Recent progress on the mathematical theory of plasmas

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Resumo: O sistema de Navier–Stokes–Maxwell incompressível é um modelo clássico que descreve a evolução de um plasma. Embora se saiba que existem pequenas soluções suaves para esse sistema (no espírito de Fujita–Kato), a existência de grandes soluções fracas (no espírito de Leray) no espaço de energia permanece desconhecida. Esse defeito pode ser atribuído à dificuldade de acoplar as equações de Navier–Stokes a um sistema hiperbólico. Nós descrevemos aqui resultados recentes, com o objetivo de criar soluções fracas para os sistemas de Navier–Stokes–Maxwell em grandes espaços funcionais. Em particular, explicamos como, para quaisquer dados iniciais com energia finita, uma condição de pequenez apenas no campo electromagnético é suficiente para garantir a existência de soluções globais.

Abstract: The incompressible Navier–Stokes–Maxwell system is a classical model describing the evolution of a plasma. Although small smooth solutions to this system (in the spirit of Fujita–Kato) are known to exist, the existence of large weak solutions (in the spirit of Leray) in the energy space remains unknown. This defect can be attributed to the difficulty of coupling the Navier–Stokes equations with a hyperbolic system. We describe here recent results aiming at building weak solutions to Navier–Stokes–Maxwell systems in large functional spaces. In particular, we explain how, for any initial data with finite energy, a smallness condition on the electromagnetic field alone is sufficient to grant the existence of global solutions.

palavras-chave: Equações de Navier–Stokes; equações de Maxwell; existência de soluções fracas; estimativas parabólicas; espaços de Besov.

keywords: Navier–Stokes equations; Maxwell's equations; existence of weak solutions; parabolic estimates; Besov spaces.

1 Introduction

Consider a gas made up of charged particles interacting microscopically through elastic collisions. At the macroscopic level, this gas behaves as a conducting fluid that will interact with any existing electromagnetic field. Moreover, the motion of the charged particles will also produce an electromagnetic field, in accordance with the laws of classical electrodynamics.

The magnetohydrodynamic evolution of the gas will therefore be conditioned by the complex interaction of an electrically conducting moving fluid with a self-induced electromagnetic force.

Such fluids are typically found in the core of nuclear fusion reactors in the form of plasmas, which are ionized gases. Another typical example of an electrically conducting fluid consists in liquid metals, such as the liquid iron found in the core of the earth, which is responsible for the geodynamo effect.

We give now an account of some recent mathematical developments, mainly from [1], concerning the study of plasmas (or conducting fluids). We make here the somewhat arbitrary choice of focusing exclusively on viscous incompressible regimes, because such physical characteristics lead to interesting mathematical properties. Of course, there are numerous other relevant regimes, but we will not discuss them.

We refer to [4] or [5] for a introduction to magnetohydrodynamics from a physical viewpoint.

2 The Navier–Stokes–Maxwell systems

The behavior of a viscous incompressible fluid is described by the Navier–Stokes equations

$$\partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + F, \quad \operatorname{div} u = 0, \tag{1}$$

where $\mu > 0$ is the viscosity, $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^3$ are the time and space variables, $u(t,x)$ stands for the velocity field of the (incompressible) fluid, $F(t,x)$ is a given force field, and the scalar function $p(t,x)$ is the pressure and is also an unknown. Note that, for convenience, we ignore the effect of boundaries on the fluid by assuming the domain to be the whole space.

The validity of this model is well established at both physical and mathematical levels. We refer to [8] for a recent mathematical treatise on the incompressible Navier–Stokes equations.

In a conducting fluid, it is also important to take into account the influence of the Lorentz force produced by the charged particles. The relevant macroscopic field F is therefore the Lorentz force

$$F = nE + j \times B, \tag{2}$$

where $E(t,x)$ and $B(t,x)$ are the electric and magnetic fields respectively, $n(t,x)$ is the electric charge density and $j(t,x)$ is the electric current.

