A SHARP INEQUALITY IN FOURIER RESTRICTION THEORY

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Resumo: Recentemente, os autores provaram em [15, 16] que *as funções constantes são os únicos maximizantes reais da desigualdade $L^2 \to L^6$ de extensão de Fourier na 2-esfera.* Isto é um caso particular de [16, Teorema 1.1], cuja prova contém vários dos métodos e ideias-chave. Neste artigo, descrevemos a prova deste caso particular, e apresentamos algumas generalizações e problemas em aberto.

Abstract: We focus on the proof of the following recent result [15, 16] in Sharp Fourier Restriction Theory: *Constant functions are the unique real-valued maximizers for the $L^2 \to L^6$ adjoint Fourier restriction inequality on the 2-sphere.* This is a special case of [16, Theorem 1.1] which already relies on several of the key methods and ideas. We discuss generalizations, extensions, and present a few open problems.

palavras-chave: Teoria de restrição de Fourier; desigualdade de Tomas–Stein; constantes ótimas; maximizantes.

keywords: Sharp Fourier Restriction Theory; Tomas–Stein inequality; optimal constants; maximizers.

1 Introduction

We start with a collection of three apparently unrelated problems from geometry, probability theory, and algebra.

Question 1 *Given $d \geq 2, 0 < k < d$, what is the maximal volume of the intersection of the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$ with a k-dimensional subspace of $\mathbb{R}^d$?*

Question 2 *Given $d, n \geq 2$, what is the probability distribution of an n-step uniform random walk in $\mathbb{R}^d$?*

Question 3 *Given $d \geq 2$, what is the minimal codimension of a proper subalgebra of the special orthogonal Lie algebra $\mathfrak{so}(d)$?*

One of the goals of the present note is to describe how each of these questions played a very natural role in the recent solution of an extremal problem from harmonic analysis, to which we now turn our attention.

1.1 Fourier restriction theory

The Fourier transform is one of the most ubiquitous tools in mathematics. By decomposing a general function $f : \mathbb{R}^d \to \mathbb{C}$ into a superposition of simpler, "symmetric" functions,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, \mathrm{d}x,$$

it opens the door to powerful analytic arguments that have shaped the history of mathematics for the last two centuries. Despite its paramount importance, fundamental questions about the Fourier transform remain open.

A consequence of the classical Hausdorff–Young inequality is that the Fourier transform $\widehat{f}$ of an L^p function $f : \mathbb{R}^d \to \mathbb{C}$ is defined almost everywhere on $\mathbb{R}^d$, provided $1 \leq p \leq 2$. It is a striking observation of E. M. Stein from the late 1960s that, for a special range of p's, the function $\widehat{f}$ can be meaningfully defined on submanifolds of $\mathbb{R}^d$ possessing some degree of curvature. The simple yet fundamental observation that *curvature causes the Fourier transform to decay* links geometry to analysis, and lies at the base of Fourier restriction theory. Take, for instance, the example of the unit sphere, $\mathbb{S}^{d-1} := \{\omega \in \mathbb{R}^d : |\omega| = 1\}$, a compact manifold with positive Gaussian curvature which inherits its surface measure $\mathrm{d}\sigma_{d-1}$ from the ambient space $\mathbb{R}^d$ in the natural way. The celebrated *Fourier restriction conjecture* predicts the validity of the estimate[1]

$$\int_{\mathbb{S}^{d-1}} |\widehat{g}(\omega)|^q \,\mathrm{d}\sigma_{d-1}(\omega) \leq C\|g\|^q_{L^p(\mathbb{R}^d)}, \text{ if } 1 \leq p < \frac{2d}{d+1}, q \leq \frac{d-1}{d+1}p', \tag{1}$$

and is remarkable in its numerous connections and applications. It exhibits deep links to Bochner–Riesz summation methods and to decoupling phenomena for the Fourier transform, and is known to imply the Kakeya conjecture. Despite the great deal of attention received by this circle of problems during the past four decades, the restriction conjecture remains open

[1] Here, p' denotes the conjugate exponent to p, given by $\frac{1}{p} + \frac{1}{p'} = 1$.

in dimensions $d \geq 3$. For further details, we refer the interested reader to the classical survey [20], and the very recent, exciting account from [18].

