Concise notes on Special Holonomy with an emphasis on Calabi–Yau and G_2-Manifolds

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Abstract: These are notes for a very short introduction to some selected topics on special Riemannian holonomy with a focus on Calabi-Yau and G_2-manifolds. No material in these notes is original and more on it can be found in the papers/books of Bryant, Hitchin, Joyce and Salamon referenced during the text.

keywords: Special Holonomy, Calabi-Yau, G_2-manifold.

1 Introduction

Riemannian geometry is by now a well established and fundamental area of mathematics with most undergraduate degrees worldwide having an introductory course on it, such as one on curves and surfaces. Despite this there is still nothing like a classification of complete Riemannian manifolds and instead one attempts to understand them from secondary invariants such as their holonomy.

Given a n-dimensional Riemannian manifold (X, g) its Levi-Civita connection yields a notion of parallel transport of tangent vectors along paths. This has the property that it preserves the length and angles between parallel transported vectors. When one fixes a point p and a loop γ_p based at that point, the parallel transport along γ_p is an orthogonal linear transformation $\gamma_p : T_pX \to T_pX$ of the tangent space T_pX to X at p. The set of all such linear transformations $\mathrm{Hol}_p(X)$ is a subgroup of the group the orthogonal group $O(T_pX)$ called the holonomy group at p. If one fixes an orthogonal basis of T_pX, this may viewed as a subgroup of $O(n)$ which changes by conjugation upon changing the base point p. Thus, from now on we shall forget about the base point in the notation and simply refer to the holonomy group as $\mathrm{Hol}(X)$ which we think of as a conjugacy class in $O(n)$.

The classification of possible Riemannian holonomy groups was started by Cartan's algebraic classification of symmetric spaces [7, 8] in 1926. In the nosymmetric case one may, by a theorem of de Rham, restrict to the class of Riemannian manifolds for which the holonomy representation is irreducible, which are thus known as irreducible Riemannian manifolds. In 1953 Berger [2] compiled a set of restrictions which may be satisfied by any possible holonomy group of a simply connected, irreducible Riemannian manifold. The outcome is a list of these possible holonomy groups of these Riemannian manifolds. It is headed by $SO(n)$ which represents the generic holonomy group, and followed by some "rarer" subgroups of $SO(n)$ still acting on $\mathbb{R}^n$ in an irreducible manner. The full list is the following:

Hol	n=dim(X)	Name
SO(n)	n	Orientable manifold
U(k)	2k	Kähler manifold
SU(k)	2k	Calabi–Yau manifold
Sp(k)·Sp(1)	4k	Quaternion-Kähler manifold
Sp(k)	4k	Hyperkähler manifold
G2	7	G_2-manifold
Spin(7)	8	Spin(7) manifold

These other possible holonomy groups are known as special holonomy groups and except for G_2 and Spin(7) they all appear in infinite families. For this reason G_2 and Spin(7) are also called as the exceptional holonomy groups.

Berger's technique to cut the list down to only these groups is quite indirect and consists in transforming what is apparently an integro-differential problem of computing all the holonomies round loops into a local differential problem. The idea is to instead, classify the Lie algebra of the possible Riemannian holonomy groups which by the Ambrose-Singer theorem can be obtained from the values of the Riemann curvature tensor. Its symmetries give restrictions on the possible Lie algebras and these are then integrated by a unique simply connected Lie group. Clearly, this approach solely puts restrictions on the possible holonomy groups and, at the time Berger's list appeared, it was not known whether all groups featuring it could actually be realized as Riemannian holonomy groups. Nowadays, due to the efforts of Aubin, Bryant, Calabi, Salamon and Yau together with several contributions from many others [4, 5, 20] we know that all these groups can actually be realized as the holonomy groups of complete Riemannian metrics. However, most intricacies of their geometry and internal classification remain to be understood at present yielding one of most active areas of research in Riemannian geometry.

In a somewhat perpendicular direction several of these geometries have also appeared in the physics literature. Since the 1990's, and also more recently, Calabi–Yau and G_2-manifolds have been attracting the interest of physicists working in string and M-theory respectively. The main reason for this is the possibility of using them in compactifications of these theories which are supposed to produce realistic 4-dimensional versions of the physical world including the standard model of particle physics together with a quantization of gravity.

These notes are a selected part of topics that are supposed to serve as a modern, very quick, introduction to both these classes of manifolds from a geometric structure point of view. In this setting, calibrations and stable forms appear naturally and we use these in our approach to both these classes of special holonomy Riemannian manifolds. This approach mixes the points of view of Salamon, Harvey–Lawson and Hitchin which I find very beautiful attractive. In trying to make the material as concise as possible I have left a lot of relevant material out. This can be found in the references given and finish this introduction by admitting that, perhaps, the best contribution of this note is its brevity and mixed viewpoint.

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2 Calibrated Geometry and Holonomy

In these notes X^n will denote a smooth real n-dimensional manifold and $Fr(M)$ its principal $GL(n,\mathbb{R})$-frame bundle. When X is equipped with a Riemannian metric g we will denote by $FO(n)$ its principal $O(n)$-bundle.

2.1 Geometric Structures

Definition 1. *Let $G \subset GL(n,\mathbb{R})$ be a Lie group, a G-structure on X, denoted by P, is a principal G-subbundle of $Fr(X)$.*

Proposition 1 (weak Holonomy Principle)**.** *There is a one to one correspondence between sections of the bundle $Fr(X) \times_{GL(n,\mathbb{R})} GL(n,\mathbb{R})/G$ and G-structures on X.*

Proof. We shall only sketch the idea, for a full proof see page 11 in [18].
Let $x \in X$, then each point in the fibre P_x gives an identification $T_xX \cong V := \mathbb{R}^n$. If $\eta_0 \in V^{\otimes r} \otimes (V^*)^{\otimes s}$ is G-invariant, we can define $\eta_x \in (T_xX)^{\otimes r} \otimes (T_x^*X)^{\otimes s}$ to equal η_0 using any of the identifications $T_xX \cong V$ given by the points of P_x. This gives a well defined tensor η over the whole X.
Conversely, if η is a section of the bundle $Fr(X) \times_{GL(n,\mathbb{R})} GL(n,\mathbb{R})/G$, then one can define the G-structure P which stabilizes η. □