The electromagnetic field is determined classically through Maxwell's equations

$$\begin{cases} \partial_t E - \nabla \times B = -j, & \operatorname{div} E = n, \\ \partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0, \end{cases} \tag{3}$$

or its quasi-static approximation

$$\begin{cases} \nabla \times B = j, & \operatorname{div} E = n, \\ \partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0. \end{cases} \tag{4}$$

Generally speaking, the coupling given by combining (1), (2) and (3) (or (4)) provides now an incompressible Navier–Stokes–Maxwell system. Note, however, that such a system is not closed yet, as it contains more unknowns than equations. In fact, there remains to specify how the density n and the current j are generated by the fluid. This is performed by incorporating the so-called Ohm's law into the system.

It turns out that there is more than one way of closing the Navier–Stokes–Maxwell system, as there are several different Ohm's laws that are appropriate. We discuss now some of the available options.

2.1 Coupling I

The quasi-static system (4) is an approximation of (3) that is relevant in many physical regimes. Indeed, in many practical situations, it is physically reasonable to neglect the so-called displacement current density $\partial_t E$ in Maxwell's equations (see [5]).

Furthermore, observe that the continuity equation

$$\partial_t n + \operatorname{div} j = 0 \tag{5}$$

is expected to hold universally, for n and j respectively represent the density and the flux of the same particles. Since j is necessarily solenoidal (i.e. $\operatorname{div} j = 0$) in the quasi-static approximation due to Ampère's law $j = \nabla \times B$, one deduces that n should be constant in time. The density n is therefore fixed by the initial data and we might as well assume $n = 0$, for simplicity.

Now, recall that, in classical electrostatics, Ohm's law simply states that E and j are colinear. Here, accounting for the motion of the fluid and the effect of Galilean transformations in Faraday's equation $\partial_t B + \nabla \times E = 0$, Ohm's law becomes (see [5])

$$j = \sigma(E + u \times B), \tag{6}$$

where the electrical conductivity $\sigma > 0$ is assumed to be constant throughout the fluid.

All in all, combining (1), (2), (4) with (6), setting $n = 0$, and eliminating j and E, leads to the magnetohydrodynamic system

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + (\nabla \times B) \times B, & \operatorname{div} u = 0, \\ \partial_t B - \dfrac{1}{\sigma} \Delta B = \nabla \times (u \times B), & \operatorname{div} B = 0. \end{cases}$$

This system couples the Navier–Stokes system with a parabolic equation on the magnetic field B and has been studied extensively. As far as the existence of global weak solutions is concerned, it does not present with any additional difficulty when compared to the classical incompressible Navier–Stokes system.

Indeed, one readily computes the formal energy inequality, for any $t \geq 0$ and any initial data (u_0, B_0),

$$\begin{aligned} \frac{1}{2} \left(\|u\|_{L^2}^2 + \|B\|_{L^2}^2 \right)(t) + \int_0^t \left(\mu \|\nabla u\|_{L^2}^2 + \frac{1}{\sigma} \|\nabla B\|_{L^2}^2 \right)(s) ds \\ \leq \frac{1}{2} \left(\|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 \right). \end{aligned} \tag{7}$$

This energy inequality yields strong dissipative properties on both u and B. In particular, the ensuing a priori bounds are suitable for the application of Leray's method of construction of global weak solutions (see [8]). More precisely, it is possible to show that, for any suitable initial data $(u_0, B_0) \in L^2(\mathbb{R}^3)$, there exists a global weak solution

$$(u, B) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)).$$

The uniqueness of such solutions remains unknown, though.

2.2 Coupling II

The reduced form of Maxwell's equations (4) may not be appropriate for every physical setting, and there may be situations where one is led to consider the evolution of the electromagnetic field (E, B) governed by the full set of Maxwell's equations (3). In this case, one may combine (1), (2) and (6), with (3), which yields the Navier–Stokes–Maxwell system

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + j \times B, & \operatorname{div} u = 0, \\ \partial_t E - \nabla \times B = -j, & j = \sigma (E + u \times B), \\ \partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0, \end{cases} \tag{8}$$

where we have neglected the contribution of the Coulombian force nE in the Lorentz force for physical reasons (see [5]). This system couples now the Navier–Stokes equations with a hyperbolic wave system, which significantly changes the nature of solutions.