If $d \geq 2$ and $q \geq 2\frac{d+1}{d-1}$, then the cornerstone Tomas–Stein inequality [17, 21] states that there exists $C = C(d,q) < \infty$, such that

$$\|\widehat{f\sigma}_{d-1}\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{S}^{d-1})}, \tag{2}$$

for every function $f : \mathbb{S}^{d-1} \to \mathbb{C}$ which is square-integrable with respect to $\mathrm{d}\sigma_{d-1}$. Here, $\widehat{f\sigma}_{d-1}(x) := \mathcal{E}(f)(x) := \int_{\mathbb{S}^{d-1}} f(\omega)e^{i\omega\cdot x}\, \mathrm{d}\sigma_{d-1}(\omega)$, $x \in \mathbb{R}^d$, denotes the Fourier extension operator, which is the adjoint of the restriction operator, $\mathcal{E}^*(g) := \widehat{g}\,|_{\mathbb{S}^{d-1}}$, considered in (1). Inequality (2) finds deep applications in harmonic analysis and PDE. In particular, it underlies most of the early progress towards the Fourier restriction conjecture; see [20]. The Tomas–Stein argument directly implies some of the foundational Strichartz estimates for various dispersive partial differential equations, e.g. the Schrödinger, wave, and Klein–Gordon equations; see [19]. Moreover, inequality (2) has been generalized to a variety of contexts, and found surprising applications ranging from fractal geometry [14] to number theory [10], among many others.

1.2 Sharp Fourier Restriction Theory

A class of problems which is the subject of some exciting ongoing research goes under the name of *Sharp Fourier Restriction Theory.* For a gentle introduction to this fascinating topic, we refer the reader to the recent survey [7], and proceed to describe a few concrete examples.

Associated to (2), we have the functional

$$f \mapsto \Phi_{d,q}(f) := \frac{\|\widehat{f\sigma}_{d-1}\|^q_{L^q(\mathbb{R}^d)}}{\|f\|^q_{L^2(\mathbb{S}^{d-1})}}.$$

A very natural problem is to determine the value of the best (smallest) constant in inequality (2),

$$\mathbf{T}^q_{d,q} := \sup_{\mathbf{0}\neq f\in L^2} \Phi_{d,q}(f),$$

i.e. the operator norm of the extension operator. A related, but typically harder, problem is to characterize all the *maximizers* of $\Phi_{d,q}$, that is to say, the nonzero functions which realize the best constant $\mathbf{T}_{d,q}$. The mere *existence* of maximizers is a highly non-trivial question, which for $\Phi_{d,q}$ happens to be open at the endpoint $q = 2\frac{d+1}{d-1}$ in all dimensions $d \geq 4$; see [8] for a conditional result in this direction.

1.2.1 A sharp L^2–L^4 result

A remarkable recent result of D. Foschi [6] establishes that constant functions are the unique real-valued maximizers for the endpoint Tomas–Stein inequality in three-dimensional space,

$$\|\widehat{f\sigma_2}\|_{L^4(\mathbb{R}^3)} \leq \mathbf{T}_{3,4}\|f\|_{L^2(\mathbb{S}^2)}. \tag{3}$$

In particular, $\mathbf{T}_{3,4} = \|\widehat{\sigma}_2\|_{L^4(\mathbb{R}^3)}\|\mathbf{1}\|^{-1}_{L^2(\mathbb{S}^2)} = 2\pi$. The proof is short, simple, and relies on an elegant geometric identity,

$$|\omega+\nu|^2 + |\nu+\zeta|^2 + |\zeta+\omega|^2 = 4,$$

which holds for any triple of unit vectors $(\omega,\nu,\zeta) \in (\mathbb{S}^2)^3$ satisfying $|\omega+\nu+\zeta| = 1$. Additional ingredients that play a key role in [6] are some symmetry considerations, a natural spectral analysis, and two fortuitous coincidences.

The first coincidence is that in the three-dimensional case some calculations simplify considerably in comparison with other dimensions. Technically, this is seen at the level of the convolution measure $\sigma_{d-1} * \sigma_{d-1}$, which finds its simplest form when $d = 3$; see (23) below. The difficulties inherent to the higher dimensional cases were partially overcome in [4], thereby extending Foschi's $L^2 \to L^4$ sharp result to dimensions $4 \leq d \leq 7$. If $d = 8$, then a new phenomenon emerges, and the identification of one single maximizer of $\Phi_{d,4}$ is a challenging open problem in all dimensions $d \geq 8$.

