider the fiber product $\mathbb{L} \times_X \mathbb{L}$, which in our example is

$$\mathbb{L} \times_X \mathbb{L} = S^1 \bigcup_{p=X,Y,Z} (p \cup p^-),$$

where X, Y, Z are the self-intersections of $\mathbb{L}$. We define $\mathcal{F}_\tau(\mathbb{L}) := \Omega^*(\mathbb{L} \times_X \mathbb{L})\hat{\otimes}\Lambda$, where $\hat{\otimes}$ is the completed tensor product, with respect to the valuation induced by ν. More concretely, $\mathcal{F}_\tau(\mathbb{L})$ consists of the de Rham complex of a circle plus two generators (one even, one odd) for each of the self-intersection points. We will now equip this space with a sequence of operations $\mathfrak{m}_k$ of arity $k \geq 0$.

Let Σ be a orbifold which is topologically the closed unit disk in $\mathbb{C}$ and whose orbifold points lie in the interior. We take $k+1$ cyclically ordered marked points $z_0, \ldots, z_k$ on the boundary of Σ and m marked points $w_1, \ldots, w_m$ in the interior of Σ. We assume that each orbifold point is one of the w_j. Then we consider holomorphic maps $u : (\Sigma, \partial\Sigma) \to (X, \mathbb{L})$, with boundary on $\mathbb{L}$, in a fixed relative homology class $\beta \in H_2(X, \mathbb{L})$. We put a few more technical conditions on these maps, which in particular imply: 1) the restriction of u to the boundary can only switch branches at self-intersections of $\mathbb{L}$ at one of the z_i's; 2) orbifold points in Σ are mapped to orbifold points in X, (see [8] for details). Then we consider the space of tuples $(\Sigma, z_0, \ldots, z_k, w_1, \ldots, w_m, u)$ modulo complex automorphisms of the domain. This space can be compactified by, roughly speaking, allowing the domain of the map Σ to be a nodal disk, meaning a configuration of several disks and spheres attached at nodal points. For details see [13] for the manifold case and [7, 8] for the orbifold case. This is called the *stable map compactification* and we denote the resulting space by $\mathcal{M}_{k+1,m}(\beta)$. It follows from the work of Fukaya-Oh-Ohta-Ono [13] that the space $\mathcal{M}_{k+1,m}(\beta)$ is a compact Kuranishi space with boundary and corners. The definition of Kuranishi space is rather involved, we will use it as a black box to mean a space where we can pull-back and push-forward differential forms and the Stokes theorem works, in the same way as for manifolds.

It follows from the definition that these spaces have evaluation maps

$$\mathcal{I}X \xleftarrow{ev_{w_j}} \mathcal{M}_{k+1,m}(\beta) \xrightarrow{ev_{z_i}} \mathbb{L} \times_X \mathbb{L}.$$

For example, $ev_{z_i}(\Sigma, z_0, \ldots, z_k, w_1, \ldots, w_m, u) = u(z_i)$. We are now ready to define the A_∞ maps $\mathfrak{m}_k^\tau : \mathcal{F}_\tau(\mathbb{L})^{\otimes k} \to \mathcal{F}_\tau(\mathbb{L})$ by the formula

$$\mathfrak{m}_k^\tau(h_1, \ldots, h_k) = \sum_{\beta, m \geq 0} \frac{T^{\omega(\beta)}}{m!} (ev_{z_0})_*(ev_{w_1}^* \tau \wedge \ldots \quad \wedge ev_{w_m}^* \tau \wedge ev_{z_1}^* h_1 \wedge \ldots \wedge ev_{z_k}^* h_k).$$

The following theorem follows from the work of Fukaya–Oh–Ohta–Ono [12, 13] and further generalizations by Akaho–Joyce [3] and Cho–Poddar [8].

Theorem 2.1 *$\mathcal{F}(\mathbb{L})$ with the operations $\{\mathfrak{m}_k^\tau\}_{k\geq 0}$ is a filtered A_∞-algebra.*

What is a filtered A_∞-algebra? Let's define this.

