An Invitation to Symplectic Toric Manifolds

Ana Cannas da Silva[1]
ETH Dep. Mathematics
Raemistrasse 101
8092 Zurich, Switzerland
e-mail: `ana.cannas@math.ethz.ch`

Resumo: Este artigo é uma introdução a variedades simplécticas tóricas para não especialistas, começando com uma breve síntese de variedades simplécticas e acções hamiltonianas. As variedades simplécticas tóricas formam já um tema extenso, ao qual a modesta lista de referências abaixo não faz justiça – o objectivo deste texto não é ser exaustivo ou justo, mas simplesmente deixar entrever o que são estes espaços e a razão pela qual o leitor poderá querer adicioná-los ao seu repertório de objectos geométricos.

Abstract This is an elementary introduction to symplectic toric manifolds for nonspecialists, starting with a brief review of symplectic manifolds and hamiltonian torus actions. Symplectic toric manifolds are by now a vast subject, for which the undersized list of references below does no justice – the aim of this text is not to be exhaustive or fair, but simply to give a glimpse into what these spaces are like and why it can be a good idea to add them to your repertoire of geometric objects.

palavras-chave: Variedade simpléctica; acção hamiltoniana; polítopo.

keywords: Symplectic manifold; hamiltonian action; polytope.

1 What is Symplectic Geometry?

Geometry concerns the study and measure of space. *Symplectic* refers to an additional structure that can be put on some even-dimensional spaces. Symplectic geometry is intrinsically related to complex geometry and, just like complex geometry, is sometimes counterintuitive. Whereas local complex geometry is basically modelled on $\mathbb{C}$, $\mathbb{C}^2$, $\mathbb{C}^3$, etc, local symplectic geometry is basically modelled on $\mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{R}^6$, etc.

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A **symplectic form** ω at a point p of a manifold M is a special type of differential 2-form, i.e., a device that takes two tangent vectors $\vec{u}, \vec{v} \in T_pM$ as input and returns a real number as output, that may be interpreted as

$$\omega(\vec{u}, \vec{v}) = \text{ kind of } \textit{signed area} \text{ of parallelogram spanned by } \vec{u} \text{ and } \vec{v}.$$

By *signed area* we mean, in particular, a number that may be positive, negative, or zero, contrasting with usual (euclidean, riemannian, ...) geometries.

In the case of the basic model of $\mathbb{R}^2$ with its so-called **standard symplectic form**,

$$\omega_0 := dx \wedge dy \ ,$$

this *signed area* is

$$\omega_0(\vec{u}, \vec{v}) = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = u_1 v_2 - u_2 v_1 \ ,$$

thus actually equal to plus or minus the euclidean area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$. The sign depends on the orientation of the basis $\vec{u}, \vec{v}$ and $\omega_0(\vec{v}, \vec{u}) = -\omega_0(\vec{u}, \vec{v})$. Moreover, there is only *zero* as output in just one dimension, since $\omega_0(\vec{v}, \vec{v}) = 0$ for all $\vec{v}$.

In the next case of $\mathbb{R}^4$, the standard symplectic form,

$$\omega_0 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \ ,$$

just adds up the contributions from the projections onto the two coordinate planes x_1, y_1 and x_2, y_2. If we have vectors $\vec{u} = \vec{u}_1 + \vec{u}_2$ and $\vec{v} = \vec{v}_1 + \vec{v}_2$ (where $\vec{u}_1, \vec{v}_1$ and $\vec{u}_2, \vec{v}_2$ denote the projections onto the coordinate planes x_1, y_1 and x_2, y_2), then

$$\omega_0(\vec{u}, \vec{v}) = \underbrace{(dx_1 \wedge dy_1)\,(\vec{u}_1, \vec{v}_1)}_{\text{signed area of } A_1} + \underbrace{(dx_2 \wedge dy_2)\,(\vec{u}_2, \vec{v}_2)}_{\text{signed area of } A_2}$$

can be thought of as a sum of signed areas for the projections A_1 and A_2 onto each of the coordinate planes x_1, y_1 and x_2, y_2. Other projections are not taken into account.