The second coincidence is that $4 = 2\times 2$. In particular, since the Fourier transform intertwines multiplication and convolution,

$$|\widehat{f\sigma_2}|^4 = (\widehat{f\sigma_2}\overline{\widehat{f\sigma_2}})^2 = (\widehat{f\sigma_2 * \bar{f}\sigma_2})^2.$$

An application of Plancherel's identity, $\|\widehat{F}\|_{L^2(\mathbb{R}^3)} = (2\pi)^{3/2}\|F\|_{L^2(\mathbb{R}^3)}$, then reveals that (3) can be equivalently recast as a convolution inequality,

$$\|f\sigma_2 * \bar{f}\sigma_2\|_{L^2(\mathbb{R}^3)} \leq (2\pi)^{-3/2}\mathbf{T}^2_{3,4}\|f\|^2_{L^2(\mathbb{S}^2)}.$$

The 2-fold convolution measure $f\sigma_2 * \bar{f}\sigma_2$ turns out to be a relatively simple object of study, even though the function $f \in L^2$ may be quite rough. The situation changes dramatically if instead we consider the k-fold convolution $(f\sigma_2)^{*k}$, for $k \geq 3$. In fact, prior to our very recent work [15, 16], no sharp instance of inequality (2) was known if $q \in (4,\infty)$, in any dimension $d \geq 2$.

1.2.2 A sharp L^2–L^6 result

In [16], we proved that constant functions are the unique real-valued maximizers of the functional $\Phi_{d,2n}$, whenever $d \in \{3,4,5,6,7\}$ and $n \geq 3$ is an integer. The following particular case of [16, Theorem 1.1] will be the focus of our attention.

Theorem 1 *Constants are the unique real-valued maximizers of* $\Phi_{3,6}$.

This of course translates into a sharp inequality

$$\|\widehat{f\sigma}_2\|_{L^6(\mathbb{R}^3)} \leq \mathbf{T}_{3,6}\|f\|_{L^2(\mathbb{S}^2)}, \tag{4}$$

with $\mathbf{T}_{3,6} = \|\widehat{\sigma}_2\|_{L^6(\mathbb{R}^3)}\|\mathbf{1}\|^{-1}_{L^2(\mathbb{S}^2)} = (2\pi)^{5/6}$. We choose to delve into the proof of Theorem 1 because it already contains several of the main themes which were introduced in [15, 16]. On the other hand, the convenient choice of parameters $(d,q) = (3,6)$ causes several technicalities to disappear, and makes us hopeful that the key ideas may be conveyed in the course of this short note.

1.3 Notation

The constant function is denoted $\mathbf{1} : \mathbb{S}^{d-1} \to \{1\}$, $\mathbf{1}(\omega) \equiv 1$, and the zero function is denoted $\mathbf{0} : \mathbb{S}^{d-1} \to \{0\}$, $\mathbf{0}(\omega) \equiv 0$. If there is no danger of confusion, we sometimes write $L^2 = L^2(\mathbb{S}^{d-1})$. Since we will mostly be working in dimension $d = 3$, we simplify the forthcoming notation by setting $\Phi_q := \Phi_{3,q}$, $\mathbf{T}_q := \mathbf{T}_{3,q}$, and $\mathrm{d}\sigma := \mathrm{d}\sigma_2$. Finally, if x, y are real numbers, we write $x \lesssim y$ if there exists a finite absolute constant C such that $|x| \leq C|y|$.

1.4 Outline

We organize the exposition in five steps, each of them bringing in tools from the calculus of variations (§2), symmetrization techniques (§3), operator theory (§4), Lie theory (§5), and probability theory (§6). These ingredients are then combined in §7, yielding a short proof of Theorem 1. In §8, we discuss some extensions, generalizations, and open problems.

2 Step 1: Calculus of variations

Let f be a maximizer[2] for Φ_6, and normalize it so that $\|f\|_{L^2} = 1$. Recall the operators $\mathcal{E}, \mathcal{E}^*$ which were defined immediately after (2). The following

[2]The existence of maximizers for Φ_6 follows from [5, Theorem 1.1].