Example 1. *1. A Riemannian metric defines the $O(n)$-structure, denoted $FO(n)$.*

2. An almost complex structure defines a $GL(n/2,\mathbb{C})$-structure.

When $G \subset O(n)$ and P is a G-structure one defines the $O(n)$-bundle $FO(n) = P \times_G O(n)$. A connection ∇ on P induces one on TX whose torsion $T_\nabla \in \Omega^2(X,TX)$ is by definition

$$T_\nabla(V,W) = \nabla_V W - \nabla_W V - [V,W].$$

Given any two connections ∇, ∇' as above, $\nabla' = \nabla + a$ with $a \in \Omega^1(X,\mathfrak{g}_P)$ where $\mathfrak{g}_P = P \times_G \mathfrak{g} \subset \mathfrak{so}(TX)$. Then it is easy to compute that $T_{\nabla'} = T_\nabla + \delta(a)$, where δ is a section of $\mathrm{Hom}(T^*X \otimes \mathfrak{g}_P, \Lambda^2 X \otimes TX)$.
Notice that since $\mathfrak{g} \subset \mathfrak{so}(\mathfrak{n}) \cong \Lambda^2\mathbb{R}^n$, the map δ is injective and in order to get rid of the dependence on the connection we can define the reduced map $[T_\nabla]$ with values in $\mathrm{coker}(\delta)$. This is usually called the intrinsic torsion (or the structure function of the G-structure P). The following result is an immediate consequence of this construction.

Lemma 1. *Let (X,g) be a Riemannian manifold and $G \subset O(n)$ a G-structure $P \subset FO(n)$. Then, there is a connection ∇ on P inducing the Levi–Civita connection of g on M if and only if the reduced map $[T_\nabla]$ vanishes. Such a G-structure is said to be integrable.*

An immediate corollary of this construction is the next result, for which more details can be found in page 14 of [18].

Corollary 1 (Holonomy Principle)**.** *Let (X,g) be a Riemannian manifold and $x \in X$. Then, any $\eta_x \in \Omega^0(X,(T_xX)^{\otimes r} \otimes (T_x^*X)^{\otimes s})$ which is preserved by the holonomy at x of the Levi–Civita connection ∇^{LC} is the value at x of a ∇^{LC}-parallel tensor field η.*
Moreover, in this situation the $G = Hol$-structure P determined by η via the weak holonomy principle is equipped with a connection ∇ inducing ∇^{LC}. Equivalently, there is a ∇^{LC}-parallel embedding of P into $FO(n)$.

Remark 1. *In general, a similar principle holds for any vector bundle with a connection.*

Let $G \subset SO(n)$ and P a G-structure on (X,g), then G acts on the differential forms and splits these as irreducible representations as $\Lambda^k = \oplus_i \Lambda^k_i$. Moreover, the Hodge-$*$ is an isomorphism of G-representations $\Lambda^k_i \cong \Lambda^{n-k}_i$. These are the essential observations leading to the following Theorem of Chern, [9].

Theorem 1. *Let P be a G-structure on (X,g) as above and assume it has vanishing intrinsic torsion. Then, there is a metric g such that if $\mathcal{H}^k$ denotes the harmonic k-forms, there is a splitting*

$$\mathcal{H}^k = \oplus_i \mathcal{H}^k_i,$$

and isomorphisms $\mathcal{H}^k_i \cong \mathcal{H}^j_i$ if the corresponding $\Lambda^k_i \cong \Lambda^j_i$ are isomorphic representations.

Proof. Since P has vanishing intrinsic torsion, there is a metric g whose Levi Civita connection ∇ is induced by a connection on P. Thus, ∇ preserves the embedding $P \hookrightarrow FSO(n) = P \times_G SO(n)$ and so for $\beta \in \Omega^k_i$, we have $\nabla\beta \in \Omega^0(X, T^*X \otimes \Lambda^k_i)$ and $\nabla^*\nabla\beta \in \Omega^k_i$. Having in mind that there is a Weitzenböck type formula

$$\Delta\beta = \nabla^*\nabla\beta + \mathcal{R}(\beta),$$

where $\mathcal{R}$ is an algebraic operator computed in terms of the curvature tensor $R \in \Omega^0(X, S^2\mathfrak{hol})$. Since $\mathfrak{hol} \subset \mathfrak{g}$ and the fact that $\mathcal{R} \in \Omega^0(X, \mathfrak{hol})$, it follows that $\mathcal{R}(\beta) \in \Omega^k_i$. Hence, the Laplacian Δ preserves the splitting into irreducible representations which then passes on to the harmonic forms. Moreover, one can show that $\nabla^*\nabla$ and $\mathcal{R}$ only depend on the representation in which they are acting and not on the specific degree of the differential form which concludes the proof of the statement. □

Lemma 2. *If $G \subset SO(n)$ is simply connected any a G-structure canonically lifts to a Spin-structure.*

Proof. Since G is simply connected there is a unique lift of the inclusion of G in $SO(n)$ to an inclusion $G \hookrightarrow Spin(n)$. Using this one can construct $\hat{F} = P \times_G Spin(n)$ and the projection $Spin(n) \to SO(n)$ gives a canonical map $\hat{F} \to FSO(n) = P \times_G SO(n)$. Hence $\hat{F}$ is a $Spin$ structure on (X,g). □

2.2 Stable Forms and Calibrations

In this section we shall review the notion of a stable form following Hitchin in [12] and [14]. Then, we shall see how some calibrations yield examples of such stable forms. Finally, we relate these to special Riemannian holonomy.

Definition 2. *Let V^n be a real n dimensional vector space, $\eta \in \Lambda^p V^*$ is a stable p-form if its $GL(V)$-orbit in $\Lambda^p V$ is open.*

Example 2.