More precisely, formally computing the corresponding energy inequality, one finds that, for any initial data (u_0, E_0, B_0),

$$\begin{aligned} \frac{1}{2}\left(\|u\|_{L^2}^2 + \|E\|_{L^2}^2 + \|B\|_{L^2}^2\right)(t) + \int_0^t \left(\mu \|\nabla u\|_{L^2}^2 + \frac{1}{\sigma}\|j\|_{L^2}^2\right)(s)ds \\ \leq \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2\right). \end{aligned} \tag{9}$$

When compared to (7), this energy inequality only provides a rather weak control on the solutions, for there is no control on the regularity of the magnetic field.

This lack of compactness prevents us from applying Leray's method of construction of weak solutions, because it is impossible to show the weak stability of the non-linear term $j \times B$ solely based on the a priori bounds given by the energy inequality. As a matter of fact, it is not yet known whether, for any suitable initial data $(u_0, E_0, B_0) \in L^2(\mathbb{R}^3)$, there exists a global weak solution to the Navier–Stokes–Maxwell system (8).

It should be emphasized now that, even though the above system (8) elegantly combines the Navier–Stokes equations with the full Maxwell system, it contains a disturbing physical inconsistency. Indeed, as previously mentioned, we have neglected the term nE in (2), which suggests that n should be zero. However, in this model, the electric current j is in general not solenoidal, which violates the continuity equation (5).

This inconsistency will be resolved in the coming couplings (10) and (11) below, which achieve to combine the Navier–Stokes equations with Maxwell's equations without breaking the continuity equation (5).

2.3 Coupling III

A systematic and rigorous study of hydrodynamic limits of Vlasov–Maxwell–Boltzmann systems, in a viscous incompressible regime, has been conducted in [2], where the following incompressible Navier–Stokes–Maxwell system was derived:

$$
\begin{cases}
\partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + j \times B, & \operatorname{div} u = 0, \\
\partial_t E - \nabla \times B = -j, & \operatorname{div} B = 0, \\
\partial_t B + \nabla \times E = 0, & \operatorname{div} E = 0, \\
j = \sigma \left(-\nabla \bar{p} + E + u \times B \right), & \operatorname{div} j = 0,
\end{cases}
\tag{10}
$$

where the electromagnetic pressure $\bar{p}(t,x)$ is a new unknown. Observe that the introduction of the pressure $\bar{p}$ allows us to add a solenoidal condition on both E and j to the system. As a result, the fluid is neutral $n = 0$ and the continuity equation (5) holds.

As before, this system combines the incompressible Navier–Stokes equations with a hyperbolic system. One easily finds that solutions of (10) formally verify the same energy inequality (9), which fails to provide the necessary compactness to apply Leray's method of proof of existence of weak solutions. Again, it is unfortunately not yet known whether, for any suitable initial data $(u_0, E_0, B_0) \in L^2(\mathbb{R}^3)$, there exists a global weak solution to the Navier–Stokes–Maxwell system (10).

2.4 Coupling IV

Yet another incompressible Navier–Stokes–Maxwell system was derived in [2]. This new model turns out to be the most complete of them all, since it involves all electromagnetic variables (including a non-trivial charge density n). It takes the following form:

$$
\begin{cases}
\partial_t u + u \cdot \nabla u - \mu \Delta u = -\nabla p + nE + j \times B, & \operatorname{div} u = 0, \\
\partial_t E - \nabla \times B = -j, & \operatorname{div} B = 0, \\
\partial_t B + \nabla \times E = 0, & \operatorname{div} E = n, \\
j - nu = \sigma \left(-\nabla n + E + u \times B \right). &
\end{cases}
\tag{11}
$$

Here, again, it is to be noted that the continuity equation (5) holds true.

It is possible to show, at least formally, that solutions of the above system satisfy the energy inequality, for any initial data (u_0, E_0, B_0, n_0),

$$
\begin{aligned}
\frac{1}{2} \left(\|u\|_{L^2}^2 + \|E\|_{L^2}^2 + \|B\|_{L^2}^2 + \|n\|_{L^2}^2 \right)(t) \\
+ \int_0^t \left(\mu \|\nabla u\|_{L^2}^2 + \frac{1}{\sigma} \|j - nu\|_{L^2}^2 \right)(s) ds \\
\leq \frac{1}{2} \left(\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 + \|n_0\|_{L^2}^2 \right).
\end{aligned}
$$

As previously, the ensuing a priori bounds fail to provide enough control to apply Leray's method of proof of existence of weak solutions. It is therefore not yet known whether, for any suitable initial data $(u_0, E_0, B_0, n_0) \in L^2(\mathbb{R}^3)$, there exists a global weak solution to the Navier–Stokes–Maxwell system (11).