The higher cases $\mathbb{R}^{2n}$ are analogous. In particular, in $\mathbb{R}^6$ we have the standard symplectic form

$$\omega_0 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3 \ .$$

Physicists often regard (x_1, x_2, x_3) as position coordinates and (y_1, y_2, y_3) as momentum (kind of velocity) coordinates of a particle in 3-dimensional space. The symplectic form ω_0 encodes the mutual entanglement of position and momentum in a somewhat implausible way that actually fits reality. In Section 2, we will describe the motion of a classical mechanical system via Hamilton's equations for position and momentum in terms of a flow on a symplectic manifold.

Historical remark:

> *Symplectic geometry* is a branch of mathematics, that could be viewed as emerging in the XIX century from classical mechanics. The mathematicians William Rowan Hamilton (1805-1865) and Sofia Kovalevskaya (1850-1891) were at the onset of this field and worked on problems related to the motion of rigid bodies. Symplectic geometry experienced a vigorous expansion in the last 50 years and deals nowadays with many other geometric problems, stimulated by interactions with diverse areas of mathematics and physics. The adjective *symplectic* in mathematics is a calque[2] coined by Hermann Weyl, by substituting the Latin root in *complex* by the corresponding Greek root, in order to label the symplectic group.

In general, a **symplectic manifold** is a pair (M, ω) where M is a manifold (necessarily even-dimensional, say $\dim M = 2n$) and ω is a closed nondegenerate 2-form on M. Whereas closedness is a natural differential condition from analysis, nondegeneracy is an algebraic condition saying that at each point any nonzero tangent vector admits a nontrivial pairing with some other tangent vector – this is what forces the evenness of the dimension.

One of the fundamental theorems in symplectic geometry goes back to Darboux [6] in the late XIX century in the context of differential systems. What is now known as **Darboux's theorem** states that any symplectic manifold looks *locally* near any of its points like a neighborhood of the origin in $\mathbb{R}^{2n}$ equipped with

$$\omega_0 := dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n \ .$$

We hence refer to $(\mathbb{R}^{2n}, \omega_0)$ as the *local model.* Although this shows that there are no local invariants in symplectic geometry besides the dimension, the local symplectic geometry, i.e. the symplectic geometry of $(\mathbb{R}^{2n}, \omega_0)$, is already quite interesting and there remain deep open questions about

[2]A *calque* or *loan translation* is a word or phrase that is introduced through translation of the constituents into another language.

it. Normal form theorems like Darboux's play a central role in symplectic geometry.

On a symplectic manifold (M, ω), the top power of the symplectic form, ω^n, is necessarily a volume form, called the **symplectic volume**. This follows from the nondegeneracy of ω, and may be also seen through Darboux's theorem with $\omega_0^n = n!\, dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$. Therefore, a symplectic manifold is *symplectically* oriented, and nonorientable manifolds cannot be symplectic.

On a symplectic manifold (M, ω), we are able to integrate the symplectic form ω over a surface $A \subset M$:

$$\int_A \omega = \textit{ symplectic area } \text{of } A.$$

In the case of $(\mathbb{R}^4, \omega_0)$, this yields again a sum of contributions from the two projections onto each of the coordinate planes x_1, y_1 and x_2, y_2:

$$\int_A \omega = \underbrace{\int_{A_1} dx_1 \wedge dy_1}_{\text{signed area of } A_1} + \underbrace{\int_{A_2} dx_2 \wedge dy_2}_{\text{signed area of } A_2} .$$

Such a measurement is *anisotropic* in the sense that (multiple-dimensional) directions are not all the same. For instance, a nontrivial surface in the x_1, x_2-plane has one-dimensional projection onto the x_1, y_1 and x_2, y_2 planes, hence has *zero symplectic area.* Such a surface in a four-dimensional manifold is called *lagrangian.* On the other hand, a nontrivial surface in the x_1, y_1 plane already has a *nonzero symplectic area.* Such a surface is called *symplectic.*

In general, we distinguish different important types of submanifolds in a $2n$-dimensional symplectic manifold (M, ω). A **symplectic submanifold** is a submanifold where the restriction of the symplectic form is nondegenerate, hence still a symplectic form. Such submanifolds are again even-dimensional. When $n = 1$, these submanifolds turn out to be related to *complex curves.* An **isotropic submanifold** is a submanifold where the restriction of the symplectic form vanishes identically. Any one-dimensional submanifold is isotropic and isotropic submanifolds are at most half-dimensional; this follows from linear algebra. A **lagrangian submanifold** is an n-dimensional isotropic submanifold. Lagrangian submanifolds are thus the largest isotropic submanifolds and turn out to be related to dynamics.

