AN INVITATION TO SYMPLECTIC TORIC MANIFOLDS

Ana Cannas da Silva

ETH Dep. Mathematics
Raemistrasse 101
8092 Zurich, Switzerland
e-mail: ana.cannas@math.ethz.ch

1 What is Symplectic Geometry?

Geometry concerns the study and measure of space. Symplectic refers to an additional structure that can be put on some even-dimensional spaces. Symplectic geometry is intrinsically related to complex geometry and, just like complex geometry, is sometimes counterintuitive. Whereas local complex geometry is basically modelled on $\mathbb{C}$, $\mathbb{C}^2$, $\mathbb{C}^3$, etc, local symplectic geometry is basically modelled on $\mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{R}^6$, etc.

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A symplectic form $\omega$ at a point $p$ of a manifold $M$ is a special type of differential 2-form, i.e., a device that takes two tangent vectors $\vec{u}, \vec{v} \in T_p M$ as input and returns a real number as output, that may be interpreted as

$$\omega(\vec{u}, \vec{v}) = \text{kind of signed area of parallelogram spanned by } \vec{u} \text{ and } \vec{v}.$$  

By signed area we mean, in particular, a number that may be positive, negative, or zero, contrasting with usual (euclidean, riemannian, ...) geometries.

In the case of the basic model of $\mathbb{R}^2$ with its so-called standard symplectic form,

$$\omega_0 := dx \wedge dy ,$$

this signed area is

$$\omega_0(\vec{u}, \vec{v}) = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = u_1v_2 - u_2v_1 ,$$

thus actually equal to plus or minus the euclidean area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$. The sign depends on the orientation of the basis $\vec{u}, \vec{v}$ and $\omega_0(\vec{v}, \vec{u}) = -\omega_0(\vec{u}, \vec{v})$. Moreover, there is only zero as output in just one dimension, since $\omega_0(\vec{v}, \vec{v}) = 0$ for all $\vec{v}$.

In the next case of $\mathbb{R}^4$, the standard symplectic form,

$$\omega_0 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 ,$$

just adds up the contributions from the projections onto the two coordinate planes $x_1, y_1$ and $x_2, y_2$. If we have vectors $\vec{u} = \vec{u}_1 + \vec{u}_2$ and $\vec{v} = \vec{v}_1 + \vec{v}_2$ (where $\vec{u}_1, \vec{v}_1$ and $\vec{u}_2, \vec{v}_2$ denote the projections onto the coordinate planes $x_1, y_1$ and $x_2, y_2$), then

$$\omega_0(\vec{u}, \vec{v}) = \left( \text{signed area of } A_1 \right) (\vec{u}_1, \vec{v}_1) + \left( \text{signed area of } A_2 \right) (\vec{u}_2, \vec{v}_2)$$

can be thought of as a sum of signed areas for the projections $A_1$ and $A_2$ onto each of the coordinate planes $x_1, y_1$ and $x_2, y_2$. Other projections are not taken into account.

The higher cases $\mathbb{R}^{2n}$ are analogous. In particular, in $\mathbb{R}^6$ we have the standard symplectic form

$$\omega_0 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3 .$$

Physicists often regard $(x_1, x_2, x_3)$ as position coordinates and $(y_1, y_2, y_3)$ as momentum (kind of velocity) coordinates of a particle in 3-dimensional...
space. The symplectic form $\omega_0$ encodes the mutual entanglement of position and momentum in a somewhat implausible way that actually fits reality. In Section 2, we will describe the motion of a classical mechanical system via Hamilton’s equations for position and momentum in terms of a flow on a symplectic manifold.

Historical remark:

Symplectic geometry is a branch of mathematics, that could be viewed as emerging in the XIX century from classical mechanics. The mathematicians William Rowan Hamilton (1805-1865) and Sofia Kovalevskaya (1850-1891) were at the onset of this field and worked on problems related to the motion of rigid bodies. Symplectic geometry experienced a vigorous expansion in the last 50 years and deals nowadays with many other geometric problems, stimulated by interactions with diverse areas of mathematics and physics. The adjective symplectic in mathematics is a calque [2] coined by Hermann Weyl, by substituting the Latin root in complex by the corresponding Greek root, in order to label the symplectic group.

In general, a symplectic manifold is a pair $(M, \omega)$ where $M$ is a manifold (necessarily even-dimensional, say $\dim M = 2n$) and $\omega$ is a closed nondegenerate 2-form on $M$. Whereas closedness is a natural differential condition from analysis, nondegeneracy is an algebraic condition saying that at each point any nonzero tangent vector admits a nontrivial pairing with some other tangent vector – this is what forces the evenness of the dimension.

