

RESOLUTIONS OF SURFACES WITH BIG COTANGENT BUNDLE AND A_2 SINGULARITIES

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Resumo: Neste trabalho obtemos um novo critério para garantir que a resolução duma superfície de tipo geral com singularidades canônicas tem o fibrado cotangente grande, e um novo limite inferior para os valores de d para os quais existem superfícies lisas com o fibrado cotangente grande que são equivalentes por deformação a uma hipersuperfície lisa em \mathbb{P}^3 de grau d .

Abstract We give a new criterion for when a resolution of a surface of general type with canonical singularities has big cotangent bundle and a new lower bound for the values of d for which there is a surface with big cotangent bundle that is deformation equivalent to a smooth hypersurface in \mathbb{P}^3 of degree d .

palavras-chave: fibrado cotangente grande; superfícies algébricas de tipo geral; singularidades canônicas

keywords: big cotangent bundle; surfaces of general type; canonical singularities.

1 Introduction and general theory

Symmetric differentials, i.e. sections of the symmetric powers of the cotangent bundle $S^m\Omega_X^1$, of a projective manifold X play a role in obtaining hyperbolicity properties of X . Symmetric differentials give constraints on the existence of rational, elliptic and even entire curves in X (nonconstant holomorphic maps from \mathbb{C} to X), see for example [Dem15] and [Deb04].

The cotangent bundle of a projective manifold is said to be big if the order of growth of $h^0(X, S^m\Omega_X^1)$ with m is maximal (i.e., $= 2 \dim X - 1$). The work of Bogomolov [Bog77] and McQuillan [McQ98] gives that if a surface of general type has big Ω_X^1 , then X satisfies the Green-Griffiths-Lang conjecture, i.e., there exists a proper subvariety Z of X such that any entire curve is contained in Z .

Smooth hypersurfaces $X \subset \mathbb{P}^3$ with degree $d \geq 5$ have Ω_X^1 with strong positivity properties, such as K_X being ample, but they have trivial cotangent algebra [Brj71],

$$S(X) := \bigoplus_{m=0}^{\infty} H^0(X, S^m \Omega_X^1) = H^0(X, S^0 \Omega_X^1) = \mathbb{C}$$

see also [BDO08]. The absence of symmetric differentials on smooth hypersurfaces of \mathbb{P}^3 a priori prevents them from playing a role in obtaining hyperbolicity properties on smooth hypersurfaces of \mathbb{P}^3 .

Previous work of the 1st author and Bogomolov [BDO06] showed that there are smooth surfaces X with big Ω_X^1 that are deformation equivalent to smooth hypersurfaces in \mathbb{P}^3 . Hence symmetric differentials can still play a role in obtaining hyperbolicity properties for hypersurfaces of \mathbb{P}^3 . In [BDO06] it was shown that there are nodal hypersurfaces $X \subset \mathbb{P}^3$ whose resolutions \tilde{X} have big cotangent bundle. The simultaneous resolution result of Brieskorn [Bri70] implies that minimal resolutions \tilde{X} of hypersurfaces $X \subset \mathbb{P}^3$ with only rational double points, i.e. canonical singularities, are deformation equivalent to smooth hypersurfaces of the same degree.

The results in this presentation are:

Theorem 1. *Let X be a surface of general type with canonical singularities. Then the minimal resolution \tilde{X} of X has big cotangent bundle if*

$$\sum_{x \in \text{Sing} X} h^1(x) \geq -\frac{s_2(\tilde{X})}{3!}$$

See (2.1) for the definition of $h^1(x)$, it is an invariant of the singularity. Note that the left side encodes only information about the germs of the singularities of X , so it is local in nature. This result is stronger than the result in [RR14] stating that Ω_X^1 is big if $s_2(\tilde{X}) + s_2(\mathcal{X}) > 0$, $s_2(\tilde{X})$ and $s_2(\mathcal{X})$ respectively the 2nd Segre number of \tilde{X} and of the orbifold \mathcal{X} associated to X , see section 2 for more details.

In section 2.2 we give a method to find $h^1(x)$ where (X, x) is the germ of an A_2 -singularity. In a later work [DOW20] we show how to extend this method to calculate $h^1(x)$ for other A_n singularities. Then using theorem 1 and information on the possible number of canonical singularities of prescribed types allowed in a hypersurface $X \subset \mathbb{P}^3$ of degree d , we obtain

Theorem 2. *For $d = 9$ and $d \geq 11$, there are minimal resolutions of hypersurfaces $X \subset \mathbb{P}^3$ with canonical singularities and degree d which have big cotangent bundle.*

The condition $s_2(\tilde{X}) + s_2(\mathcal{X}) > 0$ of [RR14] gives only $d \geq 13$ and there nodes are the best singularities. The above theorem uses A_2 singularities which due to theorem 1 are unexpectedly better than nodes, see 2.2 for more details.