Examples and nonexamples:

(0) As mentioned, the examples $(\mathbb{R}^{2n}, \omega_0)$ above are the local prototypes of symplectic manifolds.

(1) Any oriented surface may be equipped with a symplectic structure by choosing any area form to take the role of symplectic form. In particular, a unit sphere in $\mathbb{R}^3$ equipped with the standard (euclidean) area form is automatically a symplectic manifold. This area form may be written away from the poles as

$$\omega_{\text{std}} := d\theta \wedge dh \ ,$$

where h is a height function and θ the angle around that height axis, giving total area 4π; cf. Section 4.

(2) Some of the simplest 4-dimensional symplectic manifolds are products of oriented surfaces, such as $S^2 \times S^2$ equipped with a sum of area forms (eventually different on each factor), and complex projective space $\mathbb{CP}^2$, that is, the space of complex lines in $\mathbb{C}^3$. The standard symplectic form in $\mathbb{CP}^2$ (or, for that matter, in $\mathbb{CP}^n$) is called *Fubini-Study* form and we will give some insight into it in Section 3. In general, products of symplectic manifolds are symplectic.

(3) The only spheres that may be symplectic are the 2-dimensional ones. Let us see why. In a sphere S^k of any other dimension, closed 2-forms are always exact (this topological fact is usually encoded as $H^2(S^k) = 0$ for $k \neq 2$). Now, by Stokes' theorem, a symplectic form cannot be exact on a compact manifold without boundary, because if it were $\omega = d\alpha$, then its top power $\omega^n = d(\alpha \wedge \omega^{n-1})$ would also be exact, which is impossible for a volume form on such a manifold:

$$\int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-1}) = \int_{\partial M} \alpha \wedge \omega^{n-1} = 0 \text{ contradicting } \int_M \omega^n > 0 \ .$$

By now there are a number of texts on symplectic geometry, a subset of which is [11, 12, 4]. For a beautiful overview geared towards symplectic topology, see McDuff's lecture [10].

2 What are Hamiltonian Torus Symmetries?

The definition of symplectic form contains exactly what is needed for the following general assertion: *On a symplectic manifold* (M, ω)*, any smooth* *function $H : M \to \mathbb{R}$ generates (in a nontrivial way) a flow that preserves both the symplectic structure ω and the function H.*

Such a flow is called the **hamiltonian flow** generated by H and then H is called a corresponding **hamiltonian function**. The asserted property refers to the *existence and uniqueness* (by nondegeneracy of ω) of a vector field X_H defined by

$$\omega(X_H, \cdot) = dH(\cdot) \ . \qquad \star$$

This vector field X_H satisfies the following equations where we use Cartan's magic formula, $\mathcal{L}_X = d\imath_X + \imath_X d$, for the Lie derivative with respect to a vector field X:

$$\mathcal{L}_{X_H}\omega = d\underbrace{\imath_{X_H}\omega}_{dH} + \imath_{X_H}\underbrace{d\omega}_{0} = 0 \qquad \text{and} \qquad \mathcal{L}_{X_H}H = \imath_{X_H}\underbrace{dH}_{\imath_{X_H}\omega} = 0 \ .$$

This vector field X_H integrates (by the theorem of Picard-Lindelöf) into a local time evolution, a.k.a. *flow*, and the equations $\mathcal{L}_{X_H}\omega = 0$ and $\mathcal{L}_{X_H}H = 0$ amount infinitesimally to this flow preserving ω and H. The vector field X_H is called the **hamiltonian vector field** of H.