Definition 2.2 *A filtered A_∞-algebra is a $\mathbb{Z}/2$-graded Λ-vector space $\mathcal{A}$ of the form $\mathcal{A} = A_0 \hat{\otimes} \Lambda$, where A_0 is a complex vector space. There are maps $\mathfrak{m}_k : \mathcal{A}^{\otimes k} \to \mathcal{A}$ of degree $k \pmod 2$, for each $k \geq 0$, satisfying*

$$\sum_{\substack{0\leq j\leq n \\ 0\leq i\leq n-j}} (-1)^{|a_1|+\ldots+|a_i|+i} \mathfrak{m}_{n-j+1}(a_1, \ldots, \mathfrak{m}_j(a_{i+1}, \ldots, a_{i+j}), \ldots, a_n) = 0.$$

Moreover $\nu(\mathfrak{m}_k(a_1, \ldots, a_k)) \geq \sum_i \nu(a_i)$ and $\nu(\mathfrak{m}_0) > 0$. We will also require that the A_∞-algebra is unital: there is an even element $\mathbb{1}$ satisfying:

$$\mathfrak{m}_2(\mathbb{1}, a) = (-1)^{|a|}\mathfrak{m}_2(a, \mathbb{1}) = a, \quad , \mathfrak{m}_k(\ldots, \mathbb{1}, \ldots) = 0, \ k \neq 2.$$

If we stare at the A_∞ equation above for $n = 1, 2, 3$, we can easily see that when $\mathfrak{m}_0$ is a multiple of the unit $\mathbb{1}$, then $\mathfrak{m}_1$ is a differential and so we can consider the cohomology of the A_∞-algebra. Moreover $\mathfrak{m}_2$ then defines an associative product on the cohomology. In general, filtered A_∞-algebras are rather complicated objects, so when we have to work with one we try to deform it (when possible) to another where this condition holds. In order to do that we need to solve the *Maurer-Cartan* equation.

Definition 2.3 *A Maurer-Cartan element in $\mathcal{A}$ is an odd element b satisfying $\sum_{k\geq 0} \mathfrak{m}_k(b, \ldots, b) = \lambda \mathbb{1}$, for some $\lambda \in \Lambda$, called the potential of b.*

Note that the sum on the left hand side of the equation is in general an infinite sum, so we need to ensure convergence. The safest way to do this is to require that $\nu(b) > 0$, but as we will see in our example, this can sometimes be relaxed. Given a Maurer-Cartan element we can define a new A_∞ structure on $\mathcal{A}$ by setting

$$\mathfrak{m}_k^b(a_1, \ldots, a_k) := \sum_{i_0,\ldots,i_k} \mathfrak{m}_{k+i_0+\ldots+i_k}(b, \ldots, b, a_1, b, \ldots, b, a_k, b, \ldots, b).$$

By construction $\mathfrak{m}_0^b = \lambda \mathbb{1}$. Maurer-Cartan elements for the Fukaya algebra $\mathcal{F}_\tau(L)$ of a Lagrangian L are called bounding cochains. Objects in the Fukaya category $Fuk(X, \tau) := \oplus_\lambda Fuk_\lambda(X, \tau)$ are pairs (L, b) where L is a Lagrangian and b is a Maurer-Cartan element in $\mathcal{F}_\tau(L)$ with potential λ. The endomorphism space of the object (L, b) is then $H^*(\mathcal{F}_\tau(L), \mathfrak{m}_1^{\tau,b})$.

3 The mirror

3.1 Potential

Like we promised in the introduction, we will construct the mirror to $X = \mathbb{P}^1_{a,b,c}$ as the moduli space of objects in the Fukaya category of X supported on the Seidel Lagrangian $\mathbb{L}$. More precisely we will construct a mirror for the pair $(\mathbb{P}^1_{a,b,c}, \tau)$, where $\tau \in H^*_{orb}(X)$. As explained in the previous section, the moduli space of these objects is exactly the space of Maurer-Cartan elements in $\mathcal{F}_\tau(\mathbb{L})$. In [4] we prove the following proposition.

Proposition 3.1 *Let X, Y, Z be the odd generators of $\mathcal{F}_\tau(\mathbb{L})$ corresponding to the three self-intersections. All elements of the form $b = T^{-3}(xX + yY + zZ)$, for elements $x, y, z \in \Lambda$ of non-negative valuation, are Maurer-Cartan elements with potential $W_\tau(x, y, z)$.*

At this point $W_\tau(x, y, z)$ is just a formal series on x, y, z, but in fact it is convergent in the following (non-archimedian) sense.