One of the fundamental theorems in symplectic geometry goes back to Darboux [6] in the late XIX century in the context of differential systems. What is now known as Darboux’s theorem states that any symplectic manifold looks locally near any of its points like a neighborhood of the origin in $\mathbb{R}^{2n}$ equipped with

$$\omega_0 := dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n .$$

We hence refer to $(\mathbb{R}^{2n}, \omega_0)$ as the local model. Although this shows that there are no local invariants in symplectic geometry besides the dimension, the local symplectic geometry, i.e. the symplectic geometry of $(\mathbb{R}^{2n}, \omega_0)$, is already quite interesting and there remain deep open questions about

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2 A calque or loan translation is a word or phrase that is introduced through translation of the constituents into another language.
it. Normal form theorems like Darboux’s play a central role in symplectic geometry.

On a symplectic manifold \((M, \omega)\), the top power of the symplectic form, \(\omega^n\), is necessarily a volume form, called the \textit{symplectic volume}. This follows from the nondegeneracy of \(\omega\), and may be also seen through Darboux’s theorem with \(\omega^n_0 = n! \, dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n\). Therefore, a symplectic manifold is \textit{symplectically} oriented, and nonorientable manifolds cannot be symplectic.

On a symplectic manifold \((M, \omega)\), we are able to integrate the symplectic form \(\omega\) over a surface \(A \subset M\):

\[
\int_A \omega = \text{symplectic area of } A.
\]

In the case of \((\mathbb{R}^4, \omega_0)\), this yields again a sum of contributions from the two projections onto each of the coordinate planes \(x_1, y_1\) and \(x_2, y_2\):

\[
\int_A \omega = \int_{A_1} \, dx_1 \wedge dy_1 + \int_{A_2} \, dx_2 \wedge dy_2.
\]

Such a measurement is \textit{anisotropic} in the sense that (multiple-dimensional) directions are not all the same. For instance, a nontrivial surface in the \(x_1, x_2\)-plane has one-dimensional projection onto the \(x_1, y_1\) and \(x_2, y_2\) planes, hence has \textit{zero symplectic area}. Such a surface in a four-dimensional manifold is called \textit{lagrangian}. On the other hand, a nontrivial surface in the \(x_1, y_1\) plane already has a \textit{nonzero symplectic area}. Such a surface is called \textit{symplectic}.

In general, we distinguish different important types of submanifolds in a \(2n\)-dimensional symplectic manifold \((M, \omega)\). A \textit{symplectic submanifold} is a submanifold where the restriction of the symplectic form is nondegenerate, hence still a symplectic form. Such submanifolds are again even-dimensional. When \(n = 1\), these submanifolds turn out to be related to \textit{complex curves}. An \textit{isotropic submanifold} is a submanifold where the restriction of the symplectic form vanishes identically. Any one-dimensional submanifold is isotropic and isotropic submanifolds are at most half-dimensional; this follows from linear algebra. A \textit{lagrangian submanifold} is an \(n\)-dimensional isotropic submanifold. Lagrangian submanifolds are thus the largest isotropic submanifolds and turn out to be related to dynamics.
Examples and nonexamples:

(0) As mentioned, the examples \((\mathbb{R}^{2n}, \omega_0)\) above are the local prototypes of symplectic manifolds.

(1) Any oriented surface may be equipped with a symplectic structure by choosing any area form to take the role of symplectic form. In particular, a unit sphere in \(\mathbb{R}^3\) equipped with the standard (euclidean) area form is automatically a symplectic manifold. This area form may be written away from the poles as

\[
\omega_{\text{std}} := d\theta \wedge dh ,
\]

where \(h\) is a height function and \(\theta\) the angle around that height axis, giving total area \(4\pi\); cf. Section 4.

(2) Some of the simplest 4-dimensional symplectic manifolds are products of oriented surfaces, such as \(S^2 \times S^2\) equipped with a sum of area forms (eventually different on each factor), and complex projective space \(\mathbb{CP}^2\), that is, the space of complex lines in \(\mathbb{C}^3\). The standard symplectic form in \(\mathbb{CP}^2\) (or, for that matter, in \(\mathbb{CP}^n\)) is called Fubini-Study form and we will give some insight into it in Section 3. In general, products of symplectic manifolds are symplectic.