1. *$n = 2m$, $m \in \mathbb{N}$ and $p = 2$. Then, (V, η) is a symplectic vector space and the stabilizer of η is $Sp(2m, \mathbb{R})$.*
2. *$n = 6$ and $p = 3$. There is an open orbit of $GL(6, \mathbb{R})$ on $\Lambda^3 V$ such than all η lying on it have stabilizer $SL(3, \mathbb{C})$. Such an η induces a complex structure on V with respect to which η is of type $(3,0)+(0,3)$.*
3. *$n = 7$ and $p = 3$. There are two open orbits of the $GL(7, \mathbb{R})$ on $\Lambda^3 V$, for η in one of those the stabilizer is compact group G_2.*
4. *$n = 8$ and $p = 3$, there is an open orbit with stabilizer $PSU(3)$.*

In all the examples above the stabilizer preserves a volume form on the respective vector space. In fact, as observed by Hitchin in [12, 14], one has the following result.

Proposition 2. *There is a $GL(V)$-equivariant homogeneous function*

$$\phi : \Lambda^p V^* \to \Lambda^n V^*,$$

of degree $\frac{n}{p}$. For each $\eta \in \Lambda^p V^*$, there is a unique $\hat{\eta}$, such that the derivative $d_\eta \phi : \Lambda^p V^* \to \Lambda^n V^*$ is given by

$$d_\eta \phi(\dot{\eta}) = \hat{\eta} \wedge \dot{\eta},$$

for $\dot{\eta} \in \Lambda^p V^*$ and moreover $\phi(\eta) = \frac{p}{n} \eta \wedge \hat{\eta}$.

Proof. The existence of the $GL(V)$ equivariant function ϕ follows from the fact that all isotropy subgroups of such η preserve a volume form on V. The $GL(V)$ invariance for scalar matrices $\lambda 1$, with $\lambda \in \mathbb{R}$, shows that $\phi(\lambda^p \eta) = \lambda^n \phi(\eta)$ and so ϕ is homogeneous of degree n/p.
The derivative $d\phi$ is linear and an element of $(\Lambda^p V^*)^* \otimes \Lambda^n V^* \cong \Lambda^{n-p} V^*$. Hence, there is a unique $\hat{\eta}$ with the properties stated, and the last statement that $\phi(\eta) = \frac{p}{n} \eta \wedge \hat{\eta}$ follows from Euler's formula

$$d\phi = \frac{n}{p} \phi$$

for homogeneous functions. □

Example 3. *1. $n = 2m$ and $p = 2$, η is a symplectic form and $\hat{\eta} = \frac{\eta^{m-1}}{(m-1)!}$.*

2. $n = 6$ and $p = 3$, $\eta + i\hat{\eta}$ is a form of type $(3,0)$, for the complex structure determined by η.

3. $n = 7$ and $p = 3$, the stabilizer of η is G_2, which is a compact group. So the volume form $\phi(\eta)$ preserved by G_2 is the volume form of an invariant metric on V. Using this metric one obtains $\hat{\eta} = *\eta$.

4. $n = 8$ and $p = 3$, the stabilizer $PSU(3)$ is also compact and the same discussion goes on with $\hat{\eta} = -*\eta$.

Definition 3. *If g is a metric on an oriented vector space V and $\{e_i\}_{i=1}^n$ an orthonormal basis, then a p-form $\theta \in \Lambda^p V^*$ is said to be a calibration if*

$$|\theta(e_{i_1}, ..., e_{i_k})| \leq 1,$$

for all $i_1, ..., i_k \in \{1, ..., n\}$, i.e. if its comass is smaller or equal than 1.

Equivalently, θ is a calibration on (V, g), if and only if for all p-dimensional oriented subspaces $W \subset V$

$$\theta|_W \leq vol_W, \tag{1}$$

where vol_W is the volume form of the metric $g|_W$ induced on W, by g.

Definition 4. *Let (V, g) be a vector space with metric and $\theta \in \Lambda^p V^*$ a calibration on V. A subspace $W \subset V$ is said to be calibrated by θ is $\theta|_W = vol_W$, i.e. if equality is attained in the inequality 1.*

This discussion can be globalized in Harvey–Lawson's notion of a calibration [13].

Definition 5. *Let X^n be a real n-dimensional smooth manifold and $\eta \in \Omega(X, \mathbb{R})$ a p-form is said to be stable if for all $p \in X$ $\eta_p \in \Lambda^p T_p X$ is a stable form.*
If (X, g) is an oriented Riemannian manifold and $\theta \in \Omega^p(X, \mathbb{R})$ is closed, then θ is called a calibration on (X, g), if for all $x \in X$, θ_x is a calibration on $(T_x X, g_x)$. A submanifold $N \subset X$ is said to be calibrated by θ is for all $x \in N$, $T_x N \subset T_x X$ is calibrated by θ_x.

The construction from proposition 2 gives a volume form on M, whose volume defines the Hitchin functional

$$\Phi(\eta) = \int_X \phi(\eta) \in \mathbb{R} \cup \infty. \tag{2}$$

Notice that the existence of a stable p form η on M^n reduces the structure group of the tangent bundle to the isotropy subgroup of the form η. A natural question is if there is any relation between these reductions and possible reductions of the holonomy group of a special metric on M, determined by η.

Proposition 3. *If X is compact, and $[\eta] \in H^p(X,\mathbb{R})$ is a fixed cohomology class. Then Hitchin's functional gives a well defined function*

$$\Phi : [\eta] \to \mathbb{R},$$

whose critical points are the $\eta \in [\eta]$ with $d\hat{\eta} = 0$.

