Dynamics of Planar Piecewise Isometries: Recent Advances

Pedro Peres, *Ana Rodrigues*
University of Exeter
Department of Mathematics
Exeter EX4 4QF, UK
e-mail: pp334@exeter.ac.uk
a.rodrigues@exeter.ac.uk

Resumo: Neste trabalho revemos alguns resultados relacionados com o estudo de isometrias por pedaços. Introduziremos *embeddings* de uma transformação de troca de intervalos numa isometria por pedaços, discutiremos a renormalização de uma isometria por pedaços particular e provaremos a existência de curvas invariantes para estas transformações.

Abstract: In this survey we review recent results on the study of the dynamics of piecewise isometries. We will introduce embeddings of an interval exchange transformation into a piecewise isometry, discuss the renormalization of a particular piecewise isometry and finally show that invariant curves exist for such transformations.

palavras-chave: Renormalização; curvas invariantes.

keywords: Renormalization; invariant curves.

1 Introduction

An *interval exchange transformation (IET)* is a bijective piecewise order preserving isometry f of an interval $I \subset \mathbb{R}$, where I is partitioned into subintervals $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, indexed over a finite alphabet $\mathcal{A}$ of $d \geq 2$ symbols, so that the restriction of f to each subinterval is a translation. IETs were studied for instance in [20, 28]. Masur [22] and Veech [28] proved independently that a typical IET is uniquely ergodic while Avila and Forni [12] established that a typical IET is either weakly mixing or an irrational rotation.

Piecewise isometries (PWIs) are higher dimensional generalizations of one dimensional IETs. They have been defined on higher dimensional spaces and Riemannian manifolds [5, 17]. In this paper we consider orientation preserving planar piecewise isometries with respect to the standard euclidean metric. Let X be a subset of $\mathbb{C}$ and $\mathcal{P} = \{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a finite partition of X into convex sets (or *atoms*), that is $\bigcup_{\alpha\in\mathcal{A}} X_\alpha = X$ and $X_\alpha \cap X_\beta = \emptyset$

for $\alpha \neq \beta$. Given a *rotation vector* $\theta \in \mathbb{T}^{\mathcal{A}}$ (with $\mathbb{T}^{\mathcal{A}}$ denoting the torus $\mathbb{R}^{\mathcal{A}}/2\pi\mathbb{Z}^{\mathcal{A}}$) and a *translation vector* $\eta \in \mathbb{C}^{\mathcal{A}}$, we say (X, T) is a *piecewise isometry* if T is such that

$$T(z) := T_\alpha(z) = e^{i\theta_\alpha} z + \eta_\alpha, \text{ if } z \in X_\alpha,$$

so that T is a piecewise isometric rotation or translation (see [16]).

For a given PWI we may partition X into a *regular* and an *exceptional set* [7]. If we consider the zero measure set given by the union $\mathcal{E}$ of all preimages of the set of discontinuities D, then its closure $\overline{\mathcal{E}}$ (which may be of positive measure) is called the *exceptional set* for the map. The complement of the exceptional set is called the *regular set* for the map and consists of disjoint polygons or disks that, if X is compact, are periodically coded by their itinerary through the atoms of the PWI. There is numerical evidence that the exceptional set may have positive Lebesgue measure for typical PWIs [5]. In [18], the author shows that this is the case for certain rectangle-exchange transformations.

Even when the exceptional set has positive Lebesgue measure, there is numerical evidence that Lebesgue measure on the exceptional set may not be ergodic - there can be invariant curves that prevent trajectories from spreading across the whole of the exceptional set [7]. In [3, 7], the existence of a large number of these invariant curves, apparently nowhere smooth, are investigated.

In [1] Adler, Kitchens and Tresser found renormalization operators for three rational rotation parameters for a non ergodic piecewise affine map of the Torus. Lowenstein and Vivaldi [21] gave a computer assisted proof of the renormalization of a family of piecewise isometries of a rhombus with one translation parameter and a fixed rational rotation parameter. Hooper [19] investigated a two dimensional parameter space of polygon exchange maps, a family of PWIs with no rotation, invariant under a renormalization operation. In [2] the authors showed how to construct minimal rectangle exchange maps, associated to Pisot numbers, using a cut-and-project method and prove that these maps are renormalizable. The maps described in these papers are PWIs with no rotational component, exhibiting very particular behaviour among more general PWIs, making it difficult to generalize their techniques.

In this survey we present recent results on the study of the dynamics of planar piecewise isometries. We introduce a new notion of renormalization to study a class of PWIs called Translation Cone Exchange Transformations. We also introduce the notion of embedding IETs into PWIs and use IET renormalization techniques to establish the existence of invariant curves for PWIs which are not the union of line segments or circle arcs.

2 Interval exchange transformations

In this section we recall some notions of the theory of interval exchange transformations following [13], [27] and [29].