3 A global existence result

We believe that the system (8) captures the essential mathematical difficulties related to the coupling of the Navier–Stokes equations with Maxwell's system. We therefore present below the main result from [1] on the existence of weak solutions to the Navier–Stokes–Maxwell system (8).

It should be mentioned here, though, that the two-dimensional case has been previously succesfully handled in [10] (some subtle questions remain open; see also [1] and [6] for some two-dimensional results). We will therefore focus now exclusively on the three-dimensional setting of (8).

The existence of three-dimensional global mild solutions to (8), for small initial data, has also been previously addressed in [6] (and some previous works), where it was shown that, for any sufficiently small initial data $(u_0, B_0, E_0) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, there exists a global mild solution $(u, E, B) \in C(\mathbb{R}^+, \dot{H}^{\frac{1}{2}})$ to (8) (uniqueness of solutions is also available in this setting).

As for weak solutions, the following theorem from [1] provides the existence of global solutions to (8), for any initial data $(u_0, B_0, E_0) \in L^2(\mathbb{R}^3)$, provided the high frequencies of the electromagnetic field are controlled in some suitable norm.

Theorem 1 ([1])**.** *There is a constant $C_* > 0$ such that, if the initial data $(u_0, E_0, B_0) \in L^2 \times (H^{\frac{1}{2}})^2$, with $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$, satisfies*

$$\|(E_0, B_0)\|_{\dot{H}^{\frac{1}{2}}} C_* e^{C_* \|(u_0, E_0, B_0)\|_{L^2}^2} \leq 1\,,$$

then there is a global weak solution to (8) *satisfying the energy inequality* (9)*.*

The strategy of proof of this result follows the usual procedure of approximating (8) with a regularized system, in order to justify all formal a priori bounds, and then passing to the limit by showing the weak stability of the system.

As previously explained, the a priori bounds provided by the energy inequality (9) are not enough to deduce the weak stability of the non-linear term $j \times B$. However, in Theorem 1, the hyperbolic structure of Maxwell's equations is used to propagate the bound on the initial electromagnetic field in $\dot{H}^{\frac{1}{2}}$. In fact, it is shown therein that the electromagnetic field is uniformly bounded in $L^\infty(\mathbb{R}^+, \dot{H}^{\frac{1}{2}})$, which is then sufficient to establish the weak stability of $j \times B$.

All relevant a priori bounds on (8) are obtained through non-linear energy estimates performed in Besov spaces. Even though the general strategy remains rather standard, these estimates are complex and sometimes technical. They rely heavily on a precise use of paraproduct estimates, a careful analysis of the damped wave flow produced by Maxwell's system (3) and, most importantly, on crucial endpoint parabolic estimates to control the Stokes flow.

These endpoint parabolic estimates provide a new fundamental tool for the analysis of partial differential equations, particularly for models from fluid dynamics. We are therefore going to give a self-contained account of the main ideas behind such estimates in the next section.

We refer to [1] for the full justification of the above theorem.

4 Endpoint parabolic estimates

We show now how to derive the crucial parabolic estimates that are used in the proof of Theorem 1. Such estimates hold in any dimension $d \geq 1$ and show that solutions to the heat equation can gain up to two derivatives with respect to the source terms in Besov spaces, without resorting to the usual Chemin–Lerner spaces (see [1] for a definition of such spaces). In fact, we believe that this is an important principle that could be useful beyond its application to the proof of Theorem 1.

We introduce now a standard dyadic decomposition

$$\mathrm{Id} = \sum_{k \in \mathbb{Z}} \Delta_k,$$

where the Fourier multiplier operators Δ_k act on a function by localizing its frequencies ξ to a domain $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$.

Recall that the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$, for any $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, is then defined by the norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \left(\sum_{k=-\infty}^{\infty} 2^{ksq} \|\Delta_k f\|^q_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{q}},$$

if $q < \infty$, and with the obvious modifications in case $q = \infty$. We refer to [1] for a precise definition of these spaces using the same notation.