Proof. Let $\eta \in [\eta]$ be a critical point, since the variation is in the fixed cohomology class $[\eta]$ all tangent vectors are exact forms $d\alpha$. So for all $\alpha \in \Omega^{p-1}(X,\mathbb{R})$

$$0 = d\Phi_\eta(d\alpha) = \int_X \hat{\eta} \wedge d\alpha = \int_X d\hat{\eta} \wedge \alpha,$$

which shows that if $\hat{\eta}$ is a critical point then $d\hat{\eta} = 0$. Conversely, the same computation also shows that if $d\hat{\eta} = 0$, then $\hat{\eta}$ is a critical point. □

Example 4.
1. *$n = 2m$ and $p = 2$, (X,η) is a symplectic manifold and $d\hat{\eta} = 0$ always. (X,ω) with $\omega = \eta$ can be equipped with a metric g and compatible almost complex structure I. Then, for all $k \le n$, $\frac{\omega^k}{k!}$ has comass ≤ 1 and is closed and so a calibration. Submanifolds N^{2k} calibrated by $\frac{\omega^k}{k!}$ are symplectic (or almost complex) submanifolds.*
   If $\nabla I = 0$ the complex structure is integrable and the metric has holonomy contained in $U(n)$. Then (X,I,η) is a Kähler manifold and the $\frac{\omega^k}{k!}$-calibrated submanifolds are complex submanifolds.

2. *$n = 6$ and $p = 3$, then $\eta + i\hat{\eta}$ equips X with an almost complex structure for which $\eta + i\hat{\eta}$ is of type $(3,0)$. If η is a critical point of Hitchin's functional, then $\overline{\partial}(\eta + i\hat{\eta}) = 0$ and so the complex structure is integrable. Since $\eta + i\hat{\eta}$ is a nonvanishing holomorphic volume form, X has trivial canonical bundle.*

   If $(X,\omega,\Omega = \Omega_1 + i\Omega_2)$ is a Calabi–Yau 3-fold, then in particular it is Kähler and choosing $\eta = \omega$ the example above gives a reduction of the holonomy to $U(3)$. Moreover, choosing $\eta = \Omega_1$, gives this precise example and $\nabla\Omega = 0$, which reduces the holonomy to $SL(3,\mathbb{C})$ and so the holonomy of the metric is contained in $SU(3) = U(3) \cap SL(3,\mathbb{C})$.

In this case both $\Omega_1 = \eta$ *and* $\Omega_2 = \hat{\eta}$ *are calibrations and submanifolds* N^3 *calibrated by them are called special Lagrangian submanifolds of phase* $0, \frac{\pi}{2}$ *respectively.*

3. $n = 7$ *and* $p = 3$*, the stable form* η *is a critical point of Hitchin's functional if*

$$d\eta = d^*\eta = 0,$$

for the metric on M *determined by* η*. Indeed, by a result of Fernández and Gray [11] this is equivalent to* $\nabla\eta = 0$*, which is to say the Holonomy of the metric is contained in* G_2*, by the Holonomy principle. In this case one usually uses the notation* $\eta = \phi$*,* $\hat{\eta} = *\phi = \psi$ *and* (X, ϕ) *is called a* G_2*-manifold. Both* ϕ *and* ψ *are calibrations and submanifolds calibrated by them are respectively called associative and coassociative.*

In the examples above only for the case $n = 7$, the stable form η determines a metric with reduced holonomy (in fact G_2 which an exceptional Lie group appearing in Berger's list). This is because the holonomy group of any oriented Riemannian manifold must be a subgroup of $SO(n)$ by the holonomy principle, and both $Sp(2m, \mathbb{R})$ and $SL(3, \mathbb{C})$ are non-compact groups.

Proposition 4. *Let* (X^n, g) *be a Riemannian manifold equipped with a calibration* $\theta \in \Omega^p(X, \mathbb{R})$*. If* $N^p \subset X$ *is compact and calibrated by* θ*, then* N *is volume minimizing in its homology class* $[N] \in H_p(X, \mathbb{R})$*.*

Proof. Let $N' \in [N]$ be cohomologous to N, then there is S^{p+1} with $\partial S = N \cup (-N')$ (with orientations) and Stokes theorem gives $\int_N \theta - \int_{N'} \theta = \int_S d\theta = 0$. Now the result follows from applying this and the definition of calibration to the following one line calculation

$$vol(N') = \int_{N'} vol_{N'} \geq \int_{N'} \theta = \int_N \theta = \int_N vol_N = vol(N). \tag{3}$$

□

Notice that the equation for a calibrated submanifold is a first order PDE, while being minimal is a second order one (the Euler Lagrange equations for critical points of the volume functional). This is an analogous situation to that of many gauge theories as for example the relation between ASD connections and the Yang Mills equations for connections on bundles over 4 manifolds. See [10, 19] for some higher dimensional gauge theories mimicking these.

We shall now change gears and focus on the more concrete cases of Calabi–Yau and G_2-manifolds. The interested reader can find a lot more about these for example in [6, 17, 18] and references therein.

3 Calabi–Yau Manifolds

On a Kähler manifold (X^n, g, ω) we shall implicitly always consider complex structure I determined by g and ω. The next proposition relates the Ricci tensor and the holomorphic triviality of the canonical bundle $K_X = \Lambda^{n,0}_{\mathbb{C}} X$ to the holonomy of the underlying Kähler metric.

Proposition 5. *A Kähler manifold (X^n, g, ω) with $n = 2m$ is Ricci flat with trivial canonical bundle K_X if and only if the holonomy of the Kähler metric on X is contained in $SU(m)$.*

In any case X is a Ricci flat Kähler manifold with trivial canonical bundle K_X and there is a unique (up to phase) holomorphic volume form Ω satisfying

$$\frac{\omega^m}{m!} = (-1)^{\frac{m(m-1)}{2}} \left(\frac{i}{2}\right)^m \Omega \wedge \overline{\Omega}, \tag{4}$$

which trivializes K_X.