1.1 Big Cotangent Bundle

The cotangent bundle Ω_X^1 on a complex manifold of dimension n is said to be big if

$$\lim_{m \rightarrow \infty} \frac{h^0(X, S^m \Omega_X^1)}{m^{2n-1}} \neq 0$$

(i.e., $h^0(X, S^m \Omega_X^1)$ has the maximal growth order possible with respect to m for $\dim X = n$). The property of Ω_X^1 being big is birational.

In the case of surfaces of general type there is a topologically sufficient condition for bigness of Ω_X^1 , $s_2(X) > 0$, where $s_2(X) = c_1^2(X) - c_2(X)$ is the 2nd Segre number of X . This follows from the asymptotic Riemann-Roch theorem for symmetric powers of Ω_X^1 :

$$h^0(X, S^m \Omega_X^1) - h^1(X, S^m \Omega_X^1) + h^2(X, S^m \Omega_X^1) = \frac{s_2(X)}{3!} m^3 + O(m^2) \quad (1.1)$$

and Bogomolov's vanishing for surfaces of general type, $h^2(X, S^m \Omega_X^1) = 0$ for $m > 2$ [Bog79].

Very few examples of minimal surfaces with $s_2(X) \leq 0$ are known to have Ω_X^1 big, they appear in [BDO06] and [RR14]. In these examples, bigness of Ω_X^1 follows from complex analytic and not topological properties of X . The complex analytic conditions are the presence of enough configurations of (-2) -curves associated with canonical singularities. In fact, these surfaces with big Ω_X^1 are diffeomorphic to surfaces with trivial cotangent algebra, $S(X) \simeq \mathbb{C}$.

If X is a smooth surface of general type, it follows from 1.1 and $h^2(X, S^m \Omega_X^1) = 0$ that Ω_X^1 is big if and only if:

$$\lim_{m \rightarrow \infty} \frac{h^1(X, S^m \Omega_X^1)}{m^3} > -\frac{s_2(X)}{3!} \quad (1.2)$$

1.2 Quotient singularities and local asymptotic Riemann-Roch for orbifold $\hat{S}^m\Omega_X^1$

In this section we present the local asymptotic Riemann-Roch for the orbifold symmetric powers of the cotangent bundle of a normal surface with only quotient singularities. For references on this topic, see [Wah93], [Bla96], [Kaw92], [Miy08].

The germ of a normal surface singularity (X, x) is a quotient singularity germ if it is biholomorphic to $(\mathbb{C}^2, 0)/G_x$, with $G_x \subset GL_2(\mathbb{C})$ finite and small, where G_x is the local fundamental group. Canonical surface singularities are quotient singularities with $G_x \subset SL_2(\mathbb{C})$. Consider

$$\begin{array}{ccc}
 & & (\mathbb{C}^2, 0) \\
 & \swarrow \varphi & \downarrow \pi \\
 (\tilde{X}, E) & \xrightarrow{\sigma} & (X, x)
 \end{array}$$

with $\pi : (\mathbb{C}^2, 0) \rightarrow (X, x)$, the quotient map by the local fundamental group, called the local smoothing of (X, x) and $\sigma : (\tilde{X}, E) \rightarrow (X, x)$ a good resolution of (X, x) where (\tilde{X}, E) is the germ of a neighborhood of the exceptional locus E with E consisting of smooth curves intersecting transversally.

A reflexive coherent sheaf \mathcal{F} , i.e. $\mathcal{F}^{\vee\vee} = \mathcal{F}$, on (X, x) is a locally free sheaf away from the singularity and satisfies $\mathcal{F} = i_*(\mathcal{F}|_{X \setminus \{x\}})$, $i : X \setminus \{x\} \hookrightarrow X$. Associated to a reflexive sheaf \mathcal{F} on the quotient surface germ (X, x) there are locally free sheaves $\tilde{\mathcal{F}}$ on (\tilde{X}, E) (not uniquely determined) and $\hat{\mathcal{F}}$ on $(\mathbb{C}^2, 0)$ (uniquely determined) satisfying $\mathcal{F} \cong (\sigma_*\tilde{\mathcal{F}})^{\vee\vee} \cong (\pi_*^{G_x}\hat{\mathcal{F}})$, where $(\pi_*^{G_x}\hat{\mathcal{F}})$ is a maximal subsheaf of $\pi_*\hat{\mathcal{F}}$ on which G_x acts trivially, ([Bla96] section 2).