chain of inequalities holds:

$$\begin{aligned}
\|\mathcal{E}\|_{L^2\to L^6}^6 &= \|\mathcal{E}(f)\|_{L^6(\mathbb{R}^3)}^6 = \langle|\mathcal{E}(f)|^4\mathcal{E}(f),\mathcal{E}(f)\rangle = \langle\mathcal{E}^*(|\mathcal{E}(f)|^4\mathcal{E}(f)),f\rangle_{L^2(\mathbb{S}^2)}\\
&\leq \|\mathcal{E}^*(|\mathcal{E}(f)|^4\mathcal{E}(f))\|_{L^2(\mathbb{S}^2)} \leq \|\mathcal{E}^*\|_{L^{6/5}\to L^2}\||\mathcal{E}(f)|^4\mathcal{E}(f)\|_{L^{6/5}(\mathbb{R}^3)}\\
&= \|\mathcal{E}^*\|_{L^{6/5}\to L^2}\|\mathcal{E}(f)\|_{L^6(\mathbb{R}^3)}^5 = \|\mathcal{E}\|_{L^2\to L^6}^6, \qquad (5)
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the $L^{6'} - L^6$ pairing in $\mathbb{R}^3$, and $\langle\cdot,\cdot\rangle_{L^2(\mathbb{S}^2)}$ denotes the L^2 pairing on $\mathbb{S}^2$. The only steps which are not entirely trivial amount to an application of the Cauchy–Schwarz inequality, and the fact that the operators norms $\|\mathcal{E}\|_{L^2\to L^6} = \|\mathcal{E}^*\|_{L^{6/5}\to L^2}$ coincide.[3] Since the first and the last terms in the chain of inequalities (5) are the same, all inequalities have to be equalities. In particular, equality holds in the application of the Cauchy–Schwarz inequality, which forces the two functions in question to be constant multiples of each other. In other words, $\mathcal{E}^*(|\mathcal{E}(f)|^4\mathcal{E}(f)) = \lambda f$, for some $\lambda\in\mathbb{C}$. Recalling the definition of the extension and restriction operators, this boils down to

$$(|\widehat{f\sigma}|^4\widehat{f\sigma})^\vee|_{\mathbb{S}^2} = \lambda f. \qquad (6)$$

By Plancherel's identity, the latter equality can be written in convolution form,

$$(f\sigma * f_\star\sigma * f\sigma * f_\star\sigma * f\sigma)|_{\mathbb{S}^2} = (2\pi)^{-3}\lambda f. \qquad (7)$$

Here, $f_\star = \bar{f}(-\cdot)$ denotes the *conjugate reflection* of f, and accounts for the complex conjugates that appear on the left-hand side of (6). Identity (7) is the *Euler–Lagrange equation* associated to the functional Φ_6, and any nonzero, square integrable solution of (7) is called a *critical point* of Φ_6.

The Euler–Lagrange equation (7) can be used to show that any maximizer of (4), and more generally any critical point of Φ_6, is an infinitely differentiable function. This is a manifestation of the general phenomenon that convolution operators are smoothing, but the actual proof entails a number of technical difficulties. We omit the details, and encourage the interested reader to take a look at [15].

3 Step 2: Symmetrization

This step is more elementary than the previous one, but plays an equally important part in the analysis. Inequality (4) can be equivalently rewritten

[3] If $T: L^p\to L^q$ is a bounded linear operator, then its adjoint T^* defines a bounded linear operator from $L^{q'}$ to $L^{p'}$, with the same operator norm. Also, $6' = \frac{6}{5}$.

in convolution form as

$$\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq (2\pi)^{-3/2}\mathbf{T}_6^3\|f\|_{L^2(\mathbb{S}^2)}^3.$$

Since $|f\sigma * f\sigma * f\sigma| \leq |f|\sigma * |f|\sigma * |f|\sigma$ holds pointwise, it follows that[4]

$$\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq \||f|\sigma * |f|\sigma * |f|\sigma\|_{L^2(\mathbb{R}^3)}. \tag{8}$$

Further define the *antipodally symmetric rearrangement* $f_\sharp$ of f via

$$f_\sharp := \sqrt{\frac{|f|^2 + |f_\star|^2}{2}},$$

where $f_\star$ denotes the conjugate reflection of f as above. Note that the L^2-norms of $f_\sharp$ and f (or $f_\star$) coincide. A straightforward application of the elementary inequality between the arithmetic and geometric means reveals that

$$\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq \|f_\sharp\sigma * f_\sharp\sigma * f_\sharp\sigma\|_{L^2(\mathbb{R}^3)},$$

with equality if and only if $f = f_\star = f_\sharp$. These considerations imply that, in the search for maximizers of Φ_6, we may limit our attention to non-negative, antipodally symmetric functions. In other words,

$$\mathbf{T}_6^6 = \max_{0 \leq f = f_\star \in L^2 \setminus \{\mathbf{0}\}} \Phi_6(f). \tag{9}$$

This is a key simplification which enables several of the subsequent steps to work.

