Quantum Cosmology: Mathematics times Physics

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Resumo: As ideias básicas de Cosmologia Quântica (QC) são apresentadas em termos gerais, indicando-se como se interligam com a Relatividade Geral. O autor apontou a ser menos técnico e focando mais em introduzir de forma simples alguns conceitos (um estudo em profundidade está disponível em [1, 2]). De forma a também apresentar alguns desenvolvimentos recentes do interesse da comunidade matemática, descrevem-se de forma muito breve resultados onde simetrias aparentes podem induzir a seleção de condições fronteira através de processos algébricos, podendo extender-se àlgebra de supersimetria [1, 2]. Detalhes mais técnicos podem encontrar-se em [3, 4, 5, 6].

Abstract We mention, in general terms, the basic ideas of quantum cosmology (QC) and how such framework can be intertwined with General Relativity (GR) . The author aimed to be less technical and focus more on introducing in simple terms some concepts [1, 2]. In order to present new advances that may be of interest to readers, we very briefly discuss recent results involving hidden symmetries, showing that specific boundary conditions can be related to the algebra of Dirac observables afterwards associated to the algebra of supersymmetry [1, 2]. More technical details are found in [3, 4, 5, 6].

palavras-chave: Cosmologia Quântica; Supersimetria; Universo Primordial.

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1 Introduction-I

Contemporary cosmology is a well-established quantitative area, where remarkable new technology has been used to get a precise chart of the universe. In particular, fundamental cosmological parameters have been recently displayed with an outstanding precision: At the dawn of the XXIst century, the cosmology community has thus entered into a *golden epoch*, where future improvements (both in quantity and quality) will allow to get an even clearer perspective of *where* (and *why* and *how*) we are.

The current paradigm in cosmology (under scrutiny but so far successfully able to pass all the major tests) is the *inflationary 'big bang'* scenario. It allowed to address some of the observational inconsistencies of the standard cosmological model:

- (Close) Regions currently observed and spatially separated would not have been (as determined by the standard cosmological dynamics) in thermal contact; But the isotropy of the cosmic microwave background radiation (CMBR) forces to consider otherwise: This is the *horizon* problem.
- For the currently observed spatial flatness, the universe had to be flat at early times with an incredible precision (with about a margin of 10^{-50}): This is the *flatness* problem.

These problems can, however, be explained by an early 'inflationary' phase of exponential-like expansion of the universe: Causal contact then becomes possible in the primeval past, at the same time as the universe enlarges so much that locally it becomes nearly flat, with topological defects becoming effectively unobservable. Inflation also provides a suitable mechanism for initial small *quantum* matter perturbations to increase and to form a fluctuations spectrum, which is consistent with observations.

But an apparent weakness emerges for this picture: For this paradigm to be realistic, it has to be generically possible. I.e., *what* is the probability for the inflationary scenario to occur? Moreover, *how* did those perturbations arise? The problem is that these questions are, not withstanding its merits, beyond the inflationary paradigm.

In fact, this is the issue of the *initial conditions* of the universe. In brief, let us first indicate *why* discussing initial conditions is crucial; Subsequently, we will focus on *how* to investigate their generality.

Classically speaking how, can we make for *any* cosmological case a choice of initial conditions over any other (choice)? Let us be more concrete. A suitable domain for an inflationary stage can be easily identified in some simple models. But, and this is the *crucial* issue, that evolution requires a *choice* of initial conditions. Are they general enough so that our Universe is general enough, or say, natural to have emerged? To subsequently assert their generality, the theory of dynamical systems applied to the, e.g., GR equations about a given cosmological model, leads to (parametrized) families of physically distinct solutions (i.e., trajectories in the corresponding phase space). The case of the spatially homogeneous and isotropic closed model (or, the Friedmann-Lemaître-Robertson-Walker (FLRW) $k = +1$ solutions to be more technical) is particulary pertinent: If initial values of matter fields and their derivatives are restricted to start away from the curve associated to spatially flat case, the universes recollapses, without any stage of sufficient inflation; Otherwise, inflation occurs. Hence, the choice of the initial values is indeed crucial for the (generality of) occurrence of satisfactory inflation.