We consider solutions of the forced heat equation

$$\partial_t w - \Delta w = f, \quad w_{|t=0} = 0. \tag{12}$$

Such solutions can be expressed by the Duhamel representation formula

$$w(t) = \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau. \tag{13}$$

Our first result provides a sharp estimate showing how the heat flow provides a gain of regularity of at most (but not equal to) two derivatives.

Lemma 2. *Let $\sigma \in \mathbb{R}$, $1 < r < m < \infty$ and $p \in [1,\infty]$. If f belongs to $L^r([0,T], \dot{B}^{\sigma+\frac{2}{r}}_{p,\infty})$, then the solution of the heat equation* (12) *satisfies*

$$\|w\|_{L^m([0,T],\dot{B}^{\sigma+2+\frac{2}{m}}_{p,1})} \lesssim \|f\|_{L^r([0,T],\dot{B}^{\sigma+\frac{2}{r}}_{p,\infty})}.$$

Proof. First, observe that, employing the representation formula (13), there is an independent constant $C > 0$ such that

$$\|\Delta_k w(t)\|_{L^p} \lesssim \int_0^t e^{-C(t-\tau)2^{2k}} \|\Delta_k f(\tau)\|_{L^p} d\tau. \tag{14}$$

In particular, we obtain that

$$\begin{aligned} \|w(t)\|_{\dot{B}^{\sigma+2+\frac{2}{m}}_{p,1}} &\lesssim \int_0^t \sum_{k\in\mathbb{Z}} e^{-C(t-\tau)2^{2k}} 2^{k(\sigma+2+\frac{2}{m})} \|\Delta_k f(\tau)\|_{L^p} d\tau \\ &\lesssim \int_0^T h(t-\tau) \|f(\tau)\|_{\dot{B}^{\sigma+\frac{2}{r}}_{p,\infty}} d\tau, \end{aligned}$$

where we denoted

$$h(\lambda) = \sum_{k\in\mathbb{Z}} \mathbb{1}_{\{\lambda>0\}} e^{-C\lambda 2^{2k}} 2^{2k(1+\frac{1}{m}-\frac{1}{r})},$$

which is a well-defined convergent series whenever $1 + \frac{1}{m} - \frac{1}{r} > 0$.

Next, for any $\lambda > 0$, choosing $j \in \mathbb{Z}$ so that $2^{2j} \leq \lambda < 2^{2(j+1)}$, observe that

$$\begin{aligned} h(\lambda) &\leq \sum_{k\in\mathbb{Z}} e^{-C2^{2(j+k)}} 2^{2k(1+\frac{1}{m}-\frac{1}{r})} \\ &= 2^{-2j(1+\frac{1}{m}-\frac{1}{r})} \sum_{k\in\mathbb{Z}} e^{-C2^{2k}} 2^{2k(1+\frac{1}{m}-\frac{1}{r})} \lesssim \lambda^{-(1+\frac{1}{m}-\frac{1}{r})}. \end{aligned}$$

It therefore follows that, since $0 < 1 + \frac{1}{m} - \frac{1}{r} < 1$ and $1 < m, r < \infty$, by virtue of the Hardy–Littlewood–Sobolev inequality,

$$\|w(t)\|_{L^m \dot{B}^{\sigma+2+\frac{2}{m}}_{p,1}} \lesssim \left\| \int_0^T |t-\tau|^{-(1+\frac{1}{m}-\frac{1}{r})} \|f(\tau)\|_{\dot{B}^{\sigma+\frac{2}{r}}_{p,\infty}} d\tau \right\|_{L^m} \lesssim \|f\|_{L^r \dot{B}^{\sigma+\frac{2}{r}}_{p,\infty}}$$

which concludes the proof of the lemma. □

Note that the gain of regularity in the preceding result corresponds to $2 - 2(\frac{1}{r} - \frac{1}{m})$. In particular, the loss of $2(\frac{1}{r} - \frac{1}{m})$ is reminiscent of Bernstein inequalities in connection with the Littlewood–Paley theory (see [3, Section 2.1.1]) and Sobolev embeddings.

Further observe that, according to the preceding proof, the constant in the main estimate of Lemma 2 blows up as r tends to m with the same behavior as the sharp constant of the Hardy–Littlewood–Sobolev inequality (see [9] for a characterization of this sharp constant). However, we do not know whether this behavior is sharp for Lemma 2.