The previous paragraph implies that reflexive coherent sheaves on normal surfaces with only quotient singularities X are orbifold vector bundles on X (also called \mathbb{Q} -vector bundles or locally V -free bundles over X). The orbifold m -symmetric power of the cotangent bundle on a normal surface X with only quotient singularities is $\hat{S}^m\Omega_X^1 := (S^m\Omega_X^1)^{\vee\vee}$ with $\Omega_X^1 = i_*(\Omega_{X_{reg}}^1)$. If $\tilde{X} \xrightarrow{\sigma} X$ is a good resolution $\hat{S}^m\Omega_X^1 = (\sigma_*S^m\Omega_{\tilde{X}}^1)^{\vee\vee}$.

In the proof of theorem 1 a lower bound for $h^1(\tilde{X}, S^m\Omega_{\tilde{X}}^1)$ is given using only information on the singularities of X . Each x_i contributes with $h^1(\tilde{U}_{x_i}, S^m\Omega_{\tilde{X}}^1)$ where \tilde{U}_{x_i} is the minimal resolution of an affine neighborhood U_{x_i} of x_i with $U_{x_i} \cap \text{Sing}(X) = \{x_i\}$. The bigness of Ω_X^1 depends on the asymptotics of $h^1(\tilde{X}, S^m\Omega_{\tilde{X}}^1)$, see section (1.2), and hence on the combined asymptotics of the $h^1(\tilde{U}_{x_i}, S^m\Omega_{\tilde{X}}^1)$.

Let $(\tilde{X}, E) \xrightarrow{\sigma} (X, x)$ be a good resolution of the germ of a quotient surface singularity and $\tilde{\mathcal{F}}, \mathcal{F}$ be sheaves such that $\tilde{\mathcal{F}}$ is locally free of rank r on \tilde{X} and $\mathcal{F} = (\sigma_* \tilde{\mathcal{F}})^{\vee\vee}$ a reflexive sheaf on X . In comparing the Euler characteristics $\chi(X, \mathcal{F})$ and $\chi(\tilde{X}, \tilde{\mathcal{F}})$ one has $\chi(X, \mathcal{F}) = \chi(\tilde{X}, \tilde{\mathcal{F}}) + \chi(x, \tilde{\mathcal{F}})$ with

$$\chi(x, \tilde{\mathcal{F}}) = \dim(H^0(\tilde{X} \setminus E, \tilde{\mathcal{F}})/H^0(\tilde{X}, \tilde{\mathcal{F}})) + h^1(\tilde{X}, \tilde{\mathcal{F}}) \quad (1.3)$$

called the modified Euler characteristic of $\tilde{\mathcal{F}}$ ([Wah93], [Bla96] 3.9). The asymptotics of 1.3 are described via a local asymptotic Riemann-Roch theorem ([Bla96] 4.1)

$$\lim_{m \rightarrow \infty} \frac{\chi(x, S^k \tilde{\mathcal{F}})}{m^{2+r-1}} = -\frac{1}{(2+r-1)!} s_2(x, \tilde{\mathcal{F}}) \quad (1.4)$$

with $s_2(x, \tilde{\mathcal{F}}) := c_1^2(x, \tilde{\mathcal{F}}) - c_2(x, \tilde{\mathcal{F}})$, the local 2nd Segre number of $\tilde{\mathcal{F}}$ and $c_i(x, \tilde{\mathcal{F}}) \in H_{dRc}^{2i}(\tilde{X}, \mathbb{C})$ the i -th local Chern class of $\tilde{\mathcal{F}}$. The local Chern classes appear when comparing the pullback of orbifold Chern classes of an orbifold vector bundle \mathcal{F} on an orbifold X and the Chern classes of the vector bundle $\tilde{\mathcal{F}}$ on \tilde{X} , a good resolution $\sigma : \tilde{X} \rightarrow X$ of X , satisfying $\mathcal{F} = (\sigma_* \tilde{\mathcal{F}})^{\vee\vee}$.