Additional arguments are therefore required. Namely, invoking *quantum* cosmological ingredients: E.g., the universe began in some sort of transition from a quantum regime, the initial classical parameters determined in a probabilistic method. The task seems then to transfer our quest into determining (*i*) the most probable[1] state (wave function) of the universe and (*ii*) its distinctive predictive signatures. Before proceeding into such issues, it must also be said that QC is basically the application of quantum mechanics to models with time reparametrization invariance (e.g, general relativity). Indeed, QC can be considered as a (toy model-like) attempt to obtain relevant information for a full quantum theory of gravity: One of the supreme challenges (surely in the XXIst century) for fundamental science.

Let us also add that on the one hand, general relativity is not perturbatively renormalisable: This means that naive efforts from, e.g., using Feynmann diagrams techniques have failed. On the other hand, general relativity is a theory appropriate to deal with the larger scales of spacetime and quantum physics applies instead to extremely small scales. Furthermore, the concept of quantum gravity also means to quantize spacetime itself and not merely quantizing the matter fields present in that spacetime background. In spite of all these (apparently) severe obstacles, the problem cannot be avoided: Overwhelming observational data seems to indicate that the universe did start out very, very small indeed (the cosmological Big Bang) and quantum mechanics (as the essential framework) ought to be applied to this very early universe.

2 From the Hamiltonian formulation of GR towards its quantization

From the previous section, it is clear that in order to advance towards a

[1] Assuming the probability interpretation still holds...

QC point of view, the essentials of a quantum formulation of gravity have to be analysed. This has usually embraced the presentation of a corresponding Hamiltonian description of general relativity, where the ADM (or $3+1$) decomposition of spacetime is mandatory. In the following, we will employ 4-dimensional general relativity in terms of a metric description.

Making a $3+1$ split of the 4-dimensional spacetime manifold, $\mathcal{M}$, is basically, to foliate it into spatial hypersurfaces, Σ_t, labeled by a *global* time parameter, t. The spacetime dimensional metric [with a $(-+++)$ Lorentzian signature] is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\omega^0 \otimes \omega^0 + h_{ij}\omega^i \otimes \omega^j, \tag{1}$$

where we use the basis

$$\omega^0 \equiv N dt, \quad \omega^i \equiv dx^i + N^i dt, \tag{2}$$

with greek indices running from 0 to 3 and latin indices from 1 to 3. A few remarks are in order:

- This decomposition requires the manifold $\mathcal{M}$ to be *globally hyperbolic*;
- $N(t, x^k)$ is called the *lapse* function and measures the difference between the *coordinate* time, t, and *proper* time, τ, on curves normal to the hypersurfaces Σ_t.
- The quantity $N^i(t, x^k)$ is the *shift* vector: It measures the difference between a spatial point, P, and the point one would reach if instead of following P from one hypersurface to the next one followed a curve tangent to the normal.
- ${}^{(3)}h_{ij}(t, x^k) \equiv h_{ij}(t, x^k)$ is the *intrinsic* 3-metric (also called *first fundamental form*), induced on the spatial hypersurfaces by the full 4-dimensional metric, $g_{\mu\nu}$;
- In components (matrix representation)

$$g_{\mu\nu} = \begin{bmatrix} -N^2 + N_i N^i & N_j \\ N_i & h_{ij} \end{bmatrix}. \tag{3}$$

In more well known terms, this embedding is described by the $(3+1)$ decomposition of the four-metric,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(N^2 - N_i N^i\right) dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j. \tag{4}$$

Let us take the action to be the (a bit simplified) standard Einstein-Hilbert action coupled to matter,

$$S = \frac{M_{\rm Pl}^2}{16\pi}\left[\int_{\mathcal{M}} d^4x(-g)^{\frac{1}{2}}(R-2\Lambda)\right] + S_{matter}. \tag{5}$$

The matter action for a scalar field Φ, is

$$S_{matter} = -\frac{1}{2}\int d^4x(-g)^{\frac{1}{2}}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)\right]. \tag{6}$$

Note that $M_{\rm Pl}^2 := \frac{1}{8\pi G}$ is the reduced Planck's mass in natural units, R is the Ricci curvature scalar of the space-time, Λ is the cosmological constant.