(3) The only spheres that may be symplectic are the 2-dimensional ones. Let us see why. In a sphere \(S^k\) of any other dimension, closed 2-forms are always exact (this topological fact is usually encoded as \(H^2(S^k) = 0\) for \(k \neq 2\)). Now, by Stokes’ theorem, a symplectic form cannot be exact on a compact manifold without boundary, because if it were \(\omega = d\alpha\), then its top power \(\omega^n = d(\alpha \wedge \omega^{n-1})\) would also be exact, which is impossible for a volume form on such a manifold:

\[
\int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-1}) = \int_{\partial M} \alpha \wedge \omega^{n-1} = 0 \text{ contradicting } \int_M \omega^n > 0 .
\]

By now there are a number of texts on symplectic geometry, a subset of which is [11, 12, 4]. For a beautiful overview geared towards symplectic topology, see McDuff’s lecture [10].

2 What are Hamiltonian Torus Symmetries?

The definition of symplectic form contains exactly what is needed for the following general assertion: On a symplectic manifold \((M,\omega)\), any smooth
function $H : M \to \mathbb{R}$ generates (in a nontrivial way) a flow that preserves both the symplectic structure $\omega$ and the function $H$.

Such a flow is called the hamiltonian flow generated by $H$ and then $H$ is called a corresponding hamiltonian function. The asserted property refers to the existence and uniqueness (by nondegeneracy of $\omega$) of a vector field $X_H$ defined by

$$\omega(X_H, \cdot) = dH(\cdot).$$

This vector field $X_H$ satisfies the following equations where we use Cartan’s magic formula, $L_X = d i_X + i_X d$, for the Lie derivative with respect to a vector field $X$:

$$L_{X_H} \omega = d i_{X_H} \omega + i_{X_H} d\omega = 0 \quad \text{and} \quad L_{X_H} H = i_{X_H} dH = 0.$$

This vector field $X_H$ integrates (by the theorem of Picard-Lindelöf) into a local time evolution, a.k.a. flow, and the equations $L_{X_H} \omega = 0$ and $L_{X_H} H = 0$ amount infinitesimally to this flow preserving $\omega$ and $H$. The vector field $X_H$ is called the hamiltonian vector field of $H$.

Examples and nonexamples:

(0) For euclidean space $(\mathbb{R}^6, \omega_0)$ and any function $H : \mathbb{R}^6 \to \mathbb{R}$, equation $\star$ for the flow generated by $H$ translates into Hamilton’s equations:

$$\begin{align*}
\frac{dx_k}{dt} &= \frac{\partial H}{\partial y_k} \\
\frac{dy_k}{dt} &= -\frac{\partial H}{\partial x_k}.
\end{align*}$$

(1) For the unit sphere $(S^2, \omega_{std} = d\theta \wedge dh)$ and hamiltonian function $H$ equal to the height function $h$, equation $\star$ yields as hamiltonian vector field

$$X_H = \frac{\partial}{\partial \theta},$$

so the corresponding flow rotates around the height axis. This clearly preserves area $\omega_{std}$ and height $H$. Notice how this contrasts with the gradient flow of $H$, which is basically perpendicular and preserves neither $\omega_{std}$ nor $H$. 
(2) For the 2-torus \((\mathbb{T}^2, \omega := d\theta_1 \wedge d\theta_2)\), we have that the rotation given by the vector field \(\frac{\partial}{\partial \theta_1}\) preserves area, yet is not hamiltonian, since the contraction
\[
\omega\left(\frac{\partial}{\partial \theta_1}, \cdot \right) = d\theta_2(\cdot)
\]
is closed yet not exact, i.e., there is no corresponding global hamiltonian function.

The flow in Example (1) is also an example of \(S^1\)-action. Indeed, the time-\(t\) evolution \(\varphi_t\) is given, with respect to these coordinates, by
\[
\varphi_t: (\theta, h) \mapsto (\theta + t, h),
\]
so it is \(2\pi\)-periodic (i.e., \(\varphi_{t+2\pi} \equiv \varphi_t\)) and satisfies the group law (i.e., \(\varphi_{t_1} \circ \varphi_{t_2} \equiv \varphi_{t_1 + t_2}\)). Because it is also hamiltonian, we call it a hamiltonian \(S^1\)-action.