We are only concerned with good resolutions $\sigma : (\tilde{X}, E) \rightarrow (X, x)$ of canonical surface singularities and $\tilde{\mathcal{F}} = \Omega_{\tilde{X}}^1$, one has $c_1^2(x, \Omega_{\tilde{X}}^1) = 0$ and:

$$s_2(x, \Omega_{\tilde{X}}^1) = -c_2(x, \Omega_{\tilde{X}}^1) = -(e(E) - \frac{1}{|G_x|}) \quad (1.5)$$

with $e(E)$ the topological Euler characteristic of the exceptional locus and $|G_x|$ the order of the local fundamental group ([Bla96] 3.18). We will use the invariant of the singularity:

$$s_2(x, X) := s_2(x, \Omega_{\tilde{X}_{min}}^1) \quad (1.6)$$

where $\sigma : (\tilde{X}_{min}, E) \rightarrow (X, x)$ is the minimal good resolution.

2 Theorems

2.1 Resolutions with big cotangent bundle

We consider minimal resolutions $\sigma : \tilde{X} \rightarrow X$ of normal surfaces X with only canonical singularities. The minimality condition has several advantages: i) the local 2nd Segre numbers $s_2(x, \Omega_{\tilde{X}}^1)$ being considered are $s_2(x, X)$ which depend only on the singularity (since the resolution is fixed); ii) in section 2.2 the simultaneous resolution results used involve minimal resolutions of

canonical singularities. Also, blowing up $b : \hat{X} \rightarrow X$ a smooth surface X at a point does not affect inequality (1.2) determining bigness of the cotangent bundle, since

$$\lim_{m \rightarrow \infty} \frac{h^1(\hat{X}, S^m \Omega_{\hat{X}}^1)}{m^3} + \frac{s_2(\hat{X})}{3!} = \lim_{m \rightarrow \infty} \frac{h^1(X, S^m \Omega_X^1)}{m^3} + \frac{s_2(X)}{3!}$$

Let $\sigma : \tilde{U}_x \rightarrow U_x$ be the minimal resolution of an affine normal surface U_x with a single canonical singularity at the point $x \in U_x$. Set:

$$h^1(x) := \lim_{m \rightarrow \infty} \frac{h^1(\tilde{U}_x, S^m \Omega_{\tilde{X}}^1)}{m^3} \tag{2.1}$$

$$h^0(x) := \lim_{m \rightarrow \infty} \frac{[H^0(\tilde{U}_x \setminus E, S^m \Omega_{\tilde{U}_x}^1) / H^0(\tilde{U}_x, S^m \Omega_{\tilde{U}_x}^1)]}{m^3} \tag{2.2}$$

The local asymptotic Riemann-Roch equation (1.4) for the local modified Euler characteristic (1.3) for \tilde{U}_x and $S^m \Omega_{\tilde{U}_x}^1$ gives:

$$h^1(x) = -\frac{1}{3!} s_2(x, X) - h^0(x). \tag{2.3}$$

with $s_2(x, \Omega_{\tilde{U}_x}^1)$ an invariant of the canonical singularity (U_x, x) , since \tilde{U}_x is its minimal resolution (and hence unique). In [DOW20] using local duality and local cohomology for the pair (\tilde{X}, E) , it is shown that $h^0(x) \leq h^1(X)$ holds, hence:

$$h^1(x) \geq -\frac{s_2(x, X)}{2 \cdot 3!} \tag{2.4}$$

Theorem 1. *Let X be a normal projective surface of general type with only canonical singularities and $\sigma : \tilde{X} \rightarrow X$ a minimal resolution. Then $\Omega_{\tilde{X}}^1$ is big if and only if:*

$$\sum_{x \in \text{Sing} X} h^1(x) \geq -\frac{s_2(\tilde{X})}{3!} \tag{2.5}$$

Proof. We saw in section 1.1 that $\Omega_{\tilde{X}}^1$ is big if and only if $\lim_{m \rightarrow \infty} \frac{h^1(\tilde{X}, S^m \Omega_{\tilde{X}}^1)}{m^3} > -\frac{s_2(\tilde{X})}{3!}$.