The Hamiltonian form of the action can be derived in a standard fashion as,

$$S = \int d^3x\ dt\left[\dot{h}_{ij}\pi^{ij} + \dot{\Phi}\pi_\Phi - N\mathcal{H} - N^i\mathcal{H}_i\right], \tag{7}$$

where π^{ij} and π_Φ are the momenta conjugate to h_{ij} and Φ respectively. In addition, a '$\cdot$' represents a derivative with respect to t, whereas $\mathcal{H}$ and $\mathcal{H}_i$ represent the Hamiltonian and momentum constraints, respectively, which are associated to specific invariance properties under coordinate transformations (see e.g. [1] for more details). The Hamiltonian is a sum of constraints, with lagrange multipliers the lapse N and shift N^i, which have arise due to the choice of slicing and time. The Hamiltonian constraint is

$$\mathcal{H} = \frac{16\pi}{M_{\rm Pl}^2}G_{ijkl}\pi^{ij}\pi^{kl} - \frac{M_{\rm Pl}^2}{16\pi}h^{\frac{1}{2}}({}^3R-2\Lambda) + \mathcal{H}^{matter} = 0, \tag{8}$$

where G_{ijkl} is the DeWitt metric and is given by

$$G_{ijkl} \equiv \frac{1}{2}h^{-\frac{1}{2}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right). \tag{9}$$

It should be noted that the classical dynamics takes place in the *superspace*, the space of all three-metrics and matter field configurations $(h_{ij}(x), \Phi(x))$ on a three-surface. A metric on superspace is the DeWitt metric (by addition of the matter field metric). Note that the signature of the DeWitt metric is independent of the signature of spacetime, its signature is hyperbolic at every point x in the three-surface.

2.1 Canonical Quantization

Following Dirac, the quantization of constrained systems is obtained and a classical constraint becomes a restriction on physically allowed wave functionals $\Psi[h_{ij}, \Phi]$ on superspace. A special feature of this wave function is the fact that it does not depend explicitly on the coordinate time label t. This is the consequence of the reparametrization-invariant feature of GR and "time" is subtended in the dynamical variables h_{ij}, Φ.

Let us be more concrete. By implementing for momenta the following canonical quantization rule,

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}} \qquad \pi_\Phi \to -i\frac{\delta}{\delta \Phi} \tag{10}$$

one obtains the equations for Ψ. The Wheeler-DeWitt (WDW) equation is

$$\mathcal{H}\Psi = \left[-G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - h^{\frac{1}{2}}({}^3R - 2\Lambda) + \mathcal{H}^{matter}\right]\Psi = 0 \tag{11}$$

where we have ignored operator ordering problems. The WDW equation (11), obtained from (**??**3.8)), denotes the reparametrization invariance of the theory and describes the dynamical evolution of the wave function in superspace and is a second order hyperbolic functional differential equation; In (10), $\frac{\delta}{\delta h_{ij}}$, e.g., denotes functional derivatives. In general, the WDW equation has many solutions, so in order to discriminate a unique solution, some boundary conditions are needed.

The above framework is indeed quite elegant and promising. Some results are rather relevant [1, 2]. But, in the end, the essential question is the following: *How* can we make any prediction within the above structure (in order to study the physical consequences for the evolution of the universe) from an initial quantum stage? These are yet *open* questions, in the sense that research is active: *How* to get solutions with physical meaning in order to proceed to make predictions.

The nature of the Wheeler–DeWitt equation requires thus that specific boundary conditions for the wavefunction of the universe must be implemented. Whereas the Hamiltonian formulation of canonical general relativity, within the ADM plus Dirac quantization guidelines, were the focus within the 60-70's, the investigation of the consequences of boundary (and initial!) conditions became the line of sight for the navigation into quantum gravity from the 80's onwards [1, 2].

