Symplectic geometry and the Alexander polynomial of a knot

Luís Diogo
Universidade Federal Fluminense
Instituto de Matemática e Estatística, Departamento de Matemática Aplicada
R. Prof. Marcos Waldemar de Freitas Reis, s/n, Bloco G - Campus do Gragoatá
São Domingos, Niterói, RJ, 24210-201, Brasil
e-mail: `ldiogo@id.uff.br`

Resumo: Apresentamos uma fórmula para o polinómio de Alexander clássico de um nó em termos de um invariante de nós introduzido recentemente, chamado polinómio de aumentação e definido a partir da homologia de contacto do nó. Damos uma ideia da prova, que parte de uma definição dinâmica do polinómio de Alexander e envolve a análise de vários espaços de moduli de curvas pseudoholomorfas.

Abstract We present a formula expressing the classical Alexander polynomial of a knot in terms of a very recent knot invariant, called the augmentation polynomial and defined using knot contact homology. We give an idea of the proof, which starts from a dynamical definition of the Alexander polynomial and involves analyzing several moduli spaces of pseudoholomorphic curves.

palavras-chave: geometria simplética, curvas pseudoholomorfas, invariantes de nós.

keywords: symplectic geometry, pseudoholomorphic curves, knot invariants.

1 Introduction

Knot theory and symplectic geometry have both seen a great development in recent years. In some instances, techniques from symplectic geometry have been successful in producing powerful new invariants of knots (like the knot Floer homology of Ozsváth–Szabó and Rasmussen [15, 16]), or in enhancing the understanding of previously known invariants (like a symplectic version of Khovanov homology [1]). In this note, we present a recent result obtained in collaboration with Tobias Ekholm, which yields a formula for the Alexander polynomial of a knot in terms of its augmentation polynomial. The former is a classical cornerstone of knot theory. The latter is a recently defined object, introduced in the context of knot contact homology. This is another very powerful new invariant of knots that was constructed using tools from symplectic geometry. Our result is saying that knot contact homology recovers the Alexander polynomial. Although this fact was already known from the work of Ng [14], the formula in terms of the augmentation polynomial appears to be new. It also has an unusual form for a relation between two polynomials. Our result will be stated as Theorem 5.1 below.

We will begin with a brief introduction to knots and the Alexander polynomial, including a dynamical definition of this invariant that will be useful for our purposes. Then, we change direction and give a quick introduction to symplectic geometry and pseudoholomorphic curves. After that, we explain how to use pseudoholomorphic curves to define knot contact homology, and how the latter yields the augmentation polynomial of a knot. Then, we state our result and give a terse presentation of the proof. Our goal will not be to convey the full logical structure of the argument (let alone its technical details), but only to give an idea of a practical and hopefully interesting application of pseudoholomorphic curves in symplectic geometry. We conclude with some directions for future work.

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2 Knots and the Alexander polynomial

2.1 Knots and links

A *knot* is a closed embedded curve in $\mathbb{R}^3$. This means that it is the image of a C^∞-smooth injective map from the circle S^1 to $\mathbb{R}^3$, with non-zero derivative at every point. A *link* is a finite collection of knots that are all pairwise disjoint. We are interested in knots and links from the point of view of topology, in the sense that we don't want to distinguish those that differ by a smooth deformation causing no self-intersections, called an *isotopy*. Formally, this is a C^∞-smooth map $f\colon [0,1]\times\mathbb{R}^3 \to \mathbb{R}^3$, such that $f_t \coloneqq f(t,.)$ is a diffeomorphism of $\mathbb{R}^3$ for every $t \in [0,1]$, and f_0 is the identity. Two links are *isotopic* if there is an isotopy f such that the image under f_1 of one link is the other link.

In Figure 1 we have the two simplest examples of knots: the *unknot* and the *trefoil*. We can think of this picture as the result of projecting our knots

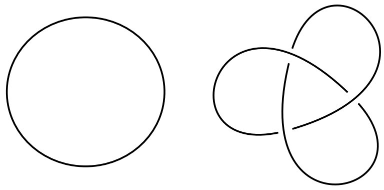


Figure 1: The unknot and the trefoil (with three crossings)

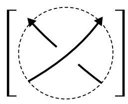
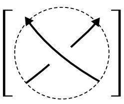
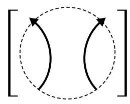
in $\mathbb{R}^3$ to a plane, so that the projection is injective at all but finitely many points, called *crossings*. A figure like this, encoding for each crossing which of the two strands is over the other, is called a *link diagram*.

It is intuitively clear that the unknot and the trefoil are not isotopic, but it is not entirely obvious how to prove this fact. The main problem in knot theory is to find an efficient way of deciding when two knots are isotopic.