From the Leray spectral sequence for σ_* and Bogomolov’s vanishing $H^2(\tilde{X}, S^m \Omega_{\tilde{X}}^1) = 0$ for $m > 2$, we obtain for $m > 2$:

$$\begin{aligned} 0 \longrightarrow H^1(X, \sigma_* S^m \Omega_{\tilde{X}}^1) &\longrightarrow H^1(\tilde{X}, S^m \Omega_{\tilde{X}}^1) \longrightarrow H^0(X, R^1 \sigma_* S^m \Omega_{\tilde{X}}^1) \\ &\longrightarrow H^2(X, \sigma_* S^m \Omega_{\tilde{X}}^1) \longrightarrow 0 \end{aligned} \tag{2.6}$$

The 1st direct image sheaf $R^1\sigma_*S^m\Omega_{\tilde{X}}^1$ has support on the zero-dimensional singularity locus $\text{Sing}(X) = \{x_1, \dots, x_k\}$ of X . Each x_i has an affine neighborhood U_{x_i} such that $U_{x_i} \cap \text{Sing}(X) = \{x_i\}$. Using the Leray spectral sequence again for each $\tilde{U}_x = \sigma^{-1}(U_x)$, $\sigma : \tilde{U}_x \rightarrow U_{x_i}$ we obtain:

$$H^0\left(X, R^1\sigma_*S^m\Omega_{\tilde{X}}^1\right) = \bigoplus_{i=1}^k H^1\left(\tilde{U}_x, S^m\Omega_{\tilde{U}_x}^1\right)$$

Hence using the notation of section 2.1:

$$\sum_{x \in \text{Sing}(X)} h^1(x) = \lim_{m \rightarrow \infty} \frac{h^0\left(X, R^1\sigma_*S^m\Omega_{\tilde{X}}^1\right)}{m^3} \quad (2.7)$$

Claim: $H^2(X, \sigma_*S^m\Omega_{\tilde{X}}^1) = 0$

Proof. Recalling that $\hat{S}^m\Omega_{\tilde{X}}^1 := (\sigma_*S^m\Omega_{\tilde{X}}^1)^{\vee\vee}$, consider the short exact sequence:

$$0 \rightarrow \sigma_*S^m\Omega_{\tilde{X}}^1 \rightarrow \hat{S}^m\Omega_{\tilde{X}}^1 \rightarrow Q_m \rightarrow 0.$$

Left injectivity holds since $\sigma_*S^m\Omega_{\tilde{X}}^1$ is torsion free. The support of $Q_m = \frac{(\sigma_*S^m\Omega_{\tilde{X}}^1)^{\vee\vee}}{\sigma_*S^m\Omega_{\tilde{X}}^1}$ is again $\text{Sing}(X)$, hence $H^2(X, \sigma_*S^m\Omega_{\tilde{X}}^1) \cong H^2(X, \hat{S}^m\Omega_{\tilde{X}}^1)$.

The surface X is an orbifold surface of general type with canonical singularities and $\hat{S}^m\Omega_{\tilde{X}}^1$ is the orbifold m -th symmetric power of the cotangent bundle of X . Bogomolov's vanishing $H^2(X, \hat{S}^m\Omega_{\tilde{X}}^1) = 0$ for $m > 2$ also holds in this setting, due to the existence of orbifold Kähler-Einstein metrics [Kob85], [TY86], see also [RR14]. \square

The vanishing of $H^2\left(X, \sigma_*S^m\Omega_{\tilde{X}}^1\right) = 0$ for $m > 0$, (2.6) and (2.7) give:

$$\lim_{m \rightarrow \infty} \frac{h^1(\tilde{X}, S^m\Omega_{\tilde{X}}^1)}{m^3} \geq \sum_{x \in \text{Sing}(X)} h^1(x) \quad (2.8)$$

and the result follows from (1.2). \square

Remark: Theorem 1 is stronger than the main theorem in [RR14] which states that $\Omega_{\tilde{X}}^1$ is big if $s_2(\tilde{X}) + s_2(X) > 0$. We have that ([Bla96] 3.14), $s_2(\tilde{X}) = s_2(X) + \sum_{x \in \text{Sing}X} s_2(x, X)$, hence condition $s_2(\tilde{X}) + s_2(X) > 0$ can be reexpressed as:

$$-\sum_{x \in \text{Sing} X} \frac{s_2(x, X)}{2} > -s_2(\tilde{X}) \tag{2.9}$$

It follows from (2.4) that the condition (2.5) in theorem 1 implies (2.9). In fact it gives much stronger results. In the next section we will show that if (X, x) is the germ of an A_2 singularity, then $h^1(x) = \frac{67}{216}$ while $-\frac{s_2(x, X)}{2 \cdot 3!} = \frac{48}{216}$. This implies that our inequality (2.5) guarantees $\Omega_{\tilde{X}}^1$ is big for surfaces of general type X with only $\frac{48}{67} \cdot \ell$ A_2 -singularities, where ℓ is the number needed to satisfy inequality (2.9).