Two boundary condition proposals[2] took most of the attention: the *no-boundary* (from J. Hartle and S. Hawking) and the *tunneling* (from A. Vilenkin). In fact, and with the exception of specific cases of simple situations, a choice of boundary condition is mandatory to solve the Wheeler–DeWitt equation. However, this statement seems to point that we also need an additional element (in the form of a *new fundamental* law of physics), which will *select* the boundary condition. How to do it?

In other known physical situations, the boundary conditions are obvious and follow, e.g., the symmetry of the system. Moreover, it seems that implementing a boundary condition for the Wheeler–DeWitt equation is just a less clarified manner to deviate (but not really solving) the issue of the arbitrary initial choice of *other* parameters, which provide the choice of the classical evolution, by leading the discussion into the choice of parameters to describe its quantum evolution. But that is not so: *If* quantum mechanics is the fundamental framework for physical interactions, then quantum dynamics precedes the classical dynamics and we must deal with that first.

3 Introduction-II

Thus, selecting an appropriate boundary condition for the wave function of the Universe has been a paramount objective of quantum cosmology; Two 'rival' approaches are the *no-boundary* proposal and the *tunneling* proposal. Notwithstanding their dominance, two other proposals have also been used, as means to address mathematically the presence of classical singularities: the wave function satisfies $\Psi(0) = 0$ (De Witt boundary condition), or regarding its derivative with respect to the scale factor, $\Psi'(0) = 0$, i.e., vanishing at the classical singularity. It is crucial to mention that all those boundary conditions were *ad hoc*[3] chosen, within a particular physical perspective.

The question that guided [3, 4, 5, 6] was the following: Can a relation between the constraints (that are present and whose algebra characterize the gravity-matter system) and the allowed boundary conditions be established? If there is such an association, then boundary conditions could be intertwined with the set of possible Dirac observables and their algebra.

[2]Other proposals have been put forward: the '*infinite wall*' by B. DeWitt , the '*all possible boundaries*' by Suen and Young and the '*symmetric initial condition*' by Conradi and Zeh.

[3]Most importantly, these *ad hoc* conditions were not formulated as part of the dynamical law. However, according to De Witt "the constraints are everything" i.e., nothing else but the constraints should be needed.

The answer to this query, requires the presence of a paramount element: Symmetry.

The most often use of a symmetry is associated to that of isometry, that is, a spacetime diffeomorphism that leaves the metric invariant; A one-parameter continuous isometry is linked to the existence of Killing vectors. Using those symmetries assists to solve Einstein's equations. However, there are other types of symmetries: Instead of looking at the symmetries of a spacetime, let us analyse the symmetries of the dynamics of a system. Being more precise, for a classical system we employ transformations in the whole phase space of the system such that the dynamics is left invariant (For a quantum system this means a set of phase space operators that commute with the Hamiltonian or with the relevant evolution operator, and transforms solutions into solutions). In the literature, some of such symmetries, have been referred to as *hidden symmetries*.

In the next section, albeit a bit more technical, where details elaborated in [3, 4, 5, 6] are somewhat summarized, we will illustrate how the constraints algebra can set-up allowed boundary conditions to be identified. Some boundary conditions can indeed be displayed as intertwined with the set of possible Dirac observables and their algebra, allowing to select them directly and explicitly from the algebra of constraints (to which the hidden symmetries are subjacent, as we will explain).

4 Quantization and Dirac Observables

Let us start with one of the simplest models in quantum cosmology. We will be making a concrete application of the content of section 2. Consider the homogeneous and isotropic FLRW minisuperspace for a closed Universe with the following line element

$$ds^2 = -N^2(\eta)d\eta^2 + a^2(\eta)d\Omega^2_{(3)}, \tag{12}$$

where $d\Omega^2_{(3)}$ is the standard line element on the unit three-sphere. Take also for this metric the corresponding action (obtained from the Einstein-Hilbert), with a specific matter content (in the form of a perfect fluid with barotropic equation of state $\rho = \gamma p$) . Moreover, we are using $\mathcal{M} = I \times S^3$ for the spacetime manifold, $\partial\mathcal{M} = S^3$ as boundary, and overdot denotes herein differentiation with respect to η. In addition, if we further redefine the lapse function N and scale factor a as