As in [13, 29], let $\mathcal{A}$ be an alphabet on $d \geq 2$ symbols, and let $I \subset \mathbb{R}$ be an interval having 0 as left endpoint. In what follows we use the notation $\mathbb{R}^{\mathcal{A}} \simeq \mathbb{R}^d$ and $\mathbb{R}^{\mathcal{A}}_+ \simeq \mathbb{R}^d_+$. We choose a partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ of I into subintervals which we assume to be closed on the left and open on the right. An *interval exchange transformation* (IET) is a bijection of I defined by

(1) A vector $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}_+$ with coordinates corresponding to the lengths of the subintervals, that is, for all $\alpha \in \mathcal{A}$, $\lambda_\alpha = |I_\alpha|$. We write $I = I(\lambda) = [0, |\lambda|)$, where $|\lambda| = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha$.

(2) A pair $\pi = \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix}$ of bijections $\pi_\varepsilon : \mathcal{A} \to \{1, ..., d\}$, $\varepsilon = 0, 1$, describing the ordering of the subintervals I_α before and after the application of the map. This is represented as

$$\pi = \begin{pmatrix} \alpha^0_1 & \alpha^0_2 & ... & \alpha^0_d \\ \alpha^1_1 & \alpha^1_2 & ... & \alpha^1_d \end{pmatrix}.$$

We call π a *permutation* and identify it, at times, with its *monodromy invariant* $\tilde{\pi} = \pi_1 \circ \pi_0^{-1} : \{1, ...d\} \to \{1, ...d\}$. In algebra literature it is common to reserve the term permutation for the monodromy invariant $\tilde{\pi}$, however, unlike the present notation, this would not be invariant under the induction and renormalization algorithms used in the study of IETs. We denote by $\mathfrak{S}(\mathcal{A})$ the set of irreducible permutations, that is $\pi \in \mathfrak{S}(\mathcal{A})$ if and only if $\tilde{\pi}(\{1, ..., k\}) \neq \{1, ..., k\}$ for $1 \leq k < d$.

Define a linear map $\Omega_\pi : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ by

$$(\Omega_\pi(\lambda))_{\alpha\in\mathcal{A}} = \sum_{\pi_1(\beta)<\pi_1(\alpha)} \lambda_\beta - \sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta. \qquad (1)$$

Given a permutation $\pi \in \mathfrak{S}(\mathcal{A})$ and $\lambda \in \mathbb{R}^{\mathcal{A}}_+$ the interval exchange transformation associated is the map $f_{\lambda,\pi}$ that rearranges I_α according to π, that is $f_{\lambda,\pi}(x) = x + \upsilon_\alpha$, for any $x \in I_\alpha$, where $\upsilon_\alpha = (\Omega_\pi(\lambda))_\alpha$. We write $f = f_{\lambda,\pi}$ and also denote an IET by the pair $(I, f_{\lambda,\pi})$.

Given $(\lambda, \pi) \in \mathbb{R}^{\mathcal{A}}_+ \times \mathfrak{S}(\mathcal{A})$ and for $\varepsilon = 0, 1$, denote by β_ε the last symbol in the expression of π_ε. Assume the intervals I_{β_0} and I_{β_1} have different lengths. Let $I^{(1)}$ be the interval obtained by removing the smallest of these intervals from I. The first return map of $f_{\lambda,\pi}$ to $I^{(1)}$ is again an IET, $f_{\lambda^{(1)},\pi^{(1)}}$. This defines a map $\mathcal{R}(\lambda,\pi) = (\lambda^{(1)},\pi^{(1)})$ called *Rauzy induction*. We assume the *infinite distinct orbit condition (IDOC)*, introduced by Keane in [20], which assures that the iterates $\mathcal{R}^n$ are defined for all $n \geq 0$. We denote $\mathcal{R}^n(\lambda,\pi) = (\lambda^{(n)},\pi^{(n)})$ and by $\{I_\alpha^{(n)}\}_{\alpha\in\mathcal{A}}$ the partition of the domain $I^{(n)}$ of $f_{\lambda^{(n)},\pi^{(n)}}$.

The *Rauzy class* (see [29]) of a permutation $\pi \in \mathfrak{S}(\mathcal{A})$, is the set $\mathfrak{R}(\pi)$ of all $\pi^{(1)} \in \mathfrak{S}(\mathcal{A})$ such that there exist $\lambda,\lambda^{(1)} \in \mathbb{R}_+^{\mathcal{A}}$ and $n \in \mathbb{N}$ such that $\mathcal{R}^n(\lambda,\pi) = (\lambda^{(1)},\pi^{(1)})$. A Rauzy class $\mathfrak{R}$ can be visualized in terms of a directed labelled graph, the *Rauzy graph* (see [27]). Its vertices are in bijection with $\mathfrak{R}$ and it is formed by edges that connect permutations which are obtained one from another by $\mathcal{R}$ and are labeled respectively by 0 or 1 according to the type of the induction. A *path* $\varrho = (\varrho_1,...,\varrho_n)$ is a sequence of compatible edges of the Rauzy graph, that is, such that the starting vertex of ϱ_{i+1} is the ending vertex of ϱ_i, $i = 1,...,n-1$. We say a path is *closed* if the starting vertex of ϱ_1 is the ending vertex of ϱ_n. The set of all paths in this graph is denoted by $\Pi(\mathfrak{R})$.