2.2 The Alexander polynomial

One of the first tools that were created to distinguish knots and links is the *Alexander polynomial*. Given a link L, its Alexander polynomial Alex_L is a Laurent polynomial in one variable μ. This means that the integer powers of μ are allowed to be negative. Define Alex_L as follows: pick an orientation for L, which is to say a direction for each of its component knots, and impose

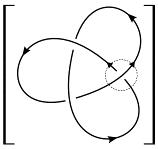
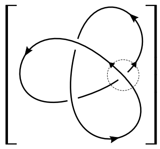
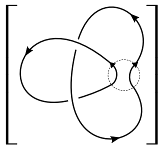
- $\mathrm{Alex}_{\mathrm{unknot}} = 1$.
- The *skein relation*:

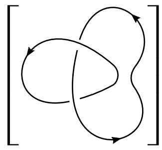
  Alex  $-$ Alex  $+ (\mu^{1/2} - \mu^{-1/2})$ Alex  $= 0$

  This is a relation between Alexander polynomials of three links with link diagrams that are equal outside the depicted neighborhood of a crossing.
- *Isotopy invariance*: $\mathrm{Alex}_L = \mathrm{Alex}_{L'}$ if L and L' are isotopic links.

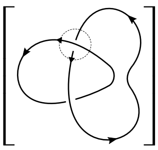
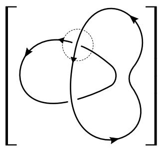
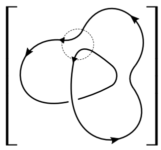
These three properties determine the Alexander polynomial for every link. Two non-obvious facts are that the Alexander polynomial is well-defined (in particular, the skein relation holds for every link diagram) and that it contains only integer powers of μ, even though the skein relation involves fractional powers.

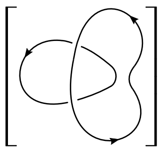
As an example, let us compute the Alexander polynomial of the trefoil. Applying the skein relation, we get

Alex  = Alex  $-\,(\mu^{1/2} - \mu^{-1/2})$ Alex 

$= 1 - (\mu^{1/2} - \mu^{-1/2})$ Alex 

where we used the fact that $\mathrm{Alex}_{\mathrm{unknot}} = 1$. The link we obtained on the right is called *Hopf link*. Let us apply the skein relation on a crossing of this link:

Alex  = Alex  $-\,(\mu^{1/2} - \mu^{-1/2})$ Alex 

= Alex  $-\,\mu^{1/2} + \mu^{-1/2}$

where we used again that $\mathrm{Alex}_{\mathrm{unknot}} = 1$. To finish the computation, we need to determine the Alexander polynomial of the link with two components on the right. Since the two components can be moved by an isotopy to lie in two disjoint open balls, the link is said to be trivial. We call it the *unlink with two components*. Its Alexander polynomial is zero, and we leave the proof of that to the reader as an exercise on the skein relation. We can now conclude the calculation of the Alexander polynomial of the trefoil:

$$\mathrm{Alex}_{\mathrm{trefoil}} = 1 - (\mu^{1/2} - \mu^{-1/2})(0 - \mu^{1/2} + \mu^{-1/2}) = \mu - 1 + \mu^{-1}.$$

Since this is different from the Alexander polynomial of the unknot, we can conclude that the unknot and the trefoil are not isotopic.

Exercise 1. Compute the Alexander polynomial of the *figure-eight knot* (the closure of the sailor's knot of the same name), depicted in Figure 2.

2.3 A dynamical definition of the Alexander polynomial

The definition of Alexander polynomial of a link that we gave in the previous section is very suitable for computations (at least for link diagrams without

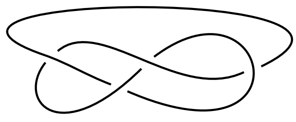


Figure 2: The figure-eight knot

too many crossings). Nevertheless, it is only one of many definitions of this invariant. We now present a different definition of the Alexander polynomial of a link, with a very different flavour. For simplicity, we will restrict our attention to the particular class of fibered knots, which we now define.

In this section, it will be convenient to think of the ambient space of a knot as the sphere S^3, instead of $\mathbb{R}^3$. This is reasonable, since we can identify $\mathbb{R}^3$ with the complement of a point in S^3. For this identification, two knots are isotopic in $\mathbb{R}^3$ if and only if they are isotopic in S^3.

We say that a knot K is *fibered* if there is a C^∞-smooth map $g\colon S^3\backslash K \to S^1$ with no critical points. This means that the knot complement $S^3\backslash K$ is a fiber bundle over S^1, with fiber a surface whose boundary is K (such a surface is called a *Seifert surface*). The differential of the function g is a 1-form dg. If we choose some Riemannian metric $\langle ., . \rangle$ on S^3, then the function g also specifies a vector field in $S^3\backslash K$, called *gradient vector field* and denoted ∇g, as follows: for every vector field v on $S^3\backslash K$,

$$\langle \nabla g, v \rangle = dg(v).$$

Since the function g has no critical points, the vector field ∇g has no zeros. A *gradient flow loop* is a path $\gamma\colon [0, R] \to S^3\backslash K$, for some $R > 0$, such that

- $\gamma(R) = \gamma(0)$ (which means that γ closes up to a loop) and
- $\frac{d}{dt}\big(\gamma(t)\big) = (\nabla g)_{\gamma(t)}$ for every $t \in [0, R]$ (that is, the time-derivative of γ coincides with ∇g at every point in γ).