$$\begin{cases} a(\eta) = x(\eta) + \frac{M}{12\pi^2 M_{\text{Pl}}^2} := x - x_0, \\ N(\eta) = 12\pi^2 M_{\text{Pl}} a(\eta)\tilde{N}, \end{cases} \tag{13}$$

this allows to obtain the Lagrangian

$$\mathcal{L} = -\frac{1}{2\tilde{N}} M_{\text{Pl}} \dot{x}^2 + \frac{\tilde{N}}{2} M_{\text{Pl}} \omega^2 x^2 - \mathcal{E}\tilde{N}, \tag{14}$$

where $\mathcal{E}$ and ω are constants. Hence, in terms of the conjugate momenta to x, Π_x, the corresponding Hamiltonian for (14) is the secondary constraint

$$H := \frac{1}{2M_{\text{Pl}}} \Pi_x^2 + \frac{1}{2} M_{\text{Pl}} \omega^2 x^2 - \mathcal{E} \approx 0. \tag{15}$$

Using the Hamiltonian constraint (15), we can easily find the well known solution of a closed Universe

$$\begin{cases} a(\eta) = \frac{a_{\text{Max}}}{1+\sec\phi} \left[1 - \sec\phi \cos(\eta + \phi)\right], \\ a_{\text{Max}} := \frac{M}{12\pi^2 M_{\text{Pl}}^2} + \left(\frac{2\mathcal{E}}{M_{\text{Pl}}\omega^2}\right)^{\frac{1}{2}}, \\ \cos\phi := \frac{M}{\sqrt{2\mathcal{E} M_{\text{Pl}}}}, \end{cases} \tag{16}$$

where a_{Max} is the maximum radius of the Universe and it is assumed that the initial singularity occurs at $\eta = 0$.

4.1 Standard quantization

The standard quantization procedure for this simple system is accomplished by using $\hat{x} = x$ and $\hat{\Pi}_x := -i\partial_x$ in the coordinate representation. Then the Hamiltonian constraint (15) becomes the WDW equation for the wave function of the Universe

$$-\frac{1}{2M_{\text{Pl}}} \frac{d^2\psi}{dx^2} + \frac{1}{2} M_{\text{Pl}} \omega^2 x^2 \psi(x) = \mathcal{E}\psi(x). \tag{17}$$

Note that the classical solution (16) has a singularity at $x = x_0$. We will assume the wave functions defined on the (x_0, ∞) domain. To have a self-adjoint Hamiltonian, this suggests us to use those wave functions which satisfy one of the following boundary conditions: either De Witt boundary condition

$$\psi(x)|_{x=x_0} = 0, \tag{18}$$

to avoid the singularity at $x = x_0$, or

$$\left(\frac{d\psi}{dx} + \alpha\psi\right)|_{x=x_0} = 0, \tag{19}$$

where α is an arbitrary constant. As α would be a new fundamental constant of the theory, so to avoid this, we set it to be zero

$$\frac{d\psi}{dx}|_{x=x_0} = 0. \tag{20}$$

Boundary conditions (18) or (20) establishes the wave function to have normalized oscillator states with eigenvalues $\mathcal{E}_n = \omega(n+1/2)$, with n is an even or odd integer, corresponding to the above boundary conditions (18) and (20), respectively. Hence, using definition (**??**), we obtain

$$\begin{cases} \left(\frac{M}{M_{\rm Pl}}\right)^2 + 24\pi^2\mathcal{N}_\gamma = 24\pi^2(n+\frac{1}{2}), \\ \psi_n = \left(\frac{\sqrt{M_{\rm Pl}\omega}}{\sqrt{\pi}2^n n!}\right)^{\frac{1}{2}} H_n(\sqrt{M_{\rm Pl}\omega}a)\exp\left(-\frac{1}{2}M_{\rm Pl}\omega a^2\right). \end{cases} \tag{21}$$

The normalized eigenfunction indicates the existence of the maximum classical radius of a closed Universe.