<u>Examples and nonexamples:</u>

(0) For euclidean space $(\mathbb{R}^6, \omega_0)$ and any function $H : \mathbb{R}^6 \to \mathbb{R}$, equation $\star$ for the flow generated by H translates into **Hamilton's equations**:

$$\begin{cases} \dfrac{dx_k}{dt} = \dfrac{\partial H}{\partial y_k} \\[2ex] \dfrac{dy_k}{dt} = -\dfrac{\partial H}{\partial x_k} \ . \end{cases}$$

(1) For the unit sphere $(S^2, \omega_{\mathrm{std}} = d\theta \wedge dh)$ and hamiltonian function H equal to the height function h, equation $\star$ yields as hamiltonian vector field

$$X_H = \frac{\partial}{\partial \theta} \ ,$$

so the corresponding flow rotates around the height axis. This clearly preserves area ω_{std} and height H. Notice how this contrasts with the gradient flow of H, which is basically perpendicular and preserves neither ω_{std} nor H.

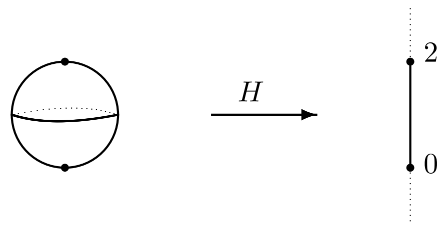

Figure 1: Hamiltonian function for the rotation of S^2 around the height axis.

(2) For the 2-torus $(\mathbb{T}^2, \omega := d\theta_1 \wedge d\theta_2)$, we have that the rotation given by the vector field $\frac{\partial}{\partial\theta_1}$ preserves area, yet is *not* hamiltonian, since the contraction

$$\omega(\tfrac{\partial}{\partial\theta_1}, \cdot) = d\theta_2(\cdot)$$

is closed yet *not* exact, i.e., there is no corresponding global hamiltonian function.

The flow in Example (1) is also an example of S^1*-action*. Indeed, the time-t evolution φ_t is given, with respect to these coordinates, by $\varphi_t : (\theta, h) \longmapsto (\theta + t, h)$, so it is 2π-periodic (i.e., $\varphi_{t+2\pi} \equiv \varphi_t$) and satisfies the group law (i.e., $\varphi_{t_1} \circ \varphi_{t_2} \equiv \varphi_{t_1+t_2}$). Because it is also hamiltonian, we call it a **hamiltonian S^1-action**.

Analogously, for a d-dimensional torus $\mathbb{T}^d = S^1 \times \ldots \times S^1$ we define a **hamiltonian $\mathbb{T}^d$-action** to be an action of $\mathbb{T}^d$ for which each of the S^1-factors acts in a hamiltonian fashion, say with hamiltonian function H_k, and each of these H_k is invariant by the rest of the action. By collecting these hamiltonian functions, we build an invariant function

$$H := (H_1, \ldots, H_d) : M \to \mathbb{R}^d \ .$$

This upgraded version of hamiltonian function is known as a (special case of) **moment map**. The concept of *moment map* for hamiltonian actions of arbitrary Lie groups has recently become central in geometry and topology.

Atiyah [2] and, independently, Guillemin and Sternberg [9] proved in the 80's, that the image of such a function $H : M \to \mathbb{R}^d$ on a compact, connected symplectic manifold (M, ω) corresponding to a hamiltonian $\mathbb{T}^d$-action is always a convex polytope. Moreover, they showed that that image is simply the convex hull of the images of the fixed points of the action. This deep and key theorem is known as the **convexity theorem**.

To get rid of lazy factors in that action, we concentrate on **faithful (i.e. effective) actions** for which only the identity group element gives rise to the identity diffeomorphism. We think of effective hamiltonian $\mathbb{T}^d$-actions as **hamiltonian torus symmetries**. Now, if a d-dimensional torus acts in a faithful and hamiltonian fashion on a $2n$-dimensional symplectic manifold, then it must be $d \le n$. This follows from the fact that the orbits are isotropic, that isotropic submanifolds are at most half-dimensional, and that Lie theory tells us that a faithful action of a d-dimensional Lie group always admits orbits equivariantly diffeomorphic to the group itself, the so-called *principal* orbits. Therefore, a maximal hamiltonian torus symmetry is of the form $\mathbb{T}^n$ acting on M^{2n}.