We are now particularly interested in the endpoint case $r = m$ of the preceding lemma, which would correspond formally to a gain of exactly two derivatives and is central to the proof of Theorem 1.

Unfortunately, the preceding proof fails miserably in this case, since it would require an endpoint application of the Hardy–Littlewood–Sobolev inequality, which is impossible. Instead, we are able to establish the following crucial endpoint lemma.

Lemma 3 ([1]). *Let $\sigma \in \mathbb{R}$, $1 \leq q \leq r < \infty$ and $p \in [1, \infty]$. If f belongs to $L^r([0,T], \dot{B}^\sigma_{p,q})$, then the solution of the heat equation* (12) *satisfies*

$$\|w\|_{L^r([0,T],\dot{B}^{\sigma+2}_{p,q})} \lesssim \|f\|_{L^r([0,T],\dot{B}^\sigma_{p,q})}.$$

The proof presented here is self-contained and is somewhat simpler than the one from [1] because it avoids abstract interpolation altogether.

Proof. By duality, it is enough to prove that, if g is a function in $L^{a'}([0,T])$ with $a = \frac{r}{q} \geq 1$ and $\frac{1}{a} + \frac{1}{a'} = 1$, then

$$\int_0^T g(t) \|w(t)\|^q_{\dot{B}^{\sigma+2}_{p,q}} dt \lesssim \|f\|^q_{L^r([0,T],\dot{B}^\sigma_{p,q})} \|g\|_{L^{a'}([0,T])}.$$

To this end, we first write

$$\int_0^T g(t) \|w(t)\|^q_{\dot{B}^{\sigma+2}_{p,q}} dt = \sum_{k\in\mathbb{Z}} \int_0^T g(t) \|\Delta_k w(t)\|^q_{L^p} 2^{k(\sigma+2)q} dt.$$

Furthermore, we deduce from (14) that

$$\|\Delta_k w(t)\|_{L^p}^q \lesssim 2^{-k(2q-2)} \int_0^t e^{-C(t-\tau)2^{2k}} \|\Delta_k f(\tau)\|_{L^p}^q d\tau,$$

which implies that

$$\begin{aligned}\int_0^T & g(t)\|w(t)\|_{\dot B^{\sigma+2}_{p,q}}^q dt \\ &\lesssim \sum_{k\in\mathbb{Z}} \int_0^T \int_0^t |g(t)| e^{-C(t-\tau)2^{2k}} \|\Delta_k f(\tau)\|_{L^p}^q 2^{k(\sigma q+2)} d\tau dt.\end{aligned}$$

Next, we introduce a maximal operator defined by

$$Mg(\tau) = \sup_{\rho>0} \int_0^T \rho \mathbb{1}_{\{t-\tau\geq 0\}} e^{-(t-\tau)\rho} |g(t)| dt.$$

Classical results from harmonic analysis (see [7, Theorems 2.1.6 and 2.1.10]) establish that M is bounded over $L^b([0,T])$, for any $1 < b \leq \infty$. One can now write that

$$\int_0^T g(t)\|w(t)\|_{\dot B^{\sigma+2}_{p,q}}^q dt \lesssim \sum_{k\in\mathbb{Z}} \int_0^T Mg(\tau) \|\Delta_k f(\tau)\|_{L^p}^q 2^{k\sigma q} d\tau,$$

whence, by definition of $\dot B^{\sigma}_{p,q}$,

$$\int_0^T g(t)\|w(t)\|_{\dot B^{\sigma+2}_{p,q}}^q dt \lesssim \int_0^T Mg(\tau) \|f(\tau)\|_{\dot B^{\sigma}_{p,q}}^q d\tau.$$

We finally conclude, by Hölder's inequality, that

$$\begin{aligned}\int_0^T Mg(\tau) \|f(\tau)\|_{\dot B^{\sigma}_{p,q}}^q d\tau &\lesssim \|Mg\|_{L^{a'}([0,T])} \|f\|_{L^r([0,T],\dot B^{\sigma}_{p,q})}^q \\ &\lesssim \|g\|_{L^{a'}([0,T])} \|f\|_{L^r([0,T],\dot B^{\sigma}_{p,q})}^q,\end{aligned}$$

which completes the proof of the lemma. □

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