Proof. The Ricci form $\rho(\cdot,\cdot) = \mathrm{Ric}(\cdot, I\cdot)$ is the curvature of the connection on K_X induced via the Levi Civita connection on the holomorphic tangent bundle. First suppose that X is Ricci flat and K_X trivial (this a necessary assumption if X is not simply connected). Ricci flatness gives that $\rho = 0$, while the triviality of K_X guarantees not only that $c_1(X) = 0 \in H^2(X, \mathbb{Z})$, but that the element in the Jacobian representing K_X is trivial. Hence, the connection has no periods and there is an holomorphic trivializing section of K_X, i.e. there is a $(n,0)$-form Ω such that $\overline{\partial}\Omega = 0$. This implies it is parallel and so by the holonomy principle (corollary 1) the Kähler metric has holonomy contained in $SU(m)$.
The converse statement also follows from the holonomy principle since if the holonomy is contained in $SU(m)$, then there are nonzero parallel forms $\omega \in \Omega^2(X, \mathbb{R})$ and $\Omega \in \Omega^{3,0}(X, \mathbb{C})$ (unique up to phase) satisfying the relation 4 in the statement. Since $\nabla\Omega = 0$, then also $\overline{\partial}\Omega = 0$ and so it is holomorphic and trivializes K_X. Then, $c_1(X) = 0$ and the definition of curvature also gives $\rho(\Omega) = d_\nabla \nabla\Omega = 0$, and as Ω is nonvanishing $\rho = 0$, i.e. the metric is Ricci flat, which is the same thing as saying that the connection on K_X induced by the Levi Civita one is flat. □

Remark 2. *If X is Ricci-flat Kähler and simply connected, then K_X is automatically trivial and the holonomy contained in $SU(m)$. This follows from the fact that $\rho = 0$ and so the Levi Civita connection equips K_X with a flat connection. These are parametrized by $Hom(\pi_1(X), U(1))$, which vanishes as X is simply connected. Then K_X is trivial and proposition 5 shows the*

holonomy is in $SU(m)$*.*
When X *is not simply connected there are counterexamples to this statement. For example an Enriques surface is a Ricci flat Kähler manifold with* $c_1(X)$ *a torsion class in* $H^2(X,\mathbb{Z})$*. In this case* K_X *is not trivial and the flat connection can be seen as an element of* $Hom(\pi_1(X),U(1)) = Hom(H_1(X),U(1)) = H^1(X,U(1))$*, uniquely determined by the Hermitian metric on* K_X *via Chern's construction.*

Definition 6. *A Calabi–Yau manifold* (X,ω,Ω) *is a Ricci flat, Kähler manifold* (X,ω) *with trivial canonical bundle and a choice of holomorphic volume form* $\Omega \in \Omega^{3,0}(X,\mathbb{C})$ *satisfying equation 4.*

According to this definition Calabi–Yau manifolds will have holonomy contained in $SU(m)$. Some authors require the holonomy to be exactly $SU(m)$ and here these will be called irreducible Calabi–Yau manifolds. The question of existence of Calabi–Yau manifolds can be attacked directly by explicitly constructing the metric as is done in several noncompact examples or by PDE methods in both cases compact and noncompact. In line with the second of these, we have Yau's proof of the Calabi conjecture, [20], which states the following.

Theorem 2. *Let* X *be a compact complex manifold with* $c_1(X) = 0$ *in* $H^2(X,\mathbb{R})$*, then in all Kähler classes in* X*, there is a unique Ricci-flat Kähler metric.*

The Calabi Conjecture, so called by having been proposed by Calabi years before Yau completed its proof in [20], asserts the existence of many compact Calabi–Yau manifolds. For example, if X is a complex manifold with $c_1(X) = 0$, $\pi_1(X) = 0$ and which admits Kähler classes, then combining the Calabi conjecture 2 with proposition 5, there is a Calabi–Yau structure on each Kähler class of X.

Remark 3. *The Enriques surface from remark 2 is not a Calabi–Yau manifold according to definition 6, as it has nontrivial canonical bundle. However, the Calabi conjecture stated as Theorem 2, proves the existence of a Ricci-flat Kähler metric on the Enriques surface.*

The next results explore some properties of Calabi–Yau manifolds.

Proposition 6. *Let* (X,ω,Ω) *be a compact Calabi–Yau manifold. Then there is a finite cover* $\tilde{X}$ *of* X*, which is biholomorphic to the product of* $T^{2k} \times Y$*, where* T^{2k} *is a real* $2k$ *dimensional torus and* Y *a complex manifold with* $c_1(Y) = 0$*.*

If (X, ω, Ω) is further assumed to be irreducible, then it has finite fundamental group.

Proof. Calabi–Yau manifolds are Ricci flat and so the Cheeger Gromoll splitting theorem applies and for each $2k$ linearly independent parallel 1 forms, there is a T^{2k} splitting off. As both the complex torus and X have vanishing first Chern class so must be for Y.
Now suppose X is irreducible, then it cannot have any parallel 1 form as this would make the Holonomy to be strictly contained in $SU(m)$. Moreover, for Ricci flat manifolds there is a Weitzenböck formula

$$\|\nabla\alpha\| = \langle \alpha, \Delta\alpha \rangle,$$

which shows that each harmonic 1 form gives rise to a parallel 1 form. Hence there can be no harmonic 1-forms and this forces the fundamental group of X to be finite. □

Remark 4. *A version of this result also holds in the noncompact case, there one may have to let some of the torus directions to be noncompact (i.e. $\mathbb{R}$) and Y may be noncompact as well.*

Proposition 7. *Let (X, ω, Ω) be a compact Calabi–Yau manifold, then for $i \in \{1, ..., m-1\}$*

$$dim(H^{i,0}(X, \mathbb{C})) \leq \frac{m!}{i!(m-i)!}.$$

If (X, ω, Ω) is further assumed to be irreducible, then

$$\begin{aligned} H^{0,0}(X,\mathbb{C}) &= 1 \\ H^{m,m}(X,\mathbb{C}) &= 1 \\ H^{i,0}(X,\mathbb{C}) &= 0, \quad i \in 1, ..., m-1. \end{aligned}$$

Proof. For $i = 1, ..., m-1$, let $\alpha \in \Omega^{i,0}$ be a representative of a class in $H^{i,0}(X, \mathbb{C})$ and recall that Calabi–Yau manifolds are Ricci flat. Then as in the proof of proposition 6, the Weitzenböck formula is $\|\nabla\alpha\| = \langle \alpha, \Delta\alpha \rangle$. Moreover, the Kähler identities imply that $\Delta\alpha = 2\overline{\partial}^*\overline{\partial}\alpha = 0$[1] and α is then parallel. In the general case, the maximum number of linearly independent of these is then the dimension of $\Lambda^{i,0}\mathbb{C}^n$, which is precisely $\frac{m!}{i!(m-i)!}$. In the irreducible case there can be no nonzero parallel $(i, 0)$ forms as this would reduce the holonomy to be strictly contained in $SU(m)$. □

[1]Notice that $\overline{\partial}^*\alpha = 0$ as α is of type $(i, 0)$.