4.2 Reduced phase space and observables

This subsection and the subsequent one, will be far (much) more technical and the reader may skip it, although it presents in a very compact manner some of the recent developments in [3, 4, 5, 6]. These bear a significant synergy framework, in a clear mathematics plus physics endeavor, assisting in shedding new light in quantum cosmology conundrums [1, 2].

As mentioned, the invariance of GR under the group of diffeomorphisms of the spacetime manifold $\mathcal{M}$ leads to the result that the Hamiltonian can be expressed as a sum of constraints and that any observable must commute with these constraints. Constraints are classified in two classes: first class constraints (which generate a gauge transformation) and second class constraints (essentially arise in the Hamiltonian formulation of a system). In a constraint Hamiltonian system, a first class constraint is a phase space function on the constraint surface (a surface of simultaneous vanishing of all constraints) whose Poisson bracket vanishes weakly with all the constraints. Accordingly, an observable is an invariant function under the gauge transformations generated by all of the first class constraints. As in GR, the momentum and Hamiltonian constraints are always first class, then a function on the phase space is an observable if it has weakly vanishing Poisson brackets with the first class constraints when the first class constraints hold. To find gauge invariant observables, we can proceed as follows. The unconstrained phase space Γ of the model is $\mathbb{R}^2$, with global canonical coordinates (x, Π_x) with Poisson structure $\{x, \Pi_x\} = 1$. More consistently, let us define on Γ the complex-valued functions

$$\begin{cases} C := \sqrt{\frac{M_{\text{Pl}}\omega}{2}}\left(x + i\frac{\Pi_x}{M_{\text{Pl}}\omega}\right), \\ C^* := \sqrt{\frac{M_{\text{Pl}}\omega}{2}}\left(x - i\frac{\Pi_x}{M_{\text{Pl}}\omega}\right). \end{cases} \tag{22}$$

The set $S = \{C, C^*, 1\}$ form a closed algebra under the Poisson bracket, $\{C, C^*\} = -i$ and every sufficiently differentiable function on Γ can be expressed in terms of S. Therefore, the Hamiltonian can be recast in

$$\mathcal{H} = -\tilde{N}\left(\omega C^* C - \mathcal{E}\right). \tag{23}$$

Moreover, consider on Γ the functions

$$\begin{cases} J_0 := \frac{1}{2}C^* C, \\ J_+ := \frac{1}{2}C^{*2}, \\ J_- := \frac{1}{2}C^2, \end{cases} \tag{24}$$

which have a closed algebra

$$\begin{cases} \{J_0, J_\pm\} = \mp i J_\pm, \\ \{J_+, J_-\} = 2iJ_0. \end{cases} \tag{25}$$

The Hamiltonian constraint implies

$$J_0 = \frac{\mathcal{E}}{2\omega}. \tag{26}$$

Furthermore, we have

$$J^2 := J_0^2 - \frac{1}{2}(J_+J_- + J_-J_+) = j(j-1), \tag{27}$$

where $j = \{1/4, 3/4\}$ denote the Bargmann indexes for the simple harmonic oscillator.

4.3 Hidden symmetry and boundary conditions

To specify boundary conditions for the evolution of subsystems of the Universe, one needs observations done out of the subsystem; they are related to the rest of the Universe. However, in quantum cosmology, the Universe as a whole, has nothing external to determine those boundary conditions. "The cosmological boundary condition must be one of the fundamental laws of physics" or, as we define herein, it can be related, at least in some specific, albeit restrictive, circumstances, to the constraint algebra of the cosmological model. In what follows, we will give some insight towards determining boundary conditions using the hidden dynamical symmetries of the model. With this aim, let us focus our attention on the Dirac observables of the cosmological model. We start by introducing the set of operators $\hat{S} = \{C, C^\dagger, 1\}$, with the following commutator algebra

$$[C, C^\dagger] = 1, \;\; [C, 1] = [C^\dagger, 1] = 0. \tag{28}$$

Thus, the set $\hat{S}$ and its commutator algebra are the quantum counterpart of the set S, terms of operators, with $C^\dagger$ being the Hermitian conjugate to operator C. The action of operators $\{C, C^\dagger\}$ on the states of the physical Hilbert space are given by

$$\begin{cases} C|n >= \sqrt{n}|n-1\rangle, \\ C^\dagger|n >= \sqrt{n+1}|n+1\rangle. \end{cases} \tag{29}$$