Analogously, for a \(d\)-dimensional torus \(\mathbb{T}^d = S^1 \times \ldots \times S^1\) we define a hamiltonian \(\mathbb{T}^d\)-action to be an action of \(\mathbb{T}^d\) for which each of the \(S^1\)-factors acts in a hamiltonian fashion, say with hamiltonian function \(H_k\), and each of these \(H_k\) is invariant by the rest of the action. By collecting these hamiltonian functions, we build an invariant function
\[
H := (H_1, \ldots, H_d): M \to \mathbb{R}^d.
\]
This upgraded version of hamiltonian function is known as a (special case of) moment map. The concept of moment map for hamiltonian actions of arbitrary Lie groups has recently become central in geometry and topology.

Atiyah \([2]\) and, independently, Guillemin and Sternberg \([9]\) proved in the 80’s, that the image of such a function \(H: M \to \mathbb{R}^d\) on a compact, connected symplectic manifold \((M, \omega)\) corresponding to a hamiltonian \(\mathbb{T}^d\)-action is always a convex polytope. Moreover, they showed that that image is simply the convex hull of the images of the fixed points of the action. This deep and key theorem is known as the convexity theorem.
To get rid of lazy factors in that action, we concentrate on **faithful (i.e. effective) actions** for which only the identity group element gives rise to the identity diffeomorphism. We think of effective hamiltonian $\mathbb{T}^d$-actions as **hamiltonian torus symmetries**. Now, if a $d$-dimensional torus acts in a faithful and hamiltonian fashion on a $2n$-dimensional symplectic manifold, then it must be $d \leq n$. This follows from the fact that the orbits are isotropic, that isotropic submanifolds are at most half-dimensional, and that Lie theory tells us that a faithful action of a $d$-dimensional Lie group always admits orbits equivariantly diffeomorphic to the group itself, the so-called *principal* orbits. Therefore, a maximal hamiltonian torus symmetry is of the form $\mathbb{T}^n$ acting on $M^{2n}$.

### 3 What are Symplectic Toric Manifolds?

A **symplectic toric manifold** is a compact connected symplectic manifold $(M,\omega)$ with a maximal hamiltonian torus symmetry, meaning, with a faithful hamiltonian action of a half-dimensional torus. If $\dim M = 2n$, then we have the $n$-dimensional torus $\mathbb{T}^n$ acting faithfully and with a moment map

$$H : M \to \mathbb{R}^n.$$ 

Examples and nonexamples:

1. **Examples**

   a) The image interval $[0, 2]$ is the orbit space, i.e., there is exactly one $S^1$-orbit per height value. The endpoints of this interval correspond to the two fixed points (singular orbits), South pole and North pole.

   b) The best coordinates to understand this system are the *angle* coordinate $\theta$ where the rotation occurs and the function $H = h$ encoding the hamiltonian *action*, valid away from the poles. Such coordinates are called *action-angle coordinates*. With respect
to such coordinates, the symplectic form is simply a product form \( d\theta \wedge dH \), just like a form in the local model space \( (\mathbb{R}^2, dx \wedge dy) \).

(c) The area of an invariant strip on \( S^2 \) corresponding to a subinterval of \([0, 2]\) of height \( \Delta h \) is equal to \( 2\pi \cdot \Delta h \). This result goes back more than two millennia; see Section 4.

(1') We revisit the previous example from a complex viewpoint. Regarding \( S^2 \) as a Riemann sphere, we denote by \([z_0 : z_1]\) the point given by the complex line in \( \mathbb{C}^2 \) through \((z_0, z_1)\) and \((0, 0)\). The South pole is \([1 : 0]\) and the North pole is \([0 : 1]\). Now we recast that example as \((\mathbb{C}P^1, \omega_{FS})\), where the Fubini-Study symplectic form \( \omega_{FS} \) is equal to \( \frac{1}{2} \omega_{std} \), the element \( e^{it} \) of the circle acts by multiplication on the coordinate \( z_1 \),

\[
e^{it} \cdot [z_0 : z_1] = [z_0 : e^{it}z_1],
\]

which, on a chart, is again a simple shift of the angle coordinate, and the corresponding hamiltonian function is

\[
H_1 := \frac{|z_1|^2}{2(|z_0|^2 + |z_1|^2)}.
\]

(2) Consider now complex projective space \( \mathbb{C}P^n \) (as a \( 2n \)-dimensional real manifold) with a diagonal action of \( T^n \) by

\[
(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : e^{i\theta_1}z_1 : \ldots : e^{i\theta_n}z_n].
\]