The *Rauzy cocycle* $B_R(\lambda,\pi)$ is a matrix function such that each entry $(B_R^{(n)}(\lambda,\pi))_{\alpha,\beta}$ of $B_R^{(n)}(\lambda,\pi)$ counts the number of visits of $I_\alpha^{(n)}$ to I_β during the Rauzy induction time.

The *projection of the Rauzy cocycle* on the Torus $\mathbb{T}^{\mathcal{A}} \simeq \mathbb{R}^{\mathcal{A}}/2\pi\mathbb{Z}^{\mathcal{A}}$ is given by

$$B_{\mathbb{T}^{\mathcal{A}}}(\lambda,\pi)\cdot\theta = B_R(\lambda,\pi)\cdot\theta \mod 2\pi,$$

for any $(\lambda,\pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R}$, $n \geq 0$ and $\theta \in \mathbb{T}^{\mathcal{A}}$.

A *translation surface* (see for instance [12], [28]), is a surface with a finite number of conical singularities endowed with an atlas such that coordinate changes are given by translations in $\mathbb{R}^2$. Given an IET it is possible to associate, via a suspension construction, a translation surface, with genus $g(\mathfrak{R})$ only depending on the combinatorial properties of the underlying IET (see [28]).

3 Translated cone exchange transformations

In this section we present a result on the renormalization of a particular family of PWIs which we designate by *Translated cone exchange transformations* following [24].

Consider a family of dynamical systems $\mathcal{F} = \{f_\mu : X \to X\}$ parametrized by $\mu \in \mathcal{P}$, where $\mathcal{P}$ is called the parameter space of $\mathcal{F}$. A *renorma-*

lization scheme for $\mathcal{F}$ is a decreasing chain of subsets of X, $X = Y_0(\mu) \supset Y_1(\mu) \supset Y_2(\mu) \supset ...$, together with a renormalization operator $\mathcal{R} : \mathcal{P} \to \mathcal{P}$ such that the first return map of a point in $Y_{n+1}(\mu)$ under iteration by $f_{\mathcal{R}^n(\mu)} : Y_n(\mu) \to Y_n(\mu)$ is given by $f_{\mathcal{R}^{n+1}(\mu)} : Y_{n+1}(\mu) \to Y_{n+1}(\mu)$. Renormalization is a powerful tool in the study of nonlinear maps (see [10]), such as diffeomorphisms of the circle [26], one-frequency Schrödinger cocycles [11] and analytic unimodal maps [15].

Set $\omega = (\omega_1, ..., \omega_d) \in \mathbb{W}$, where $\mathbb{W}$ is the open polytope defined by

$$\mathbb{W} = \left\{ \omega \in \mathbb{R}_+^d : 0 < \sum_{j=1}^{d} \omega_j < \pi \right\}, \tag{2}$$

and let $\vartheta = \dfrac{\pi}{2} - \dfrac{|\omega|}{2}$, where $|\omega|$ is the ℓ_1 norm of ω.

In order to introduce the family of TCEs, consider a partition of the upper half plane $\mathbb{H}$ into $d+2$ cones $\mathcal{P} = \{P_0, P_1, \ldots, P_d, P_{d+1}\}$, where $P_j = \{z \in \mathbb{C} : \arg(z) \in W_j\}$, and W_j for $j = 0, \ldots, d+1$ are defined as

$$W_j = \begin{cases} [0, \vartheta), & \text{for } j = 0, \\ [\vartheta, \vartheta + \omega_1], & \text{for } j = 1, \\ (\vartheta + \sum_{k=1}^{j-1} \omega_k, \vartheta + \sum_{k=1}^{j} \omega_k], & \text{for } j \in \{2, ..., d\}, \\ (\pi - \vartheta, \pi], & \text{for } j = d+1. \end{cases}$$

We set $\nu = \tan(\vartheta)$. Note that ν depends on $|\omega|$, and when necessary to stress this dependence we write $\nu = \nu(|\omega|)$.

Let $G : \mathbb{H} \to \mathbb{H}$ be the following family of translation maps

$$G(z) = \begin{cases} z - 1, & z \in P_0, \\ z - \eta', & z \in P_j,\ j \in \{1, ..., d\}, \\ z + \eta, & z \in P_{d+1}, \end{cases}$$

depending on the parameters ϑ, η and η' with $\vartheta > 0$, $\eta \in \mathbb{R}^+ \backslash \mathbb{Q}$ and $0 < \eta' < \eta$.