3 What are Symplectic Toric Manifolds?

A **symplectic toric manifold** is a compact connected symplectic manifold (M, ω) with a maximal hamiltonian torus symmetry, meaning, with a faithful hamiltonian action of a half-dimensional torus. If $\dim M = 2n$, then we have the n-dimensional torus $\mathbb{T}^n$ acting faithfully and with a moment map

$$H : M \to \mathbb{R}^n \ .$$

Examples and nonexamples:

(0) Examples with $(\mathbb{R}^{2n}, \omega_0)$ are ruled out by lack of compactness. However, most of the theory could be, and often is, extended to such examples.

(1) The unit sphere $(S^2, \omega_{\text{std}} = d\theta \wedge dh)$ together with the S^1-action generated by the height function $H = h$ is a symplectic toric manifold. We point out some of the features of this example, to which we will come back in more general set-ups:

   (a) The image interval $[0, 2]$ is the orbit space, i.e., there is exactly one S^1-orbit per height value. The endpoints of this interval correspond to the two fixed points (singular orbits), South pole and North pole.

   (b) The best coordinates to understand this system are the *angle* coordinate θ where the rotation occurs and the function $H = h$ encoding the hamiltonian *action*, valid away from the poles. Such coordinates are called **action-angle coordinates**. With respect to such coordinates, the symplectic form is simply a product form $d\theta \wedge dH$, just like a form in the local model space $(\mathbb{R}^2, dx \wedge dy)$.

(c) The area of an invariant strip on S^2 corresponding to a subinterval of $[0,2]$ of height Δh is equal to $2\pi \cdot \Delta h$. This result goes back more than two millenia; see Section 4.

(1') We revisit the previous example from a complex viewpoint. Regarding S^2 as a Riemann sphere, we denote by $[z_0 : z_1]$ the point given by the complex line in $\mathbb{C}^2$ through (z_0, z_1) and $(0,0)$. The South pole is $[1:0]$ and the North pole is $[0:1]$. Now we recast that example as $(\mathbb{CP}^1, \omega_{\text{FS}})$, where the *Fubini-Study* symplectic form ω_{FS} is equal to $\frac{1}{4}\omega_{\text{std}}$, the element e^{it} of the circle acts by multiplication on the coordinate z_1,

$$e^{it} \cdot [z_0 : z_1] = [z_0 : e^{it} z_1] \ ,$$

which, on a chart, is again a simple shift of the angle coordinate, and the corresponding hamiltonian function is

$$H_1 := \frac{|z_1|^2}{2\left(|z_0|^2 + |z_1|^2\right)} \ .$$

(2) Consider now complex projective space $\mathbb{CP}^n$ (as a $2n$-dimensional real manifold) with a diagonal action of $\mathbb{T}^n$ by

$$\left(e^{i\theta_1}, \dots, e^{i\theta_n}\right) \cdot [z_0 : z_1 : \dots : z_n] = [z_0 : e^{i\theta_1} z_1 : \dots : e^{i\theta_n} z_n] \ .$$

The **Fubini-Study symplectic form** is a globally well-defined form, which, away from the hyperplanes $z_k = 0$, is given by the Darboux-type formula

$$\omega_{\text{FS}} = d\theta_1 \wedge dH_1 + \dots + d\theta_n \wedge dH_n \ ,$$

where the component H_k of the moment map $H : \mathbb{CP}^n \to \mathbb{R}^n$ is

$$H_k := \frac{|z_k|^2}{2\left(|z_0|^2 + \dots + |z_n|^2\right)} \ .$$

For instance, when $n = 3$ we get the following picture:

We list again the earlier features, some of which now take more thought to check:

(a) The image simplex is the orbit space, i.e., there is exactly one $\mathbb{T}^n$-orbit per point on the n-simplex. The vertices of this simplex

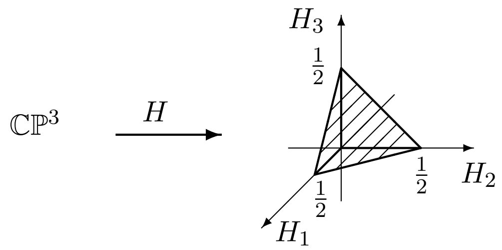

Figure 2: Moment map for the standard action on $\mathbb{CP}^3$.

correspond to the $n+1$ fixed points, $[1:0:\ldots:0], \ldots [0:\ldots:0:1]$. The interior points correspond to orbits through points of the form $[z_0 : z_1 : \ldots : z_n]$ with all coordinates z_k nonzero.