2.2 Deformations of smooth hypersurfaces with big Ω_X^1

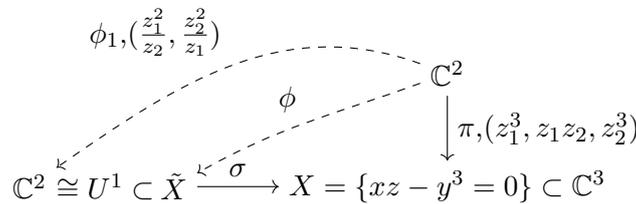
In this section we study for which d there are (smooth) surfaces with big cotangent bundle that are deformation equivalent to smooth hypersurfaces in \mathbb{P}^3 of degree d . We do this by considering minimal resolutions \tilde{X} of hypersurfaces $X \subset \mathbb{P}^3$ of degree d with only A_2 singularities. A simultaneous resolution result of Brieskorn [Bri70] gives that \tilde{X} is deformation equivalent to a smooth hypersurface of \mathbb{P}^3 of degree d . In [DOW20] other canonical singularities are also considered.

Proposition 2.1. *Let $\sigma : (\tilde{X}, E) \rightarrow (X, x)$ be the minimal resolution of the germ of an A_2 surface singularity. Then:*

$$h^0(x) := \lim_{m \rightarrow \infty} \frac{\dim[H^0(\tilde{X} \setminus E_i, S^m \Omega_{\tilde{X}}^1) / H^0(X, S^m \Omega_X^1)]}{m^3} = \frac{29}{216} \tag{2.10}$$

Proof. For the full proof see [DOW20].

We give here an extended description of what is involved in the proof. We use the affine model of an A_2 -singularity $X = \{xz - y^3 = 0\} \subset \mathbb{C}^3$ with the minimal resolution \tilde{X} obtained as the strict preimage of X under $\sigma : \hat{\mathbb{C}}^3 \rightarrow \mathbb{C}^3$, the blow up of \mathbb{C}^3 at $(0, 0, 0)$.



where $\pi : \mathbb{C}^2 \rightarrow X$ gives the smoothing as in section 1.2. Let $U^1 = \tilde{X} \cap p^{-1}(U_1)$ with $p : \hat{\mathbb{C}}^3 \rightarrow \mathbb{P}^2$ the canonical projection and $U_1 = \{y \neq 0\} \subset \mathbb{P}^2$,

$[x : y : z]$ as homogeneous coordinates of \mathbb{P}^2 . The exceptional locus of σ is $E = E_1 + E_2$, E_i (-2) -curves intersecting transversally. On U^1 put coordinates (u_1, u_2) with $\phi_1^*u_1 = \frac{z_1^2}{z_2}$ and $\phi_1^*u_2 = \frac{z_2^2}{z_1}$ and $E \cap U^1 = \{u_1u_2 = 0\}$.

The isomorphism $\phi^* : H^0(\tilde{X} \setminus E, S^m\Omega_{\tilde{X}}^1) \rightarrow H^0(\mathbb{C}^2, S^m\Omega_{\mathbb{C}^2}^1)^{\mathbb{Z}_3}$ will be used to move the setting for finding $h^0(x)$ from $\tilde{X} \setminus E$ to \mathbb{C}^2 . We need a good description of $G(m) := \phi^*(H^0(\tilde{X}, S^m\Omega_{\tilde{X}}^1))$. We use:

$$G(m) = \phi_1^*(H^0(\mathbb{C}^2, S^m\Omega_{\mathbb{C}^2}^1)) \cap H^0(\mathbb{C}^2, S^m\Omega_{\mathbb{C}^2}^1)$$

We call $z_1^{i_1}z_2^{i_2}dz_1^{m_1}dz_2^{m_2}$ a z -monomial of full type (f-type) $(i_1, i_2, m_1, m_2)_z$ and type $(i, m)_z$ with $i = i_1 + i_2$ the order and $m = m_1 + m_2$ the degree of the monomial. A monomial is holomorphic if $i_1, i_2 \geq 0$ and \mathbb{Z}_3 -invariant if $i_1 + 2i_2 + m_1 + 2m_2 \equiv 0 \pmod{3}$.

For each triple (k, i, m) with $k \equiv -(m + i) \pmod{3}$ there is a collection of z -monomials:

$$B(k, i, m)_z = \{(k - m + l, i + m - k - l, m - l, l)_z\}_{l=0, \dots, m} \quad (2.11)$$

These collections give a partition of the set of all \mathbb{Z}_3 -invariant z -monomials of type (i, m) . Set $V(k, i, m)_z = \text{Span}(B(k, i, m)_z)$.