Observe that if $\gamma\colon [0, R] \to S^3\backslash K$ is a gradient flow loop, then so is any multiple cover $\gamma_m\colon [0, mR] \to S^3\backslash K$, where m is a positive integer. Here, $\gamma_m(t) = \gamma(t')$ for $t' \in [0, R]$ such that $t' \equiv t \mod R$. We say that a flow loop is *simple* if it is not multiply covered. Given a flow loop γ, we denote by $m(\gamma)$ its multiplicity with respect to its underlying simple loop. For every knot K, the homology group $H_1(S^3\backslash K; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. If we pick a generator e for this homology group, then we can associate to a flow loop γ its degree $d(\gamma)$, such that the class of γ on homology is $d(\gamma)e$. Note that

$m(\gamma)$ divides $d(\gamma)$. To avoid flow loops too close to K, we will require the map g to "grow near K".

Theorem 2.1 (Milnor [13]). *The Alexander polynomial of a fibered knot K is given by*

$$\mathrm{Alex}_K(\mu) = (1-\mu)\exp\left(\sum_{\gamma}\frac{\sigma(\gamma)}{m(\gamma)}\mu^{d(\gamma)}\right) \tag{1}$$

where the sum is over all gradient flow loops (not only the simple ones). In the formula, $\sigma(\gamma) \in \{\pm 1\}$ is a sign (associated to the linearized return map of γ).

This formula was generalized for all knots K by Hutchings and Lee [11].

Example 2.2. Let us see how to recover the Alexander polynomial of the unknot from formula (1). The complement of the unknot in S^3 is diffeomorphic to $S^1 \times \mathbb{R}^2$ (if this is not clear, try to identify both spaces with the complement of the vertical axis in $\mathbb{R}^3$). In coordinates (θ, x, y) for $S^1 \times \mathbb{R}^2$, take $g(\theta, x, y) = \theta + x^2 + y^2$. For the standard product metric on $S^1 \times \mathbb{R}^2$, the only periodic orbits are the covers of the central circle $S^1 \times \{0\}$, and all the signs $\sigma(\gamma)$ in (1) are positive. The sum over flow loops becomes

$$\sum_{k>0}\frac{1}{k}\mu^k = -\ln(1-\mu)$$

hence

$$\mathrm{Alex}_{\mathrm{unknot}}(\mu) = (1-\mu)\exp\left(-\ln(1-\mu)\right) = 1$$

as we already knew.

3 Some symplectic geometry

3.1 Classical mechanics and symplectic geometry

Symplectic geometry is a recent area of mathematics, with its roots in classical mechanics, but with deep connections to other areas of mathematics and physics. In the Hamiltonian formulation of classical mechanics, a particle moving in $\mathbb{R}^3$ is described by its trajectory in the phase space $\mathbb{R}^6$, which keeps track of the position and momentum of the particle. If we denote position variables in $\mathbb{R}^3$ by q_1, q_2, q_3 and the corresponding momentum variables by p_1, p_2, p_3, then the trajectory of the particle in phase space satisfies Hamilton's equations

$$\begin{cases} \dot{q}_i = \partial_{p_i} H \\ \dot{p}_i = -\partial_{q_i} H \end{cases}$$

where the *Hamiltonian function* $H : \mathbb{R}^6 \to \mathbb{R}$ is the energy of the particle. This trajectory is a flow line of the *Hamiltonian vector field*, which we denote by X_H. If we define a differential 2-form on $\mathbb{R}^6$ by

$$\omega \coloneqq \sum_{i=1}^{3} dp_i \wedge dq_i, \tag{2}$$

then Hamilton's equations above tell us that the vector field X_H is given by the condition

$$\omega(., X_H) = dH. \tag{3}$$

A reader unfamiliar with differential forms may find it mildly useful to think of ω as a way of prescribing signed areas to 2-dimensional oriented surfaces in $\mathbb{R}^6$ (where the signs depend on the orientations of the surfaces).

We can interpret equation (3) as saying that the 2-form ω allows us to do Hamiltonian mechanics for any function that we choose to call energy on $\mathbb{R}^6$. We can think of symplectic geometry as generalizing this point of view on mechanics to any differentiable manifold of even dimension $2n$, equipped with a differential 2-form ω whose properties mimic those of the form (2), namely:

- ω is closed: $d\omega = 0$, and
- ω is non-degenerate: the n-fold wedge product $\omega \wedge \ldots \wedge \omega$ is a volume form (which means that it vanishes nowhere).

For the application to knot theory that we present in this text, it will mostly suffice to think of the symplectic manifold $\mathbb{R}^6$. We refer to the article by Ana Cannas da Silva in this volume [4] for more on symplectic geometry.

3.2 Pseudoholomorphic curves

In 1985, Gromov introduced the notion of *pseudoholomorphic curve* [10], which was revolutionary in symplectic geometry. It gave a powerful tool to study symplectic manifolds, and eventually led to many deep relations to algebraic geometry and theoretical physics, in particular the so called *mirror symmetry* phenomenon (see Lino Amorim's article in this volume [3] for some background on mirror symmetry). Before we state one of the striking results in Gromov's paper, let us introduce some more terminology. First, we observe that an open subset of a symplectic manifold, equipped with the restriction of the symplectic form ω, is also a symplectic manifold. Let $B^{2n}(r) \subset \mathbb{R}^{2n}$ denote the open ball of radius r and centered at the origin. Consider also the open subset $B^2(R) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$, where it is crucial that $B^2(R)$ has coordinates (p_1, q_1) (instead of (p_1, p_2) or (q_1, q_2), for instance) and $\mathbb{R}^{2n-2}$ has the remaining coordinates $p_2, \ldots, p_n, q_2, \ldots, q_n$. Given two symplectic manifolds (M_1, ω_1) and (M_2, ω_2), a smooth embedding $\varphi\colon M_1 \hookrightarrow M_2$ such that $\varphi^*\omega_2 = \omega_1$ is called a *symplectic embedding* (here, φ^* is pullback by φ).