Consider a permutation $\pi \in \mathfrak{S}(\{1, ..., d\})$ with a monodromy invariant $\tilde{\pi}$, and let $\theta_j(\omega, \tilde{\pi})$ be the angle associated to the monodromy invariant $\tilde{\pi}$ for the cone P_j for $j = 1, \ldots, d$. We have

$$\theta_j(\omega, \tilde{\pi}) = \sum_{\tilde{\pi}(k) < \tilde{\pi}(j)} \omega_k - \sum_{k<j} \omega_k. \tag{3}$$

Let $E : \mathbb{H} \to \mathbb{H}$ be the following family of maps

$$E(z) = \begin{cases} z, & z \in P_0 \cup P_{d+1}, \\ z e^{i\theta_j(\omega, \tilde{\pi})}, & z \in P_j,\ j \in \{1, ..., d\}, \end{cases}$$

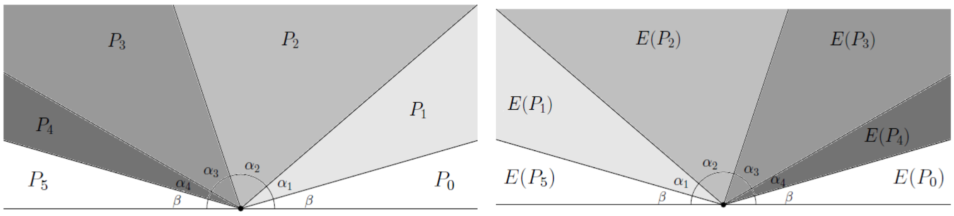

Figure 1: On the left a partition $\mathcal{P}$ with $d = 5$. On the right the action of map E on this partition with $\tilde{\pi}(1) = 4$, $\tilde{\pi}(2) = 3$, $\tilde{\pi}(3) = 2$ and $\tilde{\pi}(4) = 1$.

depending on $\theta_j(\omega, \tilde{\pi})$. This map also depends on ω and ϑ as the partition elements P_j depend on these parameters. Note that we have

$$\vartheta + \arg\left(E(z)\right) = f_{\omega,\pi}(\arg(z) - \vartheta),$$

for $z \in P_j$, $j = 1, ..., d$. Hence E exchanges these cones according to the monodromy invariant $\tilde{\pi}$.

From the translation and exchange families of maps we get our family of TCEs, $F : \mathbb{H} \to \mathbb{H}$, given by

$$F(z) = G \circ E(z).$$

We define the *central cone* P_c of F as

$$P_c = P_1 \cup ... \cup P_d,$$

the *first hitting time of* $z \in \mathbb{H}$ *to* P_c, as the map $k : \mathbb{H} \to \mathbb{N}$ given by

$$k(z) = \inf\{n \geq 1 : F^n(z) \in P_c\}, \tag{4}$$

and the *first return map of* $z \in P_c$ *to* P_c, as the map $F_c : P_c \to P_c$ such that

$$F_c(z) = F^{k(z)}(z). \tag{5}$$

The typical notion of renormalization may not capture all possible self similar behaviour in PWIs. TCEs apparently exhibit invariant regions on which the dynamics is self similar after rescaling. Thus, we say a TCE is renormalizable if F_c, the first return map to P_c described above, is conjugated to itself by a scaling map.

Theorem 3.1 ([24]) For all $\omega \in \mathbb{W}$, $\eta = 1/(k + \Phi)$ and $\eta' = 1 - k\eta$ with $k \in \mathbb{N}$, there is an open set U containing the origin such that F is renormalizable for all $z \in U$, that is

$$F_c(\Phi^2 z) = \Phi^2 F_c(z). \tag{6}$$

The proof of this theorem uses a one dimensional approach to the study of these TCEs. We define sequences coding information related to the first return map of a given line contained in the cone P_c. We are then able to relate the renormalizability of a map of this family with the periodicity of these sequences and indeed, for the parameters in the statement of the theorem, these are proved to be periodic. As a consequence of this we show that for these parameters F_c is a PWI with respect to a partition $\mathcal{P}_{F_c}$ of countably many atoms.

We say that a collection of atoms $\mathcal{B} \subseteq \mathcal{P}$ is a *barrier* for a PWI $(T, \mathcal{P})$ if $X \setminus \bigcup_{B \in \mathcal{B}} B$ is the union of two disjoint connected components A_1, A_2 such that

$$A_1 \cap T(A_2) = T(A_1) \cap A_2 = \emptyset,$$

and for any $P \in \mathcal{P}$ such that $P \subseteq A_j$ and $T(P) \cap (\bigcup_{B \in \mathcal{B}} \overline{B}) \cap \overline{A_j} = \emptyset$ then $T(P) \cap (\bigcup_{B \in \mathcal{B}} B) = \emptyset$, for $j = 1, 2$.

Denote the ray in $\mathbb{H}$ passing through the origin and with slope $a \in \mathbb{R}$ by

$$L_a = \{z \in \mathbb{H} : \mathrm{Im}(z) = a\mathrm{Re}(z)\}, \tag{7}$$

and by $\partial\mathcal{P}$ the union of the boundaries of the elements of the partition $\mathcal{P}$ and by L_ν and $L_{-\nu}$, respectively, the rays $\overline{P_0} \cap \overline{P_1}$ and $\overline{P_d} \cap \overline{P_{d+1}}$.