Subsequently, the Poisson bracket algebra of the classical J 's can be promoted into a commutator algebra version by setting

$$\begin{cases} J_0 := \frac{1}{4}(C^\dagger C + CC^\dagger), \\ J_+ := \frac{1}{2}C^{\dagger 2}, \\ J_- := \frac{1}{2}C^2, \end{cases} \tag{30}$$

With m being any non-negative integer, in this representation, $su(1,1)$ is determined by the number j and the eigenstates of J^2 and J_0. Thus, the irreducible representations of $su(1,1)$ are labelled by $|j, m\rangle$. In addition, the Hamiltonian can be presented as

$$H = -\mathcal{E} + \omega(C^\dagger C + \frac{1}{2}) = -\mathcal{E} + 2\omega J_0, \tag{31}$$

As J^2 and J_0 commute with the Hamiltonian, they leave the physical Hilbert space V_H invariant and consequently we choose $\{J_0, J^2, 1\}$ as physical operators of the model. Using definition (30), the Casimir operator of $su(1,1)$

reduces identically to $J^2 = j(j-1) = -3/16$. Hence, the Bargmann index $j = \{\frac{1}{4}, \frac{3}{4}\}$ is a gauge invariant observable of the quantum cosmological model. Subsequently, we obtain

$$\mathcal{E}_{m,j} = 2\omega(j+m). \tag{32}$$

Hence, the Bargmann index classifies the underlying states of the Hilbert space, by means of the Hamiltonian constraint $V_{H=0}$, into two invariant subspaces:

$$\begin{cases} \mathcal{E}_{\frac{3}{4},m} = \omega(\frac{3}{2}+2m); \ \ V_{H=0,j=\frac{3}{4}} = \{|\frac{3}{4},m\rangle\}, \\ \mathcal{E}_{\frac{1}{4},m} = \omega(\frac{1}{2}+2m); \ \ V_{H=0,j=\frac{1}{4}} = \{|\frac{1}{4},m\rangle\}, \end{cases} \tag{33}$$

with $V_{H=0} = V_{H=0,j=\frac{1}{4}} \oplus V_{H=0,j=\frac{3}{4}}$. Therefore, the gauge invariance of the Bargmann index allows the partition of the Hilbert space into two disjointed invariant subspaces, and this equivalently implies imposing boundary conditions (18) and (19), respectively.

5 Outlook

Within the framework for quantum cosmology, the wave function retrieved from the WDW equation with appropriate boundary conditions would describe the Universe. However, difficulties have been identified along the path towards this purpose, not merely being of a purely technical (mathematical) nature, or even model dependent. Some obstacles range from the conceptual and complex up to deep fundamental questions [1].

Our suggestion is that a very specific use of (hidden) symmetries may provide a helpful beacon to guide investigation. Let us be more precise on the motivation. It is well known that the factorization method enables us to investigate the properties of the quantum system in a easier way. It is upon to consider a pair of first order differential equations which can be obtained from a given second-order differential equation with boundary conditions. The main advantage of this method is that we may discover the hidden symmetry of the quantum system through constructing a suitable Lie algebra, which can be realized by ladder operators. As in the case of the simple harmonic operator, which can be elegantly solved using the raising and lowering operator method, the operator method for the harmonic oscillator can be generalized to the whole class of shape invariant potentials which include all the popular, analytically solvable potentials.

This can be done using the ideas of supersymmetric quantum mechanics introduced in [1, 2] and an integrability condition called the shape invariance condition. We are indeed already unveiling what can be a wider scope of perspective and understanding of our approach and its potential, but this can assist in preparing for the route of 'uncharted navigation' we took and we share herein with the readers.

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