Theorem 3.1 (Gromov's non-squeezing)**.** *If we have a symplectic embedding*

$$B^{2n}(r) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2},$$

then $r \leqslant R$.

Observe that a symplectic embedding is, by definition, volume-preserving. We can interpret Gromov's non-squeezing as saying that not all volume-preserving embeddings are symplectic.

Now that we have given a little indication of what pseudoholomorphic curves can achieve, let us define them. We need the auxiliary notion of an *almost complex structure* on an even-dimensional manifold M^{2n}, which is an endomorphism of the tangent bundle $J\colon TM \to TM$ (covering the identity map $M \to M$) such that $J^2 = -Id$. Given such a J and a Riemann surface (S, j), a pseudoholomorphic curve is a map $u\colon S \to M$ satisfying the Cauchy–Riemann equation

$$du \circ j = J \circ du.$$

If M has a symplectic form ω, then one can ask that $\omega(., J.)$ be a Riemannian metric on M, in which case J is said to be *compatible* with ω. Gromov's idea was to study (M, ω) by analyzing moduli spaces of pseudoholomorphic curves (modulo domain reparametrizations, and possibly with additional structures like fixing the homology class of the map, or equipping the domain with marked points). If J is compatible with ω, then we can control the L^2-norm of u (the *energy*) by its ω-area, which is crucial for obtaining compactness of moduli spaces. Gromov also observed that the space of ω-compatible J is contractible, which implies that the moduli spaces defined for two different J are cobordant (that is, there is a manifold whose oriented boundary is the difference of the two moduli spaces). This allows for the definition of numerical invariants counting pseudoholomorphic curves (with appropriately chosen constraints) that depend on ω but not on the choice of ω-compatible J. Those are called *Gromov–Witten invariants*, and they have many applications in symplectic and algebraic geometry.

4 Symplectic knot invariants

4.1 From knots to Lagrangians and Legendrians

Recent decades have seen many applications of pseudoholomorphic curves. We will focus on a particular application to knot theory, called *knot contact homology*. This is part of a broader packaging of pseudoholomorphic curve information that goes by the name of *symplectic field theory* [8], but we will focus on the specific case of interest to us. Let us begin with some geometric constructions.

Given a knot $K \subset \mathbb{R}^3$, we can define its *conormal Lagrangian*

$$L_K := \{(q,p) \in \mathbb{R}^6 \,|\, q \in K \text{ and } (\forall v \in T_qK)\, \langle p, v\rangle = 0\},$$

where $\langle .,. \rangle$ is the Euclidean inner product in $\mathbb{R}^3$. This is a submanifold of $\mathbb{R}^6$ that is diffeomorphic to $S^1 \times \mathbb{R}^2$, and whose intersection with $\mathbb{R}^3_q$ (the subspace of $\mathbb{R}^6$ where all $p_i = 0$) is the knot K. See Figure 3 for a geometric depiction that would greatly benefit from additional dimensions. Furthermore, L_K is *Lagrangian*, in the sense that it has half the dimension of the ambient space $\mathbb{R}^6$, and the restriction of the symplectic form ω in (2) to L_K vanishes. In addition, the Lagrangian L_K is *exact*, which means the following. The symplectic form ω in $\mathbb{R}^6$ has a primitive $\lambda = \sum_{i=1}^3 p_i dq_i$ and the restriction $\lambda|_L$ admits a primitive $f \in C^\infty(L)$ (in this case, we can take f to be any constant function). Other exact Lagrangians are $\mathbb{R}^3_q$ and $\mathbb{R}^3_p$.

We can identify $\mathbb{R}^6$ with the tangent bundle[1] of $\mathbb{R}^3_q$ and, with respect to the Euclidean inner product in $\mathbb{R}^3$, we can identify $\mathbb{R}^3 \times S^2 \subset \mathbb{R}^6$ with the unit tangent bundle of $\mathbb{R}^3_q$. Recall that the *geodesic flow* on the unit tangent bundle of a Riemannian manifold Q takes a point $q \in Q$ and a unit vector $v \in T_qQ$ and follows the geodesic starting at q in the direction prescribed by v. This is an example of what is called a *Reeb flow* in contact geometry (hence the name "knot contact homology"), but we will not go further in that direction in this note. The conormal Lagrangian L_K intersects $\mathbb{R}^3 \times S^2$ in a 2-torus Λ_K, which we call *conormal Legendrian* (again borrowing terminology from contact geometry).

It will be useful to observe that $H_2(\mathbb{R}^3 \times S^2, \Lambda_K; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^3$. We will explain this point, but a reader less familiar with homology groups might want to skip the details. Let us just mention that this is the reason why the augmentation polynomial below will have three variables.

[1] From the point of view of symplectic geometry, it would be preferable to think of the cotangent bundle, but we can ignore that point in this text.