For $\omega \in \mathbb{W}$, $\eta = 1/(k + \Phi)$ and $\eta' = 1 - k\eta$, $k \in \mathbb{N}$, we denote by $\mathfrak{A}(\eta, \eta')$ the subset of $\mathbb{W}$ such that for all $\omega \in \mathfrak{A}(\eta, \eta')$ there are $d' \geq 2$, $\lambda \in \mathbb{R}_+^{d'}$, $\pi \in \mathfrak{S}(\{1, ..., d'\})$ and a continuous embedding γ of $f_{\lambda,\pi} : I \to I$ into $F_c : P_c \to P_c$ such that

i) the collection $\mathcal{B} = \{P \in \mathcal{P}_{F_c} : P \cap \gamma(I) \neq \emptyset\}$, is a barrier for F_c,

ii) $\gamma(0) \in L_{-\nu}$ and $\lim_{a \to |\lambda|} \gamma(a) \in L_\nu$,

iii) $\gamma(I) \subset \Phi^2 U$, where U is the open set from Theorem 3.1.

In the next theorem we show, as a consequence of Theorem 3.1, that the existence of one continuous embedding of an IET into a first return map F_c of a TCE, satisfying the property that the image of the embedding is contained in a barrier, implies the existence of infinitely many embeddings of the same IET into F_c, as well as infinitely many bounded and forward invariant regions.

Theorem 3.2 ([24]) Let $\eta = 1/(k + \Phi)$, $\eta' = 1 - k\eta$ with $k \in \mathbb{N}$ and assume that $\mathfrak{A}(\eta, \eta')$ is non-empty. For all $\omega \in \mathfrak{A}(\eta, \eta')$,

i) There exist sets $V_1, V_2, ...,$ which are forward invariant for F_c and $y^* > 0$ such that for all $z \in P_c$, satisfying $0 < \mathrm{Im}(z) < y^*$, there is an $n \in \mathbb{N}$ for which $z \in V_n$.

ii) For all $n \in \mathbb{N}$ there exist constants $0 < \underline{b}_n < \bar{b}_n$ such that for all $z \in V_n$ and $k \in \mathbb{N}$,

$$\underline{b}_n < |F^k(z)| < \bar{b}_n. \tag{8}$$

iii) There exist infinitely many continuous embeddings of IETs into F_c.

The proof of Theorem 3.2 relies on the Jordan curve Theorem, and on the properties of the barrier containing the image of the embedding in order to prove the existence of one invariant set V_1. Then the renormalizability of F implies the existence of infinitely many such sets.

4 Embedding interval exchange transformations into piecewise isometries

Recently [9], we developed a new mechanism that allow us to study the dynamics of PWIs using tools from IETs - embeddings - and we used combinatorial properties of IETs to prove that in order for a PWI to realize a continuous embedding of an IET with the same permutation its parameters must satisfy a particular condition. In this section we give an overview of these mathematical tools.

It is commonly accepted that the phase space of typical Hamiltonian systems is divided into regions of regular and chaotic motion [14]. Area preserving maps which can be obtained as Poincaré sections of Hamiltonian systems, exhibit this property as well, with KAM curves splitting the domain into regions of chaotic and periodic dynamics (see for instance [23]). A general and rigorous treatment of this has been however missing. Area preserving PWIs that have been studied as linear models for the standard map (see [4]), can exhibit a similar phenomenon. Unlike IETs which are typically ergodic, there is numerical evidence, as noted in [7], that Lebesgue measure on the exceptional set is typically not ergodic in some families of PWIs - there can be non-smooth invariant curves that prevent trajectories from spreading across the whole of the exceptional set. These curves were first observed in [3] for an isolated parameter and later found in [7] to be apparently abundant for a large family of PWIs.

We now relate the existence of invariant curves to the general problem of embedding IET dynamics within PWIs. We start by introducing some definitions.

An injective map $\gamma : I \to X$ is a *piecewise continuous embedding* of (I, f) into (X, T) if $\gamma|_{I_\alpha}$ is a homeomorphism for each $\alpha \in \mathcal{A}$ such that $\gamma(I_\alpha) \subset X_\alpha$

and

$$\gamma \circ f(x) = T \circ \gamma(x), \tag{9}$$

for all $x \in I$. In this case note that $\gamma(I) \subset X$ is an invariant set for (X, T).

If γ is a piecewise continuous embedding that is continuous on I, we say it is a *continuous embedding* (or *embedding* when this does not cause any ambiguity). Otherwise we say it is a *discontinuous embedding*.