4 Step 3: Operator theory

In this section, we explore some of the compactness inherent to the problem. Given a nonzero function $f \in L^2(\mathbb{S}^2)$, normalized so that $\|f\|_{L^2} = 1$, consider the integral operator

$$T_f : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2),\ T_f(g) = g * K_f,$$

with convolution kernel given by

$$K_f(\xi) = (|\widehat{f\sigma}|^4)^\vee(\xi) = (2\pi)^3(f\sigma * f_\star\sigma * f\sigma * f_\star\sigma)(\xi). \tag{10}$$

[4] For a characterization of the cases of equality in (8), see [4, Lemma 8].

The relevance of this operator is easy to highlight. In fact, the Euler–Lagrange equation (6) is nothing but the eigenvalue problem for T_f, namely $T_f(f) = \lambda f$. Observe that λ is entirely dictated by f: From $\lambda f = T_f(f)$ one has that $\lambda \int |f|^2 = \int T_f(f)\bar{f} = \int |\widehat{f\sigma}|^6$, whence $\lambda = \Phi_6(f)$ (since $\|f\|_{L^2} = 1$).

We proceed to study T_f from the operator theoretic point of view. First of all, the function K_f from (10) satisfies $K_f(0) = \|\widehat{f\sigma}\|_{L^4}^4$. Moreover, K_f defines a bounded, continuous function on $\mathbb{R}^3$, satisfying $K_f(\xi) = \overline{K_f(-\xi)}$, for all ξ. As a consequence, the operator T_f is self-adjoint, $T_f = T_f^*$, and positive definite: $\langle T_f(g), g\rangle_{L^2} > 0$, for every nonzero $g \in L^2$. The operator T_f is also Hilbert–Schmidt (and therefore compact), since the companion kernel $K_f^\flat$ defined by $K_f^\flat(\omega,\nu) := K_f(\omega-\nu)$ belongs to $L^2(\mathbb{S}^2\times\mathbb{S}^2)$. But more is true: the operator T_f is actually *trace class*. To see this, let $\{\lambda_j\}_{j=0}^\infty \subset (0,\infty)$ denote the eigenvalues of T_f in non-increasing order, counted with multiplicity, with corresponding L^2-normalized eigenfunctions $\{\varphi_j\}_{j=0}^\infty$. By the classical theorem of Mercer (see e.g. [22, §VI.4]),

$$K_f^\flat(\omega,\nu) = \sum_{j=0}^\infty \lambda_j \varphi_j(\omega)\overline{\varphi_j(\nu)},$$

where the series converges absolutely and uniformly. The trace of T_f can be then estimated as follows:

$$\begin{aligned}\operatorname{tr}(T_f) = \sum_{j=0}^\infty \langle T_f(\varphi_j), \varphi_j\rangle_{L^2(\mathbb{S}^2)} = \sum_{j=0}^\infty \lambda_j = \int_{\mathbb{S}^2} K_f^\flat(\omega,\omega)\, \mathrm{d}\sigma(\omega) \\ = 4\pi K_f(0) = 4\pi \|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)}^4 \lesssim \|f\|_{L^2(\mathbb{S}^2)}^4 < \infty,\end{aligned} \tag{11}$$

where in the last line we invoked the endpoint Tomas–Stein inequality (3).

5 Step 4: Lie theory

It is natural to expect the symmetries of the sphere to enter the picture at some point. The symmetry group of $\mathbb{S}^2$, including reflections, is the orthogonal group, O(3). The subgroup of rotations, i.e. orthogonal 3×3 matrices with unit determinant, is the so-called *special orthogonal group*, SO(3). As a Lie group, SO(3) is compact, connected, and of dimension 3. Its Lie algebra, $\mathfrak{so}(3)$, consists of skew-symmetric 3×3 matrices with real entries. The exponential map, $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3), A \mapsto \exp(A)$, is surjective onto SO(3). For more information on the Lie group SO(3) and its Lie algebra, see [11].