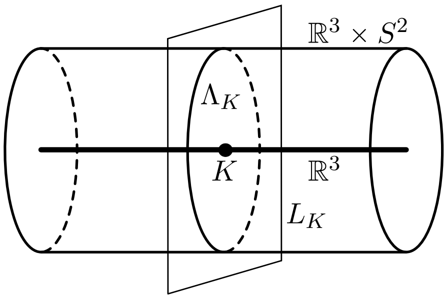


Figure 3: The Lagrangian L_K in $\mathbb{R}^6$

In this paragraph, consider all homology groups with $\mathbb{Z}$ coefficients. The long exact sequence of the pair $(\mathbb{R}^3 \times S^2, \Lambda_K)$ includes the segment

$$H_2(\Lambda_K) \to H_2(\mathbb{R}^3 \times S^2) \to H_2(\mathbb{R}^3 \times S^2, \Lambda_K) \to H_1(\Lambda_K) \to 0.$$

The first map turns out to vanish. Since $H_2(\mathbb{R}^3 \times S^2) \cong H_2(S^2) \cong \mathbb{Z}$ and $H_1(\Lambda_K) \cong \mathbb{Z}^2$ (Λ_K is a 2-torus), the sequence splits and we get the desired isomorphism with $\mathbb{Z}^3$. We get generators for this group from the choice of a generator t for $H_2(S^2)$ and of generators x, p for $H_1(\Lambda_K)$ (and a choice of splitting). It is customary to let x be a longitude curve (projecting to K under the restriction to Λ_K of the projection $\mathbb{R}^3 \times S^2 \to \mathbb{R}^3$), and to let p be a meridian curve (mapping to a constant under that same projection). Note that such a meridian curve p lies in a cotangent fiber (that is, a 3-dimensional subspace of $\mathbb{R}^6$ with constant q_i variables), hence the use of the letter associated with momentum.

4.2 Knot contact homology

We can now use pseudoholomorphic curves to associate a chain complex to the knot K. We will actually get a *differential graded algebra* (dga), which is a chain complex with a product satisfying the (graded) Leibniz rule. Our chain complex will be a tensor algebra generated by geodesic chords starting and ending in Λ_K. By this we mean paths $c\colon [a,b] \to \mathbb{R}^3 \times S^2$ that follow the geodesic flow and for which $c(a)$ and $c(b) \in \Lambda_K$. We don't want to get into details, but these chords are graded by a Maslov index (which is an integer).

Let us specify the ring over which we take the tensor algebra. This will be group ring (over $\mathbb{C}$) of $H_2(\mathbb{R}^3 \times S^2, \Lambda_K; \mathbb{Z})$, which, in light of the discussion at the end of the previous section, can be identified with the Laurent polynomial ring $R = \mathbb{C}[\lambda^{\pm 1}, \mu^{\pm 1}, Q^{\pm 1}]$, under the identifications $\lambda = e^x$, $\mu = e^p$ and $Q = e^t$.

The differential in the chain complex counts pseudoholomorphic curves in $\mathbb{R}\times(\mathbb{R}^3\times S^2)=\mathbb{R}^4\times S^2$ (which we can identify with the complement of $\mathbb{R}^3_q$ in $\mathbb{R}^6$), as follows. We define the differential for geodesic chords, and extend by linearity and the Leibniz rule. The differential of a geodesic chord x is

$$\partial x = \sum_{y_1,\ldots,y_k} \left(\sum_{u\in\mathcal{M}(x;y_1,\ldots,y_k)} r(u) \right) y_1 \otimes \ldots \otimes y_k$$

where the first sum is over finite sequences of geodesic chords and the second sum is over elements of the moduli space of pseudoholomorphic curves u in $\mathbb{R}^4 \times S^2$, whose domain is a disk with $k+1$ punctures on the boundary. The boundary components map to $\mathbb{R} \times \Lambda_K$. At the boundary punctures, u is asymptotic to the fixed geodesic chords, with x at $+\infty$ and the y_i at $-\infty$ (both infinities in the first $\mathbb{R}$ summand of the target $\mathbb{R} \times (\mathbb{R}^3 \times S^2)$). See Figure 4 for an illustration of one such u. Finally, the coefficient $r(u) \in R$ keeps track of the relative homology class of u in $H_2(\mathbb{R}^3 \times S^2, \Lambda_K; \mathbb{Z})$. We will not go into more details at this point, but the reader may have noticed that more choices are necessary, including of "capping half-disks" for the geodesic chords (to obtain a relative homology class).

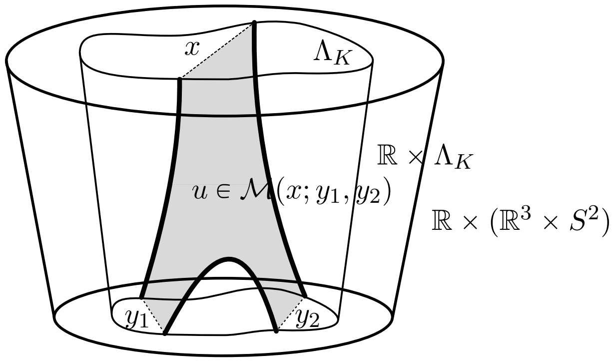


Figure 4: A contribution to ∂x

Theorem 4.1 (Ekholm–Etnyre–Ng–Sullivan [6])**.** *The differential ∂ defined above squares to zero. The homology of this dga is an invariant of the knot.*

This homology is called knot contact homology. It is sometimes useful to keep track of the dga, denoted by $\mathcal{A}_K$, instead of passing to homology.