We say γ is a *differentiable embedding* if it is a piecewise continuous embedding and $\gamma|_{I_\alpha}$ is continuously differentiable. We characterize certain differentiable embeddings as, in some sense, trivial: given $I' \subseteq I$ we say a map $\gamma : I' \to \mathbb{C}$ is an *arc map* if there exists $\xi \in \mathbb{C}$, $r, a > 0$ and $\varphi \in [0, 2\pi)$ such that for all $x \in I'$,

$$\gamma(x) = re^{i(ax+\varphi)} + \xi.$$

We say an embedding $\gamma : I \to \mathbb{C}$ of an IET into a PWI is an *arc embedding* if there exists a finite partition of I into subintervals such that the restriction of γ to each subinterval is an arc map. We say an embedding γ of an IET into a PWI is a *linear embedding* if γ is a piecewise linear map. Moreover an embedding is *non-trivial* if it is not an arc embedding or a linear embedding. Figure 4 shows an illustration of a non-trivial embedding.

From the definitions it is clear that the image $\gamma(I)$ of an embedding is an invariant curve for the underlying PWI and that if the embedding is non-trivial this curve is not the union of line segments or circle arcs. For any IET it is straightforward to construct a PWI in which it is trivially embedded. The same is not true for non-trivial embeddings, for which results have been much scarcer.

We say a d-PWI is a PWI with a partition of d atoms. Similarly, a d-IET is an IET with a partition of d subintervals. In [9] we showed that there are no non-trivial continuous embeddings of minimal 2-IETs into orientation preserving planar PWIs.

Theorem 4.1 ([9]) A minimal 2-IET has no non-trivial continuous embedding into a 2-PWI.

The next theorem states that a 3-PWI has at most one non-trivially continuously embedded minimal 3-IET with the same underlying permutation.

Theorem 4.2 ([9]) A 3-PWI has at most one non-trivially continuously embedded minimal 3-IET with the same underlying permutation.

The proofs of Theorems 4.1 and 4.2 rely on the use of combinatorial properties of IETs to prove that in order for a PWI to realize a continuous

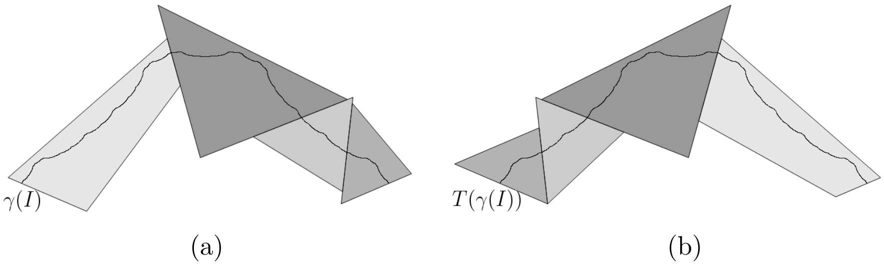

Figure 2: An illustration of the action of a PWI T with rotation vector $\theta \approx (4.85, 0.92, 1.31, 1.28)$ on its partition and on an invariant curve $\gamma(I)$. The map γ, estimated using technical tools from [25], is a non-trivial embedding of a self-inducing IET associated to the monodromy invariant $\tilde{\pi}(j) = 4 - (j-1)$, $j = 1, ..., 4$ and a translation vector of algebraic irrationals $\lambda \approx (0.43, 0.34, 0.12, 0.11)$.

embedding of an IET with the same permutation, its parameters must satisfy a necessary condition which may be found in [9].

5 Existence of invariant curves

In this section we show that almost every IET with an associated translation surface of genus $g \geq 2$ can be non-trivially and isometrically embedded in a family of piecewise isometries giving an overview of the technical tools used to prove the main results following our work in [25] .

In order to prove the main result presented in this section, we need to define the *Breaking operator* $\mathfrak{Br}$: given an ordered sequence $J = \{J_k\}_k$ of subintervals of I, an angle $\varphi \in [-\pi, \pi)$ and a piecewise linear map $\gamma : I \to \mathbb{C}$ the image of $\mathfrak{Br}(\varphi, J) \cdot \gamma$ is a piecewise linear curve, obtained from $\gamma(I)$ by rotating the segments $\gamma(J_k)$ by φ,

$$\mathfrak{Br}(\varphi, J) \cdot \gamma(x) = \begin{cases} \gamma(x) \cdot e^{i\varphi} + \overline{\epsilon}_k(\varphi, J), & x \in J_k, \\ \gamma(x) + \underline{\epsilon}_k(\varphi, J), & x \in L_k, \end{cases}$$

where $\overline{\epsilon}_k(\varphi, J)$ and $\underline{\epsilon}_k(\varphi, J)$ are determined by continuity and $L = \{L_k\}_k$ is the ordered sequence of subintervals determined by $I \backslash J$.