Remark 5. *There is an alternative argument using the maximum principle which can be used in noncompact Ricci flat manifolds. This proves for example for noncompact irreducible Calabi–Yau manifolds there can be no decaying harmonic* $(i,0)$ *forms.*

Proposition 8. *Let* (X,ω,Ω) *be a compact and irreducible Calabi–Yau manifold of real dimension* $n=2m\geq 6$. *Then* X^m *is a projective algebraic variety.*

Proof. Since $m\geq 3$ and (X,ω,Ω) is irreducible, proposition 7 gives that $h^{2,0}=0$ and so $H^2(X,\mathbb{C})=H^{1,1}(X,\mathbb{C})$. Then, the image of $H^2(X,\mathbb{Z})\rightarrow H^{1,1}(X,\mathbb{C})$ is nonempty and one can pick a positive class α there. Associated to this class there is a positive holomorphic line bundle L with $c_1(L)=\alpha$ and the Kodaira Embedding theorem provides an embedding $X\hookrightarrow\mathbb{CP}^{h^0(X,L^k)-1}$, for sufficiently large $k\in\mathbb{N}$. □

4 G_2-Manifolds

Let X^7 be a 7 dimensional manifold and denote by Λ^3_+ the bundle of stable 3 forms over X and by Ω^3_+ its sections. Given $\phi\in\Omega^3_+$, then at any point $p\in X$ the stabilizer of ϕ_p in $GL(7,\mathbb{R})$ is conjugate to G_2 (as defined in the third item of example 2). Given such a section, it determines via the weak holonomy principle a G_2 structure, which itself determines via lemma 2 a Spin-structure on (X^7,g). In fact, for G_2-structures the converse also holds.

Proposition 9. *A 7-dimensional oriented Riemannian manifold* (X^7,g) *admits a* G_2*-structure if and only if it is Spin.*

Proof. Since G_2 is simply connected, given a G_2-structure lemma 2 guarantees the existence of a Spin structure $\hat{F}$. To prove the converse let $\hat{F}$ denote a Spin bundle and Δ the standard irreducible $Spin(7)$ representation, then $\mathrm{rk}_{\mathbb{R}}\Delta=8$. Moreover, $Spin(7)$ acts transitively on $\mathbb{S}^7$ with stabilizer G_2, so it is enough to find a unit section of the bundle of spinors $\mathcal{S}=\hat{F}\times_{Spin(7)}\Delta$. Since this bundle has rank $8>7$ there is a nowhere vanishing section of $\mathcal{S}$, which we can normalize to have norm 1. Then the weak holonomy principle determines a G_2-structure. □

Definition 7. *Let* (X^7,ϕ) *be as above and* $\phi\in\Omega^3_+$. *Then* ϕ *and* g *are compatible if for all vector fields* V,W, $\iota_V\phi\wedge\iota_W\phi\wedge\phi=6g(V,W)_g dvol_g$.

Definition 8. *A G_2-manifold (X,ϕ) is a real 7 dimensional Riemannian manifold (X^7,g), equipped with a compatible G_2-structure $\phi \in \Omega^3_+$ such that $\nabla\phi = 0$.*

From the Holonomy Principle a G_2-manifold has holonomy contained in G_2, when the holonomy is the full G_2 one says that (X,ϕ) is an irreducible G_2-manifold. We shall now go on to investigate some topological and geometric properties of (irreducible) G_2-manifolds starting with the following of Fernández and Gray [11].

Theorem 3. *Let (X^7,g) be a Riemannian 7 dimensional manifold equipped with a stable 3 form ϕ compatible with g, the following are equivalent*

1. $\nabla\phi = 0$,
2. $d\phi = d^*\phi = 0$,
3. *The holonomy of g is contained in G_2.*

Proof. The holonomy principle (corollary 1) implies that the holonomy of g is in G_2 if and only if $\nabla\phi = 0$; and so it is enough to prove that the first two items are equivalent. In one direction this is obvious since $\nabla\phi \in \Omega^0(X, T^*X \otimes T^*X)$ and both $d\phi$ and $d^*\phi$ are obtained from $\nabla\phi$ by composition with algebraic operators, respectively the anti-symmetrization map $\wedge \in \mathrm{Hom}(T^*X \otimes T^*X, \Lambda^2 X)$ and the trace with respect to metric g, $\mathrm{tr}_g \in \mathrm{Hom}(T^*X \otimes T^*X, \Lambda^0)$. So if $\nabla\phi = 0$, then both $d\phi$ and $d^*\phi$ vanish.
In the opposite direction, suppose $d\phi = d^*\phi = 0$, and to proceed we need to investigate $\nabla\phi$ with more scrutiny. The intrinsic torsion of the G_2-structure determined by ϕ is $\nabla\phi$, seen as a section of $\mathrm{coker}(\delta)$, where δ is the map defined in the discussion preceeding Lemma 1. Recall that this bundle is modeled on $V^* \otimes \mathfrak{g}_2^\perp$, where $\mathfrak{g}_2^\perp \subset \mathfrak{so}(7)$ and $V \cong \mathbb{R}^7$ is the standard 7 dimensional representation of $\mathfrak{g}_2$. By an abuse of language we shall say $\nabla\phi$ is modeled on $V^* \otimes \mathfrak{g}_2^\perp$. Notice that $\mathfrak{so}(7) \cong \Lambda^2 V \cong \Lambda^2_7 \oplus \Lambda^2_{14}$ with $\Lambda^2_7 \cong V^*$ and $\Lambda^2_{14} \cong \mathfrak{g}_2$. We conclude that $\mathfrak{g}_2^\perp \cong V^*$ and so $\nabla\phi$ is a section of $V^* \otimes V^* \cong \Lambda^2 V \oplus S^2 V$. In fact this further decomposes into