(b) Best to understand this system are the *action-angle coordinates*, $H_1, \ldots, H_n$ and $\theta_1, \ldots, \theta_n$. With respect to these coordinates, and in points mapping by H to the interior of the simplex, the symplectic form is just like a form in the local model space $(\mathbb{R}^{2n}, dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n)$.

(c) The (symplectic) volume of a $\mathbb{T}^n$-invariant subset $H^{-1}(S)$ is simply equal to $(2\pi)^n \cdot |S|$, where $|S|$ is the (euclidean-)volume of the subset S of the simplex.

By the convexity theorem, we already know that the moment map image of a $2n$-dimensional symplectic toric manifold is a polytope in $\mathbb{R}^n$. One can show that such a polytope enjoys special properties: it is *simple*, i.e., there are n edges meeting at each vertex, it is *rational*, i.e., the edges meeting at each vertex τ are of the form $\tau + tu_j$, $t \geq 0$, with each $u_j \in \mathbb{Z}^n$, and it is *smooth*, i.e., for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$; see, for instance, [5].

As first proved by Delzant [7], it turns out that this polytope encodes enough information to reconstruct its originating symplectic toric manifold, and that all such *simple, rational, smooth* polytopes occur as moment map images of symplectic toric manifolds. **Delzant's theorem** is a celebrated result classifying symplectic toric manifolds in terms of polytopes:

$$\left\{ \begin{array}{c} 2n\text{-dim'l symplectic} \\ \text{toric manifolds} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{simple rational smooth} \\ \text{polytopes in } \mathbb{R}^n \end{array} \right\}$$

where this one-to-one correspondence takes a symplectic toric manifold, (M, ω, H) where the $\mathbb{T}^n$-action admits $H : M \to \mathbb{R}^n$ as moment map, to the polytope which is the *image of this moment map*:

$$(M, \omega, H) \quad \longleftrightarrow \quad H(M) \; .$$

For such a correspondence, there are underlying notions of equivalence of the objects involved. In the simplest version, polytopes in $\mathbb{R}^n$ are identified up to translation, and symplectic toric manifolds are identified up to equivariant diffeomorphism preserving the symplectic forms: (M_1, ω_1, H_1) and (M_2, ω_2, H_2) with actions of $\mathbb{T}^n$ are *equivalent* if and only if there is a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^* \omega_2 = \omega_1$ and $\varphi(g \cdot p) = g \cdot \varphi(p)$ for all $g \in \mathbb{T}^n$ and $p \in M_1$.

Note that the problem of classifying compact symplectic manifolds in dimension 4 or higher is completely open. The presence of a hamiltonian torus symmetry significantly helps.

Since there is just one 1-dimensional polytope of length ℓ up to translation, we see that the only 2-dimensional symplectic toric manifolds are scaled spheres $(S^2, \frac{\ell}{2}\,\omega_{\text{std}})$ with rotation action as above. The panorama for 2-dimensional polytopes is much more rich. Still, up to translation, the 2-dimensional simple, rational, smooth polytopes with only three vertices are the triangles with vertices $(0,0)$, $(\ell, 0)$ and $(0, \ell)$ or their transforms by $\mathrm{GL}(2; \mathbb{Z})$. This is saying that the corresponding symplectic toric manifolds are $(\mathbb{CP}^2, 2\ell\,\omega_{\text{FS}})$ with standard $\mathbb{T}^2$-action or their transforms by an isomorphism of $\mathbb{T}^2$.

The upshot is that any such symplectic toric manifold is given *combinatorially* in terms of a polytope in an euclidean space of half the dimension that of the manifold. Hence, all questions pertaining to such manifolds should admit an answer in terms of polytopes – a mathematician's dream! In particular, the earlier properties admit generalizations to all symplectic toric manifolds (M, ω, H) as follows:

(a) The polytope image is the orbit space, so H is also the point-orbit projection, and the vertices of the polytope correspond to the fixed points. There are precise descriptions of the isotropy subgroups in terms of the face-stratification.