Given $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{S}(\mathcal{A})$, consider the sequence $J^{(n)} = \{J_k^{(n)}\}_{k<r(n-1)}$ obtained by ordering the collection of sets $\{f_{\lambda,\pi}^k(I^{(n-1)} \backslash I^{(n)})\}_{k<r(n-1)}$, where $r(n-1)$ is the smallest $r \geq 1$ such that $f_{\lambda,\pi}^k(I^{(n-1)} \backslash I^{(n)}) \subset I^{(n)}$. Recall that

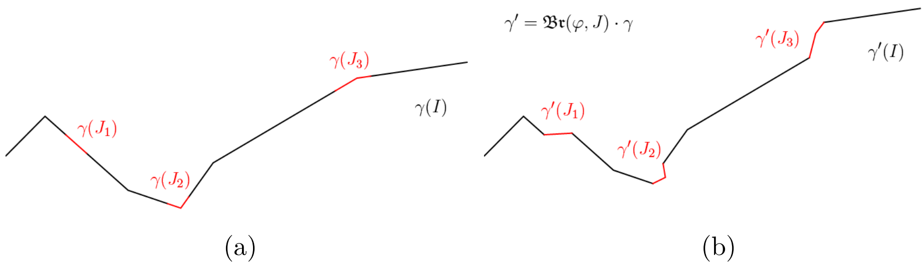

Figure 3: Action of the operator $\mathfrak{Br}$: (a) shows the image $\gamma(I)$ of a piecewise linear curve; (b) shows the image $\mathfrak{Br}(\varphi, J)\cdot\gamma(I)$, with $\varphi = \frac{\pi}{4}$ and $J = \{J_1, J_2, J_3\}$.

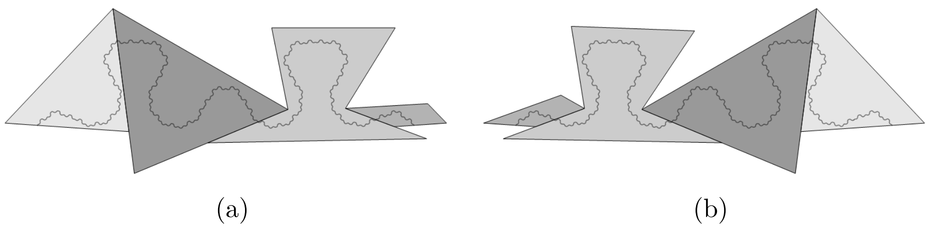

Figure 4: Action of a 4-PWI θ-adapted to a self-similar 4-IET. The curve depicted is $\gamma_\theta(I)$ and it is the image of a non-trivial embedding of the IET into this PWI.

$B_{\mathbb{T}^\mathcal{A}}$ is the projection of the Rauzy cocycle on the Torus defined in the introduction. Given $\theta \in \mathbb{T}^\mathcal{A}$ let

$$\theta^{(0)} = \theta, \quad \theta^{(n)} = B^{(n)}_{\mathbb{T}^\mathcal{A}}(\lambda, \pi)\cdot\theta,$$

With $\beta_{1,m} = (\pi_1^{(m)})^{-1}(d)$, we define the **breaking sequence** of curves $\{\gamma_\theta^{(n)}(x)\}$, by

$$\gamma_\theta^{(0)}(x) = x, \quad \gamma_\theta^{(n)}(x) = \mathfrak{Br}\left(\theta^{(n-1)}_{\beta_{1,n-1}}, J^{(n)}\right)\cdot\gamma_\theta^{(n-1)}(x), \quad x \in I.$$

Denote by $\Theta'_{\lambda,\pi}$ the set of all $\theta \in \mathbb{T}^\mathcal{A}$ such that:

- for all $n \geq 0$, $\gamma_\theta^{(n)} : I \to \mathbb{C}$ is an injective map;
- there exists a topological embedding $\gamma_\theta : I \to \mathbb{C}$ such that

$$\gamma_\theta(x) = \lim_{n\to+\infty} \gamma_\theta^{(n)}(x), \quad x \in I.$$

Given $\theta \in \Theta'_{\lambda,\pi}$, we say that a PWI $T : X \to X$ together with a partition $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is **θ-adapted to** (λ, π) if for all $\alpha \in \mathcal{A}$,

1. $X_\alpha \supseteq \gamma_\theta(I_\alpha)$;
2. For any $z \in \mathbb{C}$, we have $T(z) = T_\alpha(z)$, for all $z \in X_\alpha$, where

$$T_\alpha(z) = e^{i\theta_\alpha}\left(z - \gamma_\theta\left(\textstyle\sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta\right)\right) + \gamma_\theta\left(f_{\lambda,\pi}(\textstyle\sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta)\right).$$

Given $(\lambda, \pi) \in \mathbb{R}_+^{\mathcal{A}} \times \mathfrak{R}$ it is possible to associate, via a suspension construction, a translation surface, with genus $g(\mathfrak{R}) \geq 1$ depending only on the Rauzy class $\mathfrak{R}$.

The next theorem states the existence of invariant curves for PWIs which are not unions of circle arcs or line segments.

Theorem 5.1 ([25]) For almost every IET $(I, f_{\lambda,\pi})$ with a Rauzy class $\mathfrak{R}$ satisfying $g(\mathfrak{R}) \geq 2$, there exists a set $\mathcal{W} \subseteq \mathbb{T}^{\mathcal{A}}$ of dimension $g(\mathfrak{R})$ such that for all $\theta \in \mathcal{W}$ there exists a map $\gamma_\theta : I \to \mathbb{C}$, which is a non-trivial embedding of $(I, f_{\lambda,\pi})$ into any PWI that is θ-adapted to (λ, π).