$$V^* \otimes V^* \cong (V^* \oplus \mathfrak{g}_2) \oplus \left(S^2_0 V \oplus \mathbb{R} \right), \tag{5}$$

where $\mathbb{R}$ is the trivial representation, and it follows from highest weight theory that $S^2_0 V$ is irreducible of dimension 27. Hence, the decomposition above is irreducible.
Next, $d\phi$ is modeled on $\Lambda^4 V \cong \Lambda^3 V$, which decomposes into irreducible

components as $\mathbb{R}\oplus V\oplus S^2_0V$. Since $0=d\phi=\wedge\circ\nabla\phi$ and $\wedge$ is a morphism of representations and is surjective, it follows that $\nabla\phi$ has values in the the $\mathfrak{g}_2$ component of the decomposition in 5.
Next we analyse the vanishing of $d^*\phi$ which is modeled on $\Lambda^2V\cong V\otimes\mathfrak{g}_2$. In the same way as before $0=d^*\phi=\mathrm{tr}_g(\nabla\phi)$ and since tr_g is also a surjective morphism of representations the component of $\nabla\phi$ in $\mathfrak{g}_2$ also vanishes. Combined with $d\phi=0$ this shows that $\nabla\phi=0$ and completes the proof of the statement. □

Comparing the second point above with proposition 3, more specifically the third item in example 4 shows that G_2-manifolds are (in the compact case) critical points of Hitchin's functional. In fact, they have maximal volume with respect to local variations of the 3 form ϕ. Next, we shall give a modern proof of the following Theorem of Bonan [3].

Theorem 4. *Let (X^7,g) be a G_2-manifold, then g is Ricci flat.*

Proof. Denote by $P\subset FSO(n)$ the G_2 structure and by $R\in\Omega^0(X,S^2\mathfrak{g}_P)$ the Riemann curvature tensor of g. Using highest weight theory we can decompose the space of algebraic curvature tensors into irreducible representations. We start by decomposing

$$S^2\mathfrak{g}_2\cong W_{0,0}\oplus W_{2,0}\oplus W_{0,2}, \tag{6}$$

where $W_{0,0}\cong\mathbb{R}$ is the trivial irreducible representation and the $(n,k)\in\mathbb{Z}^2$ are labeling the weights, so that $W_{1,0}\cong V$ and $W_{0,1}\cong\mathfrak{g}_2$. Moreover, the first Bianchi identity states that $R\in\ker(b)$, where

$$b:S^2(V^*)\to\Lambda^3V^*\otimes V^*$$

is the Bianchi map which antisymmetrizes the first three entries. However $\ker(b)=\ker(b:S^2(\mathfrak{g}_2)\to\Lambda^4V)$. Decompose the right hand side into irreducibles $\Lambda^4V\cong W_{0,0}\oplus W_{1,0}\oplus W_{2,0}$ and compare with the relation 6. In fact, the Bianchi map is a morphism of G_2-representations and is injective on $W_{0,0}$ and $W_{2,0}\cong S^2_0V^*$, so we conclude that the kernel of the Bianchi map is the 77 simensional piece $W_{0,2}$. Hence the Riemannian curvature tensor has values on $W_{0,2}$ (this result is attributed to Alexeevski [1]).
We now use this information in order to analyze the Ricci tensor Ric, which has values on $S^2(V^*)$. It is obtained from R via $Ric=r(R)$, where

$$r:W_{0,2}\to S^2(V^*)$$

is the Ricci contraction, mapping a curvature tensor to a symmetric, bilinear form. This is also a morphism of G_2 representations and since $S^2(V^*)$ decomposes into irreducible components as $W_{0,0} \oplus W_{2,0}$, r must vanish identically and so does Ric. □

G_2-manifolds are Ricci flat (theorem 4) and a similar application of the Cheeger-Gromoll splitting theorem and the Böchner technique, to the one used for Calabi–Yau manifolds in Prposition 6 gives the following two propositions

Proposition 10. *Let (X,g) be a compact G_2-manifold. Then, there is a finite cover $\tilde{X}$ of X, which is isometric to $T^{7-k} \times Y^k$, where T^{7-k} is a torus and Y^k is k dimensional manifold equipped with a Ricci flat metric. Moreover, if (X,ϕ) is further supposed to be irreducible, then it has finite fundamental group.*

Proposition 11. *Let (X,g) be a simply connected G_2-manifold, then (X,g) is irreducible, i.e. $Hol = G_2$ if and only if there are no parallel 1-forms.*

Proof. Since (X,g) is a G_2-manifold the holonomy is contained in G_2 and g is Ricci flat. Hence, if there is a parallel one form one can use the flow of the associated Killing field, which is parallel by the Bochner formula, to find a line and use the Cheeger-Gromoll splitting theorem to write $X = \mathbb{R}_t \times Y^6$ with the cylindrical metric $g = dt^2 + g_6$. In this case $Hol(g) = Hol(g_6) \subset G_2 \cap (1 \times SO(6)) \cong 1 \times SU(3)$, which is properly contained in G_2.
In the opposite direction we prove that if the holonomy Hol is properly contained in G_2 then there is a parallel 1-form. First we analyze the case where (X,g) is locally symmetric. If this is the case, then since from Bonan's theorem 4 is Ricci flat and locally symmetric it must actually be flat. If (X,g) is not locally symmetric and Hol is a proper subgroup of G_2 we can invoke Berger's theorem [2] to conclude that Hol is either $1 \times SU(3)$, $SO(3) \times SU(2)$, $1_3 \times SU(2)$ or 1_7. In each of these cases there is a local splitting $U = U_1 \times U_2$ and $g|_U = g_1 + g_2$, where U_1 is at most 3 dimensional and Ricci flat and so flat. So the case $SO(3) \times SU(2)$ actually has to reduce to $1_3 \times SU(2)$ and in all the cases there is a locally flat factor, then since X is simply connected there is a global parallel one form. □

Remark 6. *Notice that in the first direction the condition that X is simply connected is not used. Hence it is true that for (X,g) an irreducible G_2-manifold there are no parallel 1-forms.*