The Fubini-Study symplectic form is a globally well-defined form, which, away from the hyperplanes \( z_k = 0 \), is given by the Darboux-type formula

\[
\omega_{FS} = d\theta_1 \wedge dH_1 + \ldots + d\theta_n \wedge dH_n,
\]

where the component \( H_k \) of the moment map \( H : \mathbb{C}P^n \to \mathbb{R}^n \) is

\[
H_k := \frac{|z_k|^2}{2(|z_0|^2 + \ldots + |z_n|^2)}.
\]

For instance, when \( n = 3 \) we get the following picture:

We list again the earlier features, some of which now take more thought to check:

(a) The image simplex is the orbit space, i.e., there is exactly one \( T^n \)-orbit per point on the \( n \)-simplex. The vertices of this simplex
Figure 2: Moment map for the standard action on $\mathbb{CP}^3$.

correspond to the $n + 1$ fixed points, $[1 : 0 : \ldots : 0], \ldots, [0 : \ldots : 0 : 1]$. The interior points correspond to orbits through points of the form $[z_0 : z_1 : \ldots : z_n]$ with all coordinates $z_k$ nonzero.

(b) Best to understand this system are the action-angle coordinates, $H_1, \ldots, H_n$ and $\theta_1, \ldots, \theta_n$. With respect to these coordinates, and in points mapping by $H$ to the interior of the simplex, the symplectic form is just like a form in the local model space $(\mathbb{R}^{2n}, dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n)$.

(c) The (symplectic) volume of a $\mathbb{T}^n$-invariant subset $H^{-1}(S)$ is simply equal to $(2\pi)^n |S|$, where $|S|$ is the (euclidean-)volume of the subset $S$ of the simplex.

By the convexity theorem, we already know that the moment map image of a $2n$-dimensional symplectic toric manifold is a polytope in $\mathbb{R}^n$. One can show that such a polytope enjoys special properties: it is simple, i.e., there are $n$ edges meeting at each vertex, it is rational, i.e., the edges meeting at each vertex $\tau$ are of the form $\tau + tu_j$, $t \geq 0$, with each $u_j \in \mathbb{Z}^n$, and it is smooth, i.e., for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$; see, for instance, [5].

As first proved by Delzant [7], it turns out that this polytope encodes enough information to reconstruct its originating symplectic toric manifold, and that all such simple, rational, smooth polytopes occur as moment map images of symplectic toric manifolds. Delzant’s theorem is a celebrated result classifying symplectic toric manifolds in terms of polytopes:

\[
\begin{align*}
\{ \text{2n-dim’l symplectic} & \text{toric manifolds} \} \leftrightarrow \{ \text{simple rational smooth} \\
& \text{polytopes in } \mathbb{R}^n \}
\end{align*}
\]

where this one-to-one correspondence takes a symplectic toric manifold, $(M, \omega, H)$ where the $\mathbb{T}^n$-action admits $H : M \to \mathbb{R}^n$ as moment map, to
the polytope which is the image of this moment map:

\[(M, \omega, H) \leftrightarrow H(M)\].

For such a correspondence, there are underlying notions of equivalence of the objects involved. In the simplest version, polytopes in \(\mathbb{R}^n\) are identified up to translation, and symplectic toric manifolds are identified up to equivariant diffeomorphism preserving the symplectic forms: \((M_1, \omega_1, H_1)\) and \((M_2, \omega_2, H_2)\) with actions of \(T^n\) are equivalent if and only if there is a diffeomorphism \(\varphi : M_1 \to M_2\) such that \(\varphi^* \omega_2 = \omega_1\) and \(\varphi(g \cdot p) = g \cdot \varphi(p)\) for all \(g \in T^n\) and \(p \in M_1\).

Note that the problem of classifying compact symplectic manifolds in dimension 4 or higher is completely open. The presence of a hamiltonian torus symmetry significantly helps.

Since there is just one 1-dimensional polytope of length \(\ell\) up to translation, we see that the only 2-dimensional symplectic toric manifolds are scaled spheres \((S^2, \frac{\ell}{2} \omega_{std})\) with rotation action as above. The panorama for 2-dimensional polytopes is much more rich. Still, up to translation, the 2-dimensional simple, rational, smooth polytopes with only three vertices are the triangles with vertices \((0,0)\), \((\ell,0)\) and \((0,\ell)\) or their transforms by \(\text{GL}(2; \mathbb{Z})\). This is saying that the corresponding symplectic toric manifolds are \((\mathbb{C}P^2, 2\ell \omega_{fs})\) with standard \(T^2\)-action or their transforms by an isomorphism of \(T^2\).