Given a matrix $A \in \mathfrak{so}(3)$, define the vector field ∂_A acting on sufficiently smooth functions $f : \mathbb{S}^2 \to \mathbb{C}$ via $\partial_A f := \lim_{t\to 0} t^{-1}(f(\exp(tA)\cdot) - f)$. The functional Φ_6 enjoys the following symmetries:

$$\Phi_6(f \circ \exp(tA)) = \Phi_6(f) = \Phi_6(e_\xi f),$$

for all $t \in \mathbb{R}$, $A \in \mathfrak{so}(3)$, and $\xi \in \mathbb{R}^3$, where e_ξ stands for the character $e_\xi(\omega) = e^{i\xi\cdot\omega}$. These symmetries naturally give rise to new eigenfunctions for the operator T_f considered in §4, as the following result indicates. We write $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$, and by $\omega_j f$ we mean the function defined via $(\omega_j f)(\omega) = \omega_j f(\omega)$.

Lemma 1 *Let $f : \mathbb{S}^2 \to \mathbb{R}$ be non-constant, continuously differentiable, antipodally symmetric, and such that $\|f\|_{L^2} = 1$. Assume $T_f(f) = \lambda f$. Then:*

$$T_f(\omega_j f) = \tfrac{\lambda}{5}\, \omega_j f, \quad \textit{for every } j \in \{1, 2, 3\}, \tag{12}$$

$$T_f(\partial_A f) = \tfrac{\lambda}{5}\, \partial_A f, \quad \textit{for every } A \in \mathfrak{so}(3). \tag{13}$$

Moreover, there exist $A, B \in \mathfrak{so}(3)$, such that the set $\{\omega_1 f, \omega_2 f, \omega_3 f, \partial_A f, \partial_B f\}$ is linearly independent over $\mathbb{C}$.

Sketch of proof. We omit the derivation of the identities (12), (13), and instead refer the reader to the proof of [16, Prop. 5.2]. The functions $\omega_1 f, \omega_2 f, \omega_3 f$ are linearly independent[5] – this is elementary.

Since $f \in C^1(\mathbb{S}^2)$ is non-constant, there exist $A, B \in \mathfrak{so}(3)$, such that $\partial_A f, \partial_B f$ are linearly independent. To see why this is necessarily the case, consider the linear map $D : \mathfrak{so}(3) \to C^0(\mathbb{S}^2)$, $D(A) = \partial_A f$. Let $r := \dim\ker D$. By the Rank-Nullity Theorem, the image of D has dimension $\dim\mathfrak{so}(3) - \dim\ker D = 3 - r$, and so it suffices to show that $r \leq 1$. Aiming at a contradiction, suppose that $r \geq 2$. In this case, there exist linearly independent matrices $X, Y \in \mathfrak{so}(3)$, such that $\partial_X f = \partial_Y f \equiv 0$. The matrices X, Y correspond to infinitesimal rotations around certain unit vectors $\omega, \nu \in \mathbb{S}^2$, respectively. Since X, Y are linearly independent, then so are ω, ν. But $\partial_X f \equiv 0$ implies that f is constant along all ω-*latitudes*, i.e. circles determined by intersecting $\mathbb{S}^2$ with the translates of a 2-plane orthogonal to ω. In a similar way, f is constant along all ν-*latitudes*. Since ω, ν are linearly independent, it follows that any two points on $\mathbb{S}^2$ can be joined by a path consisting of the alternating concatenation of a certain (finite)

[5] Since f is real-valued, linear independence of the set $\{\omega_1 f, \omega_2 f, \omega_3 f, \partial_A f, \partial_B f\}$ over $\mathbb{C}$ is equivalent to that over $\mathbb{R}$.

number of ω-latitudes and ν-latitudes. Since f is constant along each such latitude, it is constant along the whole path. It follows that f is identically constant, which is absurd.

An alternative approach, which is perhaps less intuitive but has the advantage of generalizing to higher $d \geq 3$, uses the fact[6] that the dimension of a proper, nontrivial subalgebra of $\mathfrak{so}(3)$ is equal to 1 (think of the embedding $\mathfrak{so}(2) \subseteq \mathfrak{so}(3)$). As a consequence, if $r \geq 2$, then the Lie algebra generated by $\ker D$ equals the whole of $\mathfrak{so}(3)$. In turn, this together with the fact that the action of $\mathrm{SO}(3)$ on $\mathbb{S}^2$ is transitive, can be used to show that f is constant, which again yields the desired contradiction.