Although the technical details of the proof of Theorem 4.1 are quite involved, the idea is by now standard in symplectic geometry. To prove an algebraic identity like $\partial^2 = 0$, one interprets the contributions to ∂^2 as elements in the boundary of a suitably defined moduli space of pseudoholomorphic curves. By showing that this moduli space is a compact 1-dimensional oriented manifold, one concludes that the signed count of the elements in its boundary is zero.

Knot contact homology appears to be a strong knot invariant, but it is not yet clear just how strong. A recent result shows that a small but non-trivial enhancement of knot contact homology is a complete knot invariant (that is, two knots are isotopic if and only if their enhanced knot contact homologies are isomorphic) [7].

4.3 Augmentations

Although the definition of the dga $\mathcal{A}_K$ involves pseudoholomorphic curves, which can be very difficult to analyze, the dga turns out to admit a combinatorial model, which can be written down explicitly given a braid presentation for the knot K [6]. Nevertheless, since the chain complex (a tensor algebra) is very large, it can be difficult to extract useful information from its homology. One way of obtining more treatable information about the dga is via its *augmentations*. An augmentation is a unital dga map

$$\varepsilon\colon \mathcal{A}_K \to \mathbb{C},$$

where the field $\mathbb{C}$ is thought of as a dga supported in degree zero and with trivial differential. In other words, ε is a graded unital ring map (so, it is only non-trivial on the degree zero part of $\mathcal{A}_K$) satisfying $\varepsilon \circ \partial = 0$.

Example 4.2. One important source of augmentations is given by exact Lagrangians in $\mathbb{R}^6$ "which look like Λ_K near infinity" (in some precise sense). A key example is the conormal L_K. Given such a Lagrangian, we can define an augmentation by assigning to each geodesic chord of degree 0 the count of pseudoholomorphic disks in $\mathbb{R}^6$ with boundary on the Lagrangian and one puncture on the boundary, where the disk is asymptotic to the geodesic chord "at infinity". The value of the augmentation on the coefficient ring R is constrained by the topology of the Lagrangian and its ambient space. In the case of L_K, since the meridian p-curve is contractible in L_K and the t-sphere is null-homologous in $\mathbb{R}^6$, it turns out that $\mu = 1 = Q$, but λ is not constrained. So, for each value $\lambda \in \mathbb{C}\backslash\{0\}$ we get an augmentation of $\mathcal{A}_K$.

It turns out to be useful to also think of the space of augmentations geometrically. We define the *augmentation variety* of K, denoted by V_K, to consist of the union of maximal dimensional components of the Zariski closure of the set

$$\{(\varepsilon(\lambda), \varepsilon(\mu), \varepsilon(Q)) \in (\mathbb{C}\backslash\{0\})^3 \,|\, \varepsilon \text{ is an augmentation}\}.$$

The existence of the augmentations associated to L_K in Example 4.2 implies that, for every knot K, the augmentation variety V_K contains the line $\{(\lambda, 1, 1)\}$ (where λ can be any element in $\mathbb{C}\backslash\{0\}$).

Theorem 4.3 (Diogo–Ekholm [5]). *For every knot K, the augmentation variety V_K is an affine algebraic subvariety of $(\mathbb{C}\backslash\{0\})^3$ of complex dimension at least 2.*

Conjecturally, V_K is always 2-dimensional (so it is not all of $(\mathbb{C}\backslash\{0\})^3$). Define the *augmentation polynomial* of K (denoted by $\mathrm{Aug}_K(\lambda, \mu, Q)$) as a polynomial with no repeated factors that generates the vanishing ideal of this variety: $V_K = V(\mathrm{Aug}_K)$.

Example 4.4. The augmentation polynomial of the unknot is

$$\mathrm{Aug}_U = 1 - \lambda - \mu + \lambda\mu Q$$

and that of the trefoil is

$$\mathrm{Aug}_T = \lambda^2(\mu - 1) + \lambda(\mu^4 - \mu^3 Q + 2\mu^2 Q^2 - 2\mu^2 Q - \mu Q^2 + Q^2) + (\mu^3 Q^4 - \mu^4 Q^3).$$

The augmentation polynomial has deep and surprising connections to string theory and to other knot invariants. It is conjecturally the same as the so-called *Q-deformed A-polynomial*, which is relevant for mirror symmetry and is related in a deep way with another important knot invariant called the *colored HOMFLYPT polynomial* [2, 9].

5 The Alexander polynomial from the augmentation polynomial

As we have seen, the Alexander polynomial and the augmentation polynomial are knot invariants defined in very different ways. Nevertheless, they are related in the following surprising manner.

Theorem 5.1 (Diogo–Ekholm [5]). *Recall that $\lambda = e^x$, $\mu = e^p$ and $Q = e^t$. We have*

$$\mathrm{Alex}_K(\mu) = (1 - \mu) \exp\left(\int -\frac{\partial_Q \mathrm{Aug}_K}{\partial_\lambda \mathrm{Aug}_K}\bigg|_{(\lambda,Q)=(1,1)} dp\right) \tag{4}$$

if the denominator $\partial_\lambda \mathrm{Aug}_K \,|_{(\lambda,Q)=(1,1)}$ is not identically zero.