The upshot is that any such symplectic toric manifold is given combinatorially in terms of a polytope in an euclidean space of half the dimension that of the manifold. Hence, all questions pertaining to such manifolds should admit an answer in terms of polytopes – a mathematician’s dream! In particular, the earlier properties admit generalizations to all symplectic toric manifolds \((M, \omega, H)\) as follows:

(a) The polytope image is the orbit space, so \(H\) is also the point-orbit projection, and the vertices of the polytope correspond to the fixed points. There are precise descriptions of the isotropy subgroups in terms of the face-stratification.

(b) There are action-angle coordinates, \(H_1, \ldots, H_n\) and \(\theta_1, \ldots, \theta_n\), valid at points mapping to the interior of the polytope, which are the best coordinates to understand this system. With respect to them, the symplectic form is \(\omega = d\theta_1 \wedge dH_1 + \ldots + d\theta_n \wedge dH_n\).

(c) The (symplectic) volume of a \(T^n\)-invariant subset is equal to \((2\pi)^n\)
times the (euclidean) volume of the corresponding subset in the polytope.

A lot of the geometry of symplectic toric manifolds has already been understood, yet many interesting questions remain. Currently, these manifolds are used as test grounds for theories or conjectures in topology, geometry and mathematical physics, such as mirror symmetry.

Many open questions for these manifolds relate to their lagrangian submanifolds. We can see that connected lagrangian submanifolds invariant by $\mathbb{T}^n$ are principal $\mathbb{T}^n$-orbits, i.e., those corresponding to the interior points of the image polytope. We might now ask about other lagrangian submanifolds that fit nicely with respect to the torus action, in the sense that they are invariant by some subgroup of $\mathbb{T}^n$ and they intersect $\mathbb{T}^n$-orbits in a clean way. The image under the moment map of such a lagrangian submanifold of $(M, \omega, H)$ lies in the intersection of the polytope $H(M)$ with an affine subspace. Examples are all principal $\mathbb{T}^n$-orbits, the standard real part submanifolds like $\mathbb{RP}^n$ in $\mathbb{CP}^n$, lagrangian submanifolds like the one presented in [3], and many lagrangian submanifolds sitting in level sets of components of the moment map.

4 Epilogue – all the way from Archimedes

We close by going back more than two millenia to Archimedes’ supposedly favourite work on measuring spheres and cylinders. In around 200 BC, Archimedes was the first to realize that the surface area of a sphere between two parallel planes intersecting it depends only on the distance between those planes and not on the height where they intersect the sphere. Moreover, Archimedes asserted that the surface area on the sphere is the same as that of a cylinder with the radius of that sphere and height given by the distance between the planes, as the following figure illustrates. This is exactly the feature that allows us to write the standard area form as $\omega_{std} = d\theta \wedge dh$.

Nowadays, if you know first-year calculus, you may check Archimedes result by computing an appropriate surface integral using, for instance, cylindrical coordinates $(\theta, z)$ to write points on the sphere as $(x, y, z) = (\sqrt{R^2 - z^2} \cos \theta, \sqrt{R^2 - z^2} \sin \theta, z)$:

$$\text{Area} = \int_0^{2\pi} \int_h^{h+\Delta h} R \, dz \, d\theta = 2\pi R \cdot \Delta h,$$

or else use some approximation method and then take the limit [1]. However, Archimedes did not know calculus. It seems that he used an approximation
Figure 3: Spherical and cylindrical strips all with the same area: $2\pi R \cdot \Delta h$; image kindly reproduced from [1].

argument, for which a relevant reference is the palimpsest discovered in the XX century after some quite adventurous history.

In the 80’s, Duistermaat and Heckman [8] showed powerful results for symplectic manifolds with hamiltonian torus actions, which may be viewed as a vast generalization of Archimedes’ theorem for the 2-sphere. Just like Archimedes might have had no idea that, more than two millennia later, his spirit would be at the origin of new mathematics, one wonders what other leaps await mankind starting from symplectic toric manifolds.

References

[1] Abreu, M., Cannas da Silva, A., 22 séculos a medir área (22 centuries measuring area), paper contributed for the Klein project in Portuguese (2011) available at klein.sbm.org.br/?s=cannas

3A palimpsest is a manuscript page, either from a scroll or a book, from which the text has been scraped or washed off so that the page can be reused for another document.