References

[1] Adler, R., Kitchens, B., Tresser, C. (2001). *Dynamics of non-ergodic piecewise affine maps of the torus.* Ergodic Theory and Dynamical Systems **21**, 959–999.

[2] Alevy, I., Kenyon, R., Yi, R., (2018). *A family of minimal and renormalizable rectangle exchange maps.* arXiv:1803.06369

[3] Ashwin, P. (1996). *Non-smooth invariant circles in digital overflow oscillations.* Proceedings of the 4th Int. Workshop on Nonlinear Dynamics of Electronic Systems, Sevilla, 417–422.

[4] Ashwin, P., (1997). *Elliptic behaviour in the sawtooth standard map*, Phys. Lett. A, 232, pp. 409–416.

[5] Ashwin, P., Fu, (2002). *On the Geometry of Orientation-Preserving Planar Piecewise Isometries.* Journal of Nonlinear Science, 12, no. 3, 207–240.

[6] Ashwin, P. , Goetz, A. (2006). *Polygonal invariant curves for a planar piecewise isometry.* Trans. Amer. Math. Soc. **358** no. 1, 373–390.

[7] Ashwin, P. , Goetz, A. (2005). *Invariant curves and explosion of periodic islands in systems of piecewise rotations*. SIAM J. Appl. Dyn. Syst., 4(2), 437–458.

[8] Ashwin, P., Goetz, A. (2010). *Cone exchange transformations and boundedness of orbits*. Cambridge University Press, 30(5), 1311–1330.

[9] Ashwin P., Goetz A., Peres P., Rodrigues A., (2018). *Embeddings of Interval Exchange Transformations into Planar Piecewise Isometries*, Ergodic Theory and Dynamical Systems, 1-27. doi:10.1017/etds.2018.112

[10] Avila, A. (2010) *Dynamics of Renormalization Operators*. Proceedings of the International Congress of Mathematicians, Hyderabad, India.

[11] Avila, A. (2015) *Global theory of one-frequency Schrödinger operators*. Acta Math. Volume 215, Number 1, 1–54.

[12] Avila, A., Forni, G. (2007) *Weak mixing for interval exchange transformations and translation flows*, Ann. of Math. (2), 165(2), 637–664.

[13] Avila, A., Leguile, M. (2016) *Weak mixing properties of interval exchange transformations and translation flows*, Bulletin de la Societe mathematique de France, **146**.

[14] Chirikov B.V. (1983) *Chaotic dynamics in Hamiltonian systems with divided phase space*. In: Garrido L. (eds) Dynamical System and Chaos. Lecture Notes in Physics, vol 179. Springer, Berlin, Heidelberg.

[15] Coullet, P., Tresser, C. (1978). *Itérations d?endomorphismes et groupe de renormalisation*. C. R. Acad. Sci. Paris 287A, 577–580.

[16] Goetz, A. (2000). *Dynamics of piecewise isometries*. Illinois journal of mathematics, **44**, 465–478.

[17] Gutkin, E., Haydn, N., (1997) *Topological entropy of polygon exchange transformations and polygonal billiards*. Ergod. Th. & Dynam. Sys. 17, 849–867.

[18] Haller, H. (1981). *Rectangle Exchange Transformations*. Monatsh. Math. **91**(3), 215–232.

[19] Hooper, W. Patrick (2013) *Renormalization of polygon exchange maps arising from corner percolation*. Inventiones mathematicae, 191(2), 255-320.

[20] Keane, M. S., (1975) *Interval exchange transformations*, Math. Z. 141, 25–31.

[21] Lowenstein, J., Vivaldi, F., (2016). *Renormalization of a one-parameter family of piecewise isometries*, Dynamical Systems, 31:4, 393–465.

[22] Masur, H., (1982) *Interval exchange transformations and measured foliations*, Ann. of Math. 115, 169–200.

[23] MacKay, R., Percival, I., (1985). *Converse KAM - theory and practice*, Comm. Math. Phys. 98(4) 469–512.

[24] Peres, P., Rodrigues, A. (Preprint 2019) *On the dynamics of translated cone exchange transformations* https://arxiv.org/pdf/1809.05496.pdf

[25] Peres, P., Rodrigues, A. (Preprint 2019) *Existence of non-trivial embeddings of interval exchange transformations into piecewise isometries* https://arxiv.org/pdf/1901.10406.pdf

[26] Sinai, Ya. G., Khanin, K. M. (1989). *Smoothness of conjugacies of diffeomorphisms of the circle with rotations.* Uspekhi Mat. Nauk 44, 69–99.

[27] Sinai, Y., Ulcigrai, C. (2005). *Weak mixing in interval exchange transformations of periodic type.* Letters in Mathematical Physics, 74, 2, 111–133.

[28] Veech, W. A., (1982) *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. 115, 201–242.

[29] Viana, M., (2006) *Ergodic Theory of interval exchange maps*, Revista Matematica Complutense 19, 7–100.