In formula (4), the integral symbol represents an antiderivative. We will give a brief idea of why one might expect the formula to hold, at least for fibered knots. We will be very imprecise and will not justify most of our claims. Our goal is to illustrate how the study of moduli spaces of pseudoholomorphic curves can lead to meaningful algebraic identities (we already saw that this is also the idea of the proof that $\partial^2 = 0$ in the dga $\mathcal{A}_K$). Note that, according to Milnor's formula (1), we only need to argue that

$$\frac{d}{dp}\left(\sum_{\gamma \text{ in } S^3\backslash K} \frac{\sigma(\gamma)}{m(\gamma)}\mu^{d(\gamma)}\right) = -\frac{\partial_Q \operatorname{Aug}_K}{\partial_\lambda \operatorname{Aug}_K}\bigg|_{(\lambda,Q)=(1,1)}. \tag{5}$$

Exercise 2. Apply formula (4) to Example 4.4 to recover the Alexander polynomials of the unknot and the trefoil. The Alexander polynomial is often defined up to a power of μ, and (4) should also be allowed that ambiguity.

5.1 From flow loops to pseudoholomorphic annuli

The left side of (5) involves orbits in $S^3\backslash K$, whereas the right side involves pseudoholomorphic curves in $\mathbb{R}^4 \times S^2$. To get a reformulation of the left side also in terms of pseudoholomorphic curves, we need another geometric ingredient. Recall that the conormal Lagrangian $L_K \subset \mathbb{R}^6$ intersects $\mathbb{R}^3_q$ in the knot K. There is a procedure called *Lagrangian surgery*, which produces another Lagrangian submanifold by smoothing out the union of L_K with $\mathbb{R}^3_q$ (the version we need is described in [12]). Denote the new Lagrangian in $\mathbb{R}^6$ by M_K. This submanifold is diffeomorphic to $\mathbb{R}^3\backslash K$. Since L_K and $\mathbb{R}^3_q$ are exact Lagrangians, one can ensure that M_K is also exact. In particular, it has an associated family of augmentations ε_{M_K}, sending both generators λ and Q of the coefficient ring R to 1, and the generator μ to any element of $\mathbb{C}\backslash\{0\}$. Hence, the line $\{(1,\mu,1)\}$ is also contained in the augmentation variety V_K for every K. The key role of these augmentations is the reason behind taking $\lambda = Q = 1$ in formula (4).

In $\mathbb{R}^6$, we can consider *pseudoholomorphic annuli* between $\mathbb{R}^3_q$ and M_K. These are pseudoholomorphic maps $u\colon S^1 \times [0, A] \to \mathbb{R}^6$ (for some $A \geqslant 0$) such that the restriction of u to $S^1 \times \{0\}$ maps to $\mathbb{R}^3_q$ and the restriction to $S^1 \times \{A\}$ maps to M_K. Denote the moduli space of such pseudoholomorphic annuli by $\mathcal{M}(\mathbb{R}^3_q; M_K)$.

The following result is stated in an overly simplified and somewhat imprecise manner.

Proposition 5.2. *For suitable choices of $g\colon S^3\backslash K \to S^1$, metric on S^3 and J on $\mathbb{R}^6$, gradient flow orbits in $S^3\backslash K$ can be identified with pseudoholomor-*

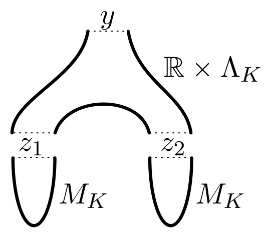


Figure 5: Definition of $F(\lambda, \mu, Q)$. The curve at the top could have arbitrarily many negative punctures capped by disks with boundary in M_K.

phic annuli in $\mathcal{M}(\mathbb{R}^3_q; M_K)$. Therefore, the sum on the left side of (5) *can be rewritten as*

$$A(\mu) \coloneqq \sum_{u \in \mathcal{M}(\mathbb{R}^3_q; M_K)} \frac{\sigma(u)}{m(u)} \mu^{d(u)} \tag{6}$$

for suitable signs $\sigma(u)$ and integers $m(u)$ and $d(u)$.

Equation (5) is thus equivalent to

$$\frac{d}{dp}\left(A(\mu)\right) = - \left.\frac{\partial_Q \operatorname{Aug}_K}{\partial_\lambda \operatorname{Aug}_K}\right|_{(\lambda,Q)=(1,1)}, \tag{7}$$

where we recall again that $\mu = e^p$.

5.2 From pseudoholomorphic annuli to knot contact homology

Instead of showing equation (7) directly, we show that

$$\frac{d}{dp}\left(A(\mu)\right) = - \left.\frac{\partial_Q F}{\partial_\lambda F}\right|_{(\lambda,Q)=(1,1)}, \tag{8}$$

for a suitable holomophic function $F(\lambda, \mu, Q)$ such that

$$\left.\frac{\partial_Q F}{\partial_\lambda F}\right|_{(\lambda,Q)=(1,1)} = \left.\frac{\partial_Q \operatorname{Aug}_K}{\partial_\lambda \operatorname{Aug}_K}\right|_{(\lambda,Q)=(1,1)}. \tag{9}$$

The function F is defined as follows. For an appropriately chosen generator y of degree 1 of the dga $\mathcal{A}_K$ (actually, an R-linear combination of such generators), take its dga differential ∂, which is an expression in λ, μ, Q and other generators $z_1, \ldots, z_n$. Then, send the z_i to their images under the augmentation ε_{M_K}. See Figure 5.