Definition 3.2 *A convergent power series is an expression of the form $\sum_{i,j,k\in\mathbb{Z}_{\geq 0}} c_{i,j,k} x^i y^j z^k$, with $c_{i,j,k} \in \Lambda$ and $\lim_{i+j+k\to\infty} \nu(c_{i,j,k}) = +\infty$. The set of all convergent power series naturally forms a ring which we denote by $\Lambda\langle\langle x, y, z\rangle\rangle$.*

Let us explain the terminology here. We can define a non-archimedian norm on Λ by setting $|v| := e^{-\nu(v)}$. Then one can see that $\Lambda\langle\langle x, y, z\rangle\rangle$ is exactly the ring of regular functions on the unit polydisk, see [5].

Proposition 3.3 ([4]) *W_τ is a convergent power series. Moreover*

$$W_\tau = T^{-8}xyz + x^a + y^b + z^c + \textit{positive valuation in } T.$$

It turns out that when $\chi(\mathbb{P}^1_{a,b,c}) \geq 0$, W_τ is actually a polynomial. An explicit description of W_τ for arbitrary τ seems out of reach, but for our purposes knowing the leading order term in the above proposition is enough.

We are finally ready to define the mirror partner to $\mathbb{P}^1_{a,b,c}$.

Definition 3.4 *The mirror to $(\mathbb{P}^1_{a,b,c}, \tau)$ is the Landau-Ginzburg model*

$$\check{X} = \mathcal{B} = \{(x,y,z), |x|, |y|, |z| \leq 1\} \subseteq \Lambda^3, \quad W_\tau : \mathcal{B} \to \Lambda.$$

One might ask why this is the correct mirror. Even assuming our philosophy that the mirror should be given as the moduli of objects in the Fukaya category supported in a certain family of Lagrangians in X, why is the Seidel Lagrangian the correct family? And even assuming that, how do we know we have "enough" bounding cochains? I don't believe there is a completely satisfactory answer to these questions. The short answer is that it works, meaning the closed-string mirror symmetry conjecture, that we will state in the next subsection, holds for this pair. Once we have established closed mirror symmetry, Abouzaid's generation criterion [1] tells us that, loosely speaking, our family of objects of the Fukaya category is "large" enough and therefore we have constructed the correct mirror and should expect open mirror symmetry to also hold for this pair.

3.2 Closed mirror symmetry

The closed mirror symmetry conjecture is an isomorphism of Frobenius manifolds. We will not define Frobenius manifold (see [20] for the complete definition), instead we will work at a more elementary level and consider it as a family of commutative algebras with a compatible inner product. In our situation, the families (on both sides of the mirror) are parameterized by $\tau \in H^*_{orb}(X)$.

On the symplectic side, the Frobenius manifold is the orbifold quantum cohomology, defined by Chen-Ruan [7]. The construction is similar to the construction of the Fukaya algebra. For each homology class $\alpha \in H_2(X)$, one constructs $\mathcal{M}^{sph}_{\ell+3}(\alpha)$ the moduli space of stable holomorphic *orbi-spheres* in X with $\ell + 3$ marked points $w_1, \ldots, w_{\ell+3}$. Then we fix τ as before and define a product $\bullet_\tau$ on $H^*(\mathcal{I}X, \Lambda)$ as follows

$$A \bullet_\tau B := \sum_{\alpha,\ell\geq 0} \frac{T^{\omega(\alpha)}}{\ell!}(ev_{w_1})_*(ev^*_{w_2}A \wedge ev^*_{w_3}B \wedge ev^*_{w_4}\tau \wedge \ldots \wedge ev^*_{w_{\ell+3}}\tau).$$

Theorem 3.5 ([7]) *The map $\bullet_\tau$ defines a commutative, associative product on $H^*_{orb}(X, \Lambda)$, compatible with the Poincaré pairing. We denote it by $QH^*_{orb}(X, \bullet_\tau)$.*

On the mirror, things are somewhat easier to define.

Definition 3.6 *The Jacobian of W_τ is the ring obtained by taking the quotient of $\Lambda\langle\langle x, y, z\rangle\rangle$ by the ideal generated by the partial derivatives of W_τ.*

$$Jac(W_\tau) = \frac{\Lambda\langle\langle x, y, z\rangle\rangle}{< \partial_x W_\tau, \partial_y W_\tau, \partial_z W_\tau >}.$$

In order to define an inner product in $Jac(W_\tau)$, one has to fix a volume form and then take the residue pairing. This is related to the choice of a primitive form as defined by Saito [21].