Proposition 12. *Let (X,ϕ) be a G_2-manifold, then the exterior bundle splits orthogonally as*

$$\begin{aligned} \Lambda^1 &= \Lambda^1_7 \\ \Lambda^2 &= \Lambda^2_7 \oplus \Lambda^2_{14} \\ \Lambda^3 &= \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}, \end{aligned}$$

where the subscript indicates the rank of the component and these components are such that for $\psi = *\phi$

$$\begin{aligned} \Lambda^2_7 &= \{\iota_V \phi, V \in \Gamma(TX)\} = \{\beta \mid *(\beta \wedge \phi) = 2\omega\} \\ \Lambda^2_{14} &= \{\beta \mid \beta \wedge \psi = 0\} = \{\beta \mid *(\beta \wedge \phi) = -\beta\} \\ \Lambda^3_1 &= \langle \phi \rangle \\ \Lambda^2_7 &= \{\iota_V \psi, V \in \Gamma(TX)\} \\ \Lambda^3_{27} &= \{\beta \mid \beta \wedge \psi = 0 \ \ \textit{and} \ \ \beta \wedge \phi = 0\}. \end{aligned}$$

Moreover if β is a 2-form and π_7, π_{14} denote the respective projections on the irreducible components, then the following algebraic identities hold

$$*(*(\beta \wedge \psi) \wedge \psi) = 3\pi_7(\beta) \quad (7)$$

$$*(\beta \wedge \phi) = 2\pi_7(\beta) - \pi_{14}(\beta). \quad (8)$$

It follows from Chern's theorem 1 that on a G_2-manifold the Laplacian Δ_ϕ preserves the decomposition of the spaces of differential forms into irreducible G_2 representations. Hence, the decomposition above still holds at the level of Harmonic forms.

Corollary 2. *Let (X,ϕ) be a G_2-manifold, then the spaces of harmonic forms $\mathcal{H}^*$ decompose into irreducible representations as*

$$\begin{aligned} \mathcal{H}^1 &= \mathcal{H}^1_7 \\ \mathcal{H}^2 &= \mathcal{H}^2_7 \oplus \mathcal{H}^2_{14} \\ \mathcal{H}^3 &= \mathcal{H}^3_1 \oplus \mathcal{H}^3_7 \oplus \mathcal{H}^3_{27}, \end{aligned}$$

and there are isomorphisms $\mathcal{H}^1 \cong \mathcal{H}^2_7 \cong \mathcal{H}^3_7$. In particular, if X is compact this induces a splitting of the de Rham cohomology.

We can combine corollary 2 to Chern's theorem with proposition 11 to investigate further the topology of irreducible G_2-manifolds.

Proposition 13. *Let (X^7, g) be an irreducible G_2-manifold, then the spaces of harmonic forms $\mathcal{H}^*$ decompose into irreducible representations as*

$$\begin{aligned}\mathcal{H}^1 &= 0\\ \mathcal{H}^2 &= \mathcal{H}^2_{14}\\ \mathcal{H}^3 &= \mathcal{H}^3_1 \oplus \mathcal{H}^3_{27}.\end{aligned}$$

In particular, if X is compact then $b^1 = 0$, $b^2 = b^2_{14}$ and $b^3 = 1 + b^3_{27}$.

Proof. The irreducibility condition, i.e. that $Hol = G_2$ implies via remark 6 that there are no parallel 1 forms. Since $Ric = 0$ by Bonan's theorem 4, there is a Weitzenböck formula $\nabla^*\nabla\alpha = \Delta\alpha$ for all 1-forms α. Combining this with corollary 2 gives the decomposition of the harmonic forms in the statement. In the particular case when X is compact, the result follows from Hodge theory. □

Remark 7. *In particular, this further proves that a compact, irreducible G_2-manifold has finite fundamental group.*

Now we will focus on compact G_2-manifolds which were first constructed by Dominic Joyce [15, 16], see also [17] for a summary of this first construction. On these we shall construct a quadratic form on the second cohomology which can be used to identify a constraint on the first Pontryagin class $p_1(X) \in H^4(X, \mathbb{R})$ of a compact, irreducible G_2-manifold.

Definition 9. *Let (X, g) be a compact G_2-manifold and define the bilinear form Q on $H^2(X, \mathbb{R})$ given by*

$$Q(\alpha, \beta) = \langle \alpha \cup \beta \cup [\phi], [X] \rangle.$$

Lemma 3. *Let (X, g) be a compact, irreducible G_2-manifold. Then, the quadratic form on $H^2(X, \mathbb{R})$ given by $\alpha \mapsto Q(\alpha, \alpha)$ is negative definite.*

Proof. Let $a \in \alpha \neq 0$ be the harmonic representative, then by proposition 13 it follows that $a = \pi_{14}(a)$, i.e. $\pi_7(a) = 0$. Moreover, using equation 8 one has

$$a \wedge a \wedge \phi = -\pi_{14}(a) \wedge *\pi_{14}(a) = -|a|^2 \,\mathrm{dvol},$$

hence $Q(\alpha, \alpha) = -\int_X |a|^2 \,\mathrm{dvol} < 0$. □

Proposition 14. *Let (X, g) be a compact, irreducible G_2-manifold, then $\langle p_1(X) \cup [\phi], [X] \rangle \leq 0$.*

Proof. Let R denote the curvature of the Levi-Civita connection of g. In a local trivialization $R \in \Omega^2_{14} \otimes \mathfrak{g}_2$ and $\mathfrak{g}_2 \subset \mathfrak{so}(4)$, i.e. it is represented by an antisymmetric matrix R_{ij} of forms in Ω^2_{14}. Then, $p_1(X) \cup [\phi]$ is represented by

$$\operatorname{tr}(R \wedge R) \wedge \phi = \sum_{i,j} R_{ij} \wedge R_{ji} \wedge \phi = -\sum_{i,j} R_{ij} \wedge R_{ij} \wedge \phi = -|R|^2 \,\mathrm{dvol}.$$

Hence as in the previous lemma (or rather as in its proof) $\langle p_1(X) \cup [\phi], [X] \rangle = -\int_X |R|^2 \,\mathrm{dvol} \leq 0$. □

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