Now, consider the moduli space of pseudoholomorphic annuli in $\mathbb{R}^6$, with one boundary component in $\mathbb{R}^3_q$ and another in M_K (as in $\mathcal{M}(\mathbb{R}^3_q; M_K)$

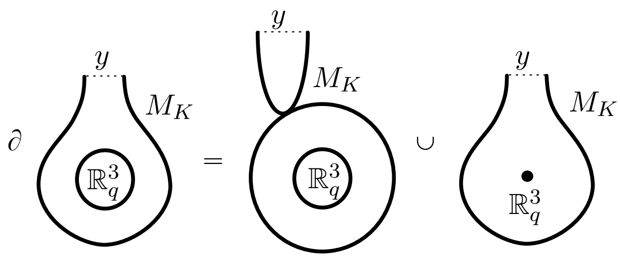


Figure 6: The boundary of a 1-dimensional moduli space

above), but with a puncture on the boundary component mapping to M_K. At this puncture, the curve is asymptotic to y. See the left side of Figure 6.

This moduli space is compact and 1-dimensional (since y has degree 1) and (if y is chosen carefully) its boundary has components of two types, which are depicted on the center and right in Figure 6. In the center configuration, the curve develops a node and breaks into a pseudoholomorphic plane asymptotic to y and an annulus in $\mathcal{M}(\mathbb{R}^3_q; M_K)$. The boundaries of the plane and annulus intersect. In the rightmost configuration, the boundary loop in $\mathbb{R}^3_q$ shrinks to a point, so the punctured annulus becomes a plane.

A further study of the pseudoholomorphic planes in the center configuration reveals that the count of such broken curves (using μ to keep track of the homology of boundaries mapping to M_K) is given by

$$\frac{dA}{dp} \cdot \frac{\partial F}{\partial x}\bigg|_{(\lambda,Q)=(1,1)} = \frac{dA}{dp} \cdot \frac{\partial F}{\partial \lambda}\bigg|_{(\lambda,Q)=(1,1)},$$

recalling once more that $\lambda = e^x$. The derivatives in the formula keep track of the intersection of the boundaries of the disk and annulus. Similarly, the counts of curves in the configuration on the right turn out to be encoded by

$$\frac{\partial F}{\partial t}\bigg|_{(\lambda,Q)=(1,1)} = \frac{\partial F}{\partial Q}\bigg|_{(\lambda,Q)=(1,1)},$$

where $Q = e^t$. This time, the derivative keeps track of the fact that the disk intersects $\mathbb{R}^3_q$. Since these two configurations are the boundaries of a compact 1-dimensional manifold, the sum of their contributions (with appropriate signs) vanishes. This implies that

$$\frac{dA}{dp} \cdot \frac{\partial F}{\partial \lambda}\bigg|_{(\lambda,Q)=(1,1)} + \frac{\partial F}{\partial Q}\bigg|_{(\lambda,Q)=(1,1)} = 0,$$

which gives equation (8), as wanted.

For a brief justification of equation (9), let us just say that one can argue that F vanishes on the augmentation variety V_K, so it should be of the form

$$F = g\ \mathrm{Aug}_K$$

for some analytic function $g(\lambda, \mu, Q)$. Equation (9) now follows from the product rule for derivatives and the fact that Aug_K vanishes along the line $\{(1, \mu, 1)\} \subset V_K$ (at least if we assume that $g|_{(\lambda,Q)=(1,1)}$ is not identically zero, which as it turns out we can).

5.3 Outlook

Theorem 5.1 should not be thought of as an efficient way of computing the Alexander polynomial of a knot, but rather as an unexpected relation between two very different knot invariants. It also suggests further investigation in a few directions. For example, one might not set $Q = 1$ in equation (4) and get a Q-deformed version of Alex_K.

Question 5.3. *What is the significance of this deformation of the Alexander polynomial? Is it related to other deformations, coming for instance from knot Floer homology [15]?*

One might also wonder about the condition of non-vanishing of the denominator in the theorem. As it turns out, this condition cannot be neglected, as it does not hold, for instance, for the 8_{20} knot (as pointed out to us by Lenny Ng).

Question 5.4. *Is there an analogue of equation* (4) *when the denominator in the formula vanishes?*

It is likely that along some branch of the variety V_K, corresponding to the augmentation M_K, one could find such an analogue.

As a final note, the reader may have wondered about interpreting the integrand in formula (4) via implicit differentiation. Indeed, since V_K is the vanishing locus of Aug_K, that integrand is the partial derivative $\frac{\partial \lambda}{\partial Q}$ along the line $\{(1, \mu, 1)\} \subset V_K$. This leads to an alternative interpretation of the right side in the formula, related to curve counts in the resolved conifold (the total space of the bundle $\mathcal{O}_{\mathbb{C}P^1}(-1) \oplus \mathcal{O}_{\mathbb{C}P^1}(-1)$), in the spirit of [2]. That is another interesting story, but unfortunately it is beyond the scope of this discussion.

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