Let $B_h(k, i, m)_z$ be the subcollection of holomorphic z -monomials of $B(k, i, m)_z$. Set $V_h(k, i, m)_z := \text{Span}(B_h(k, i, m)_z) = H^0(\mathbb{C}^2, S^m\Omega_{\mathbb{C}^2}^1) \cap V(k, i, m)$. Set $h_z(k, i, m) := \dim V_h(k, i, m)_z = \#B_h(k, i, m)_z$, from (2.11) it follows that $h_z(k, i, m) = \min(m + 1, k + 1, i + 1, m - k + i + 1)$. Note that $h_z(k, i, m) = 0$ unless $0 \leq k \leq m + i$.

Set $G(k, i, m) := G(m) \cap V(k, i, m) = G(m) \cap V_h(k, i, m)$. All the above gives (we will see below that $I(m) = 2m$):

$$\begin{aligned} \dim[H^0(\tilde{X} \setminus E, S^m\Omega_{\tilde{X}}^1)/H^0(X, S^m\Omega_X^1)] &= \dim[H^0(\mathbb{C}^2, S^m\Omega_{\mathbb{C}^2}^1)^{\mathbb{Z}_3}/G(m)] \\ &= \sum_{i=0}^{I(m)} \sum_{\substack{0 \leq k \leq m+i \\ k \equiv -(m+i) \pmod{3}}} h_z(k, i, m) - \dim G(k, i, m) \end{aligned} \quad (2.12)$$

The reason to consider the collections $B(k, i, m)$ will now be examined. The rational map $\phi_1 : (\mathbb{C}^2, z_1, z_2) \dashrightarrow (\mathbb{C}^2, u_1, u_2)$ pulls back holomorphic u -monomials of type (i, m) to rational \mathbb{Z}_3 -invariant z -monomials of type (i, m) :

$$\phi_1^*(p, i-p, q, m-q)_u = \sum_{l=0}^m c_{ql} (3(p+q) - (i+2m) + l, -3(p+q) + 2(i+m) - l, m-l, l)_z \quad (2.13)$$

with the c_{ql} given by $(2x - y)^q(-x + 2y)^{m-q} = \sum_l c_{ql}x^{m-l}y^l$.

From (2.13) and (2.11) it follows that the pullback of a u -monomial of type (i, m) lies in a single $V(k, i, m)$ and that the u -monomials whose pullback lie in $V(k, i, m)$ themselves form the collection $B(k, i, m)_u := \{(k' - m + l, i + m - k' - l, m - l, l)_u\}_{l=0, \dots, m}$ with $k' = \frac{i+m+k}{3}$. Let $B_h(k, i, m)_u$ be the subcollection of holomorphic u -monomials of $B(k, i, m)_u$ and set $V_h(k, i, m)_u = \text{Span}(B_h(k, i, m)_u)$. Set $h_u(k, i, m) := \dim V_h(k, i, m)_u$, we have $h_u(k, i, m) = \min(m + 1, \frac{k+(i+m)}{3} + 1, i + 1, \frac{2(i+m)-k}{3} + 1)$.

We proceed to find $I(m)$ and $\dim G(k, i, m)$ and calculate (2.12). We have that $G(k, i, m) = \phi_1^*(V_h(k, i, m)_u) \cap V_h(k, i, m)_z$. By using information on the rank of relevant subblocks of matrix $[c_{ql}]$, with c_{ql} as in (2.12) (see [DOW20] for details), we obtain that:

$$\dim G(k, i, m) = \max(h_z(k, i, m) + h_u(k, i, m) - (m + 1), 0)$$

From the formula for $h_u(k, i, m)$ above, it follows that $h_u(k, i, m) = m + 1$ and hence $G(k, i, m) = h_z(k, i, m)$ for all $0 \leq k \leq m + 1$ if $i \geq 2m$. This implies that all the terms in (2.12) for $i \geq 2m$ vanish, hence by setting $I(m) = 2m$ we can write the full sum and obtain:

$$h^0(x) = \lim_{m \rightarrow \infty} \frac{1}{m^3} \sum_{i=0}^{2m} \sum_{\substack{0 \leq k \leq m+i \\ k \equiv -(m+i) \pmod{3}}} \min(m+1-h_u(k, i, m), h_z(k, i, m)) = \frac{29}{216}$$