There is a natural map $\mathsf{KS}_\tau : QH^*(X, \bullet_\tau) \to Jac(W_\tau)$, called the Kodaira-Spencer map. We fix a basis $\{e_i\}_i$ of $H^*(\mathcal{I}X, \Lambda)$ and write $\tau = \sum_i \tau_i e_i$. We define the map by the formula $\mathsf{KS}_\tau(e_i) = \dfrac{\partial}{\partial \tau_i} W_\tau$.

This map was originally constructed by Fukaya–Oh–Ohta–Ono [14] for toric manifolds. They show that this is a well-defined, unital ring map. In fact, this is expected to be the case for a very wide class of symplectic manifolds/orbifolds. In [4], we extend their construction to our example and prove the following.

Theorem 3.7 *The Kodaira-Spencer $\mathsf{KS}_\tau : QH^*(X, \bullet_\tau) \to Jac(W_\tau)$ is an unital, ring isomorphism.*

This theorem is not the complete closed mirror symmetry statement. The full-fledged statement requires an identification of the Euler vector fields, which we prove in [4]:

$$\mathsf{KS}_\tau \left(c_1(X) + \sum_i (1 - \frac{\deg e_i}{2})\tau_i e_i \right) = [W_\tau].$$

Moreover the Kodaira-Spencer should intertwine the Poincaré pairing with the residue pairing on $Jac(W_\tau)$ determined by some volume form ω_τ. A complete description of ω_τ is still work in progress by Cho, Hong, Lau and myself.

3.3 Open mirror symmetry

The open (or homological) mirror symmetry conjecture in this example asserts that the derived categories of $Fuk(X, \tau)$ and $MF(W_\tau)$ the category of matrix factorizations of W_τ are equivalent. The matrix factorizations category is a dg-category, which captures some information about the singularities of W_τ. We refer the reader to [10] for the definition.

One of the main advantages of this formalism, is that $\mathbb{L}$ determines, for each τ, an A_∞-functor $\mathcal{M}^{\mathbb{L}} : Fuk(X, \tau) \to MF(W_\tau)$. This is a version of the Yoneda embedding, see [9] for a full description. We expect the following to hold.

Conjecture 3.8 *The functor $\mathcal{M}^{\mathbb{L}}$ induces an equivalence*

$$D^\pi Fuk_\lambda(X, \tau) \to D^\pi MF(W_\tau),$$

where D^π stands for the split-closed derived category.

This conjecture was proved in some cases in [9] when $\tau = 0$ and is work in progress by Cho, Hong, Lau and myself. But we are not that far off from proving this. First note that the closed mirror symmetry statement that we saw in the previous subsection implies that $Jac(W_\tau)$ is finite dimensional, which implies that the critical points of W_τ are isolated. It then follows from Dyckerhoff [10] that $MF(W_\tau)$ has finitely many generators P^η, one for each critical point $\eta \in Crit(W_\tau)$. We prove in [4] that each η also determines a bounding cochain b_η for $\mathbb{L}$. Therefore it is not unreasonable to expect that $\mathcal{M}^{\mathbb{L}}$ sends the object $(\mathbb{L}, b_\eta)$ to P^η and it is fully faithful (on cohomology) when restricted to these objects.

Assuming we can prove this, the only thing left to show is that the objects $(\mathbb{L}, b_\eta)$ split-generate the Fukaya category. This should follow from a suitable generalization of Abouzaid's generation criterion [1]. Let's explain how this criterion works. Let $\mathcal{A}$ be the subcategory of $Fuk(X, \tau)$ generated by the $(\mathbb{L}, b_\eta)$. There is a ring map

$$\mathcal{CO} : HH^*(\mathcal{A}) \to QH^*(X, \bullet_\tau),$$

whose domain is the Hochschild cohomology of $\mathcal{A}$. The generation criterion asserts that if the map $\mathcal{CO}$ is injective then $\mathcal{A}$ is derived equivalent to $Fuk(X, \tau)$. The reason this should hold in our example is the following. We expect the Hochschild cohomology $HH^*(\mathcal{A})$ to be isomorphic to the Jacobian $Jac(W_\tau)$, and under this isomorphism, the map $\mathcal{CO}$ should agree with the Kodaira-Spencer map. Therefore the condition needed for the generation criterion follows from the fact that KS is an isomorphism, in other words, it follows from closed mirror symmetry.

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