Finally, the linear independence of the set $\{\omega_1 f, \omega_2 f, \omega_3 f, \partial_A f, \partial_B f\}$ follows from the facts that $\omega_1 f, \omega_2 f, \omega_3 f$ are real-valued, antipodally anti-symmetric functions, whereas $\partial_A f, \partial_B f$ are real-valued, antipodally symmetric functions. $\square$

The conclusion is that, given a sufficiently smooth, non-constant eigenfunction $f = f_\star$ of T_f with eigenvalue λ, we can always find *five* further eigenfunctions of T_f, each with eigenvalue $\frac{\lambda}{5}$, and with the crucial property that the set $\{\omega_1 f, \omega_2 f, \omega_3 f, \partial_A f, \partial_B f\}$ is linearly independent over $\mathbb{C}$.

6 Step 5: Probability theory

Consider three independent, identically distributed random variables X_1, X_2, X_3, taking values on $\mathbb{S}^2$ with uniform distribution. In this case, the random variable $Y_3 = X_1 + X_2 + X_3$ corresponds to the so-called *uniform 3-step random walk* in $\mathbb{R}^3$, and is distributed according to the 3-fold convolution of the normalized surface measure on $\mathbb{S}^2$. In other words, if $\bar{\sigma} := \sigma(\mathbb{S}^2)^{-1}\sigma$ and $\Omega \subseteq \mathbb{R}^3$ is a Borel subset, then

$$\mathbb{P}(Y_3 \in \Omega) = \int_\Omega (\bar{\sigma} * \bar{\sigma} * \bar{\sigma})(\xi)\,\mathrm{d}\xi.$$

Let p_3 denote the probability density associated to the random variable $|Y_3|$. For any measurable subset $E \subseteq (0, \infty)$, we then have that

$$\mathbb{P}(|Y_3| \in E) = \int_E p_3(r)\,\mathrm{d}r.$$

A straightforward computation in spherical coordinates further reveals that $(\sigma * \sigma * \sigma)(r) = \sigma(\mathbb{S}^2)^2 p_3(r) r^{-2}$. Similar considerations apply to the simpler

[6] Incidentally, this provides an answer to Question 3 when $d = 3$.

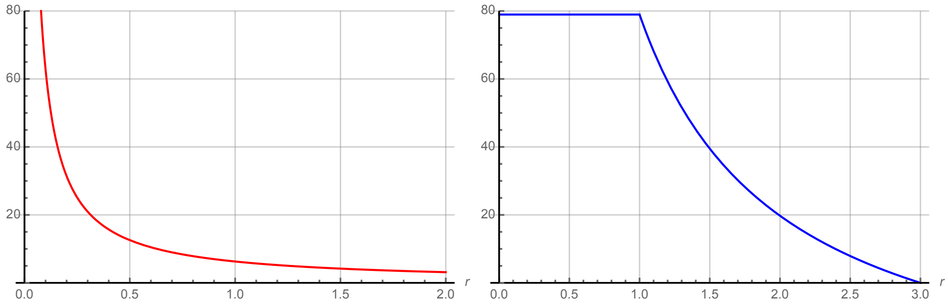

Figura 1: Left: Plot of the function $r \mapsto (\sigma * \sigma)(r)$ for $0 \leq r \leq 2$. Pairs of antipodal points on $\mathbb{S}^2$ contribute towards the singularity at $r = 0$. Right: Plot of the function $r \mapsto (\sigma * \sigma * \sigma)(r)$ for $0 \leq r \leq 3$.

uniform 2-step random walk in $\mathbb{R}^3$, $Y_2 = X_1 + X_2$, in which case we let p_2 denote the density of $|Y_2|$.

Random walks have been the subject of active investigation for more than a century, and as such it comes as no surprise that explicit formulae for p_2, p_3 are well-known, thereby providing an answer to Question 2 when $d = 3$ and $n \in \{2, 3\}$; see [3, 9]. They translate into the following result for convolutions; see also Figure 1.