(b) There are *action-angle coordinates*, $H_1, \ldots, H_n$ and $\theta_1, \ldots, \theta_n$, valid at points mapping to the interior of the polytope, which are the best coordinates to understand this system. With respect to them, the symplectic form is $\omega = d\theta_1 \wedge dH_1 + \ldots + d\theta_n \wedge dH_n$.

(c) The (symplectic) volume of a $\mathbb{T}^n$-invariant subset is equal to $(2\pi)^n$

times the (euclidean) volume of the corresponding subset in the polytope.

A lot of the geometry of symplectic toric manifolds has already been understood, yet many interesting questions remain. Currently, these manifolds are used as test grounds for theories or conjectures in topology, geometry and mathematical physics, such as *mirror symmetry*.

Many open questions for these manifolds relate to their lagrangian submanifolds. We can see that connected lagrangian submanifolds invariant by $\mathbb{T}^n$ are principal $\mathbb{T}^n$-orbits, i.e., those corresponding to the interior points of the image polytope. We might now ask about other lagrangian submanifolds that *fit nicely* with respect to the torus action, in the sense that they are invariant by some subgroup of $\mathbb{T}^n$ and they intersect $\mathbb{T}^n$-orbits in a *clean* way. The image under the moment map of such a lagrangian submanifold of (M, ω, H) lies in the intersection of the polytope $H(M)$ with an affine subspace. Examples are all principal $\mathbb{T}^n$-orbits, the standard *real part* submanifolds like $\mathbb{RP}^n$ in $\mathbb{CP}^n$, lagrangian submanifolds like the one presented in [3], and many lagrangian submanifolds sitting in level sets of components of the moment map.

4 Epilogue – all the way from Archimedes

We close by going back more than two millenia to Archimedes' supposedly favourite work on measuring spheres and cylinders. In around 200 BC, Archimedes was the first to realize that *the surface area of a sphere between two parallel planes intersecting it depends only on the distance between those planes* and not on the height where they intersect the sphere. Moreover, Archimedes asserted that the surface area on the sphere is the same as that of a cylinder with the radius of that sphere and height given by the distance between the planes, as the following figure illustrates. This is exactly the feature that allows us to write the standard area form as $\omega_{\text{std}} = d\theta \wedge dh$.

Nowadays, if you know first-year calculus, you may check Archimedes result by computing an appropriate surface integral using, for instance, cylindrical coordinates (θ, z) to write points on the sphere as $(x, y, z) = (\sqrt{R^2 - z^2}\cos\theta, \sqrt{R^2 - z^2}\sin\theta, z)$:

$$\text{Area} = \int_0^{2\pi} \int_h^{h+\Delta h} R\, dz\, d\theta = 2\pi R \cdot \Delta h \ ,$$

or else use some approximation method and then take the limit [1]. However, Archimedes did not know calculus. It seems that he used an approximation

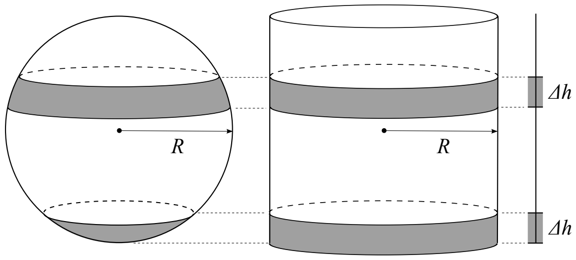

Figure 3: Spherical and cylindrical strips all with the same area: $2\pi R \cdot \Delta h$; image kindly reproduced from [1].

argument, for which a relevant reference is the *palimpsest*[3] discovered in the XX century after some quite adventurous history.

In the 80's, Duistermaat and Heckman [8] showed powerful results for symplectic manifolds with hamiltonian torus actions, which may be viewed as a vast generalization of Archimedes' theorem for the 2-sphere. Just like Archimedes might have had no idea that, more than two millennia later, his spirit would be at the origin of new mathematics, one wonders what other leaps await mankind starting from symplectic toric manifolds.

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