Remark: For A_1 singularities using the set up described in the 1st author's article [BDO08] the method to find $h^0(x)$ is substantially simpler and $h^0(x) = \frac{11}{108}$, see Jordan Thomas' thesis [Tho13]. For an approach in the lines of proposition 2.1 and valid for all A_n singularities see [DOW20]. □

Theorem 2. *For $d = 9$ and $d \geq 11$ there are minimal resolutions of hypersurfaces in \mathbb{P}^3 with canonical singularities and degree d which have big cotangent bundle.*

Proof. Let $X_{d,\ell} \subset \mathbb{P}^3$ denote a hypersurface of degree d with ℓ A_2 -singularities as its only singularities and $\tilde{X}_{d,\ell}$ its minimal resolution. The Brieskorn simultaneous resolution theorem, [Bri70] and Ehresmann's fibration theorem give that $\tilde{X}_{d,\ell}$ is diffeomorphic to a smooth hypersurface of degree d in \mathbb{P}^3 , hence $s_2(\tilde{X}_{d,\ell}) = -4d^2 + 10d$.

From sections 1.2 and 2.1 we have that $h^1(x) = -\frac{1}{3!} s_2(x, X) - h^0(x) = \frac{1}{3!} (e(E) - \frac{1}{|\mathbb{Z}_3|}) - h^0(x)$, where (\tilde{X}, E) is a minimal resolution of the germ of the A_2 -singularity (X, x) ($e(E) = 3$). Using proposition 2.1, it follows that:

$$h^1(x) = \frac{67}{216} \tag{2.14}$$

In Labs [Lab06] it is shown how to construct hypersurfaces in \mathbb{P}^3 with only A_n singularities with n fixed using Dessins d'Enfants. For A_2 singularities one has that there are hypersurfaces $X_{d,\ell}$ if:

$$\ell = \begin{cases} \frac{1}{2}d(d-1) \cdot \lfloor \frac{d}{3} \rfloor + \frac{1}{3}d(d-3)(\lfloor \frac{d-1}{2} \rfloor) - \lfloor \frac{d}{3} \rfloor & d \equiv 0 \pmod{3} \\ \frac{1}{2}d(d-1) \cdot \lfloor \frac{d}{3} \rfloor + \frac{1}{3}(d(d-3)+2)(\lfloor \frac{d-1}{2} \rfloor) - \lfloor \frac{d}{3} \rfloor & \text{otherwise} \end{cases} \quad (2.15)$$

Theorem 1 and 2.14 give that $\Omega_{\tilde{X}_{d,\ell}}^1$ is big if $\frac{67}{216}\ell > s_2(\tilde{X}_{d,\ell})$ or equivalently if:

$$\ell > \frac{72}{67}(2d^2 - 5d) \quad (2.16)$$

By 2.15 there are hypersurfaces $X_{d,\ell} \subset \mathbb{P}^3$ with d and ℓ satisfying (2.16) if $d = 9$ or $d \geq 11$. \square

Remark: 1) In Theorem 2 we can see the strength of theorem 1 when compared to the criterion for the cotangent bundle $\Omega_{\tilde{X}_{d,\ell}}^1$ to be big of [RR14], $s_2(\tilde{X}_{d,\ell}) + s_2(X_{d,\ell}) > 0$. The criterion of [RR14] needs $\ell > \frac{3}{2}(2d^2 - 5d)$ instead of (2.16). The known upper bounds by Miyaoka or Varchenko, see [Lab06], for the number of A_2 singularities possible on a hypersurface in \mathbb{P}^3 of degree d prevent $\ell > \frac{3}{2}(2d^2 - 5d)$ for $d \leq 11$. Moreover, one has to go to degree $d = 14$ for the known constructions to give enough A_2 singularities for the criterion of [RR14].

2) Following the method of theorem 2, if instead of using hypersurfaces in \mathbb{P}^3 with only A_2 singularities, one used hypersurfaces with only A_1 singularities (nodes), then one would need $\ell > \frac{9}{4}(2d^2 - 5d)$ nodes for the minimal resolution of an hypersurface with ℓ nodes to have big cotangent bundle. This would give surfaces with big cotangent bundle deformation equivalent to smooth hypersurfaces in \mathbb{P}^3 of degree $d \geq 10$. The known upper bounds for the number of nodes possible in hypersurfaces of a given degree, see [Lab06], give that for degree 9 you can not have more than 246 nodes, our criterion needs 264. So A_2 singularities give a better result.

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