Lemma 2 *The following identities hold:*

$$(\sigma * \sigma)(\xi) = \frac{2\pi}{|\xi|}, \quad \text{if } |\xi| \leq 2, \tag{14}$$

$$(\sigma * \sigma * \sigma)(\xi) = \begin{cases} 8\pi^2, & \text{if } |\xi| \leq 1, \\ 4\pi^2\Big(-1 + \dfrac{3}{|\xi|}\Big), & \text{if } 1 \leq |\xi| \leq 3. \end{cases} \tag{15}$$

As an immediate consequence of Lemma 2, we may compute the quantities

$$\Phi_4(\mathbf{1}) = (2\pi)^3 \|\mathbf{1}\|_{L^2(\mathbb{S}^2)}^{-4} \|\sigma * \sigma\|_{L^2(\mathbb{R}^3)}^2 = 16\pi^4, \tag{16}$$

$$\Phi_6(\mathbf{1}) = (2\pi)^3 \|\mathbf{1}\|_{L^2(\mathbb{S}^2)}^{-6} \|\sigma * \sigma * \sigma\|_{L^2(\mathbb{R}^3)}^2 = 32\pi^5, \tag{17}$$

which will be of use in the next section.

7 Proof of Theorem 1

Armed with the tools developed in §2–§6, the proof of Theorem 1 is now quite short.

Proof of Theorem 1. It will suffice to prove that any real-valued, continuously differentiable, non-constant critical point f of Φ_6 satisfies $\Phi_6(f) < \Phi_6(\mathbf{1})$. In view of (9), we may further assume that $f = f_\star$, and naturally that $\|f\|_{L^2} = 1$. Multiplying both sides of the Euler–Lagrange equation, $T_f(f) = \lambda f$, by f, and then integrating, one checks as in §4 that $\lambda = \Phi_6(f)$. It then follows that

$$\Phi_6(f) = \lambda = \tfrac{1}{2}(\lambda + 5 \times \tfrac{\lambda}{5}) < \frac{1}{2}\sum_{j=0}^{\infty} \lambda_j = \frac{1}{2}\int_{\mathbb{S}^2} K_f^\flat(\omega,\omega)\, \mathrm{d}\sigma(\omega) = 2\pi K_f(0), \tag{18}$$

where the strict inequality is a consequence of Lemma 1, together with the fact that all eigenvalues of T_f are positive. The remaining identities in (18) have already appeared in (11). On the other hand, we have that

$$K_f(0) = \|\widehat{f\sigma}\|^4_{L^4(\mathbb{R}^3)} = \Phi_4(f) \leq \Phi_4(\mathbf{1}), \tag{19}$$

where the last inequality follows from Foschi's result [6], discussed in §1.2.1. From (16), (17), we further have that

$$2\pi\Phi_4(\mathbf{1}) = \Phi_6(\mathbf{1}), \tag{20}$$

and so from (18) and $2\pi\times$ (19), it then follows that $\Phi_6(f) < \Phi_6(\mathbf{1})$. This completes the proof of the theorem. □

8 Extensions, generalizations, and open problems

In the last section, we discuss the extension of Theorem 1 to other exponents $q \geq 6$, its generalization to higher dimensions $d \geq 3$, and the corresponding questions for complex-valued maximizers. We conclude with a list of open problems.

8.1 Other exponents.

We have already hinted at the very special role played by even integers. It is reassuring to observe that all the steps from §2–§6 work, *mutatis mutandis*, whenever $q \geq 6$ is an even integer. In fact, the whole proof strategy can be made to work, for any $q \in \{6, 8, 10, \dots\}$. However, one encounters some difficulties along the way. Perhaps most significantly, the natural substitute of (20) boils down to the inequality

$$\Phi_q(\mathbf{1}) \leq \frac{1}{\sigma(\mathbb{S}^2)}\frac{q+6}{q+1}\Phi_{q+2}(\mathbf{1}), \tag{21}$$

which needs to be checked for each of the relevant values of q. If $q \geq 4$ is an even integer, then

$$\sigma(\mathbb{S}^2)^{q/2}\Phi_q(\mathbf{1}) = (2\pi)^3\|\sigma^{*(q/2)}\|^2_{L^2(\mathbb{R}^3)} = \|\widehat{\sigma}\|^q_{L^q(\mathbb{R}^3)}.$$

The Fourier transform of the surface measure σ on $\mathbb{S}^2$ is given by[7] $\widehat{\sigma}(x) = 4\pi\,\mathrm{sinc}(|x|)$, and so (21) holds if and only if

$$\int_0^\infty |\mathrm{sinc}(r)|^q\, r^2\,\mathrm{d}r \leq \frac{q+6}{q+1}\int_0^\infty |\mathrm{sinc}(r)|^{q+2}\, r^2\,\mathrm{d}r. \tag{22}$$

Three natural paths to tackle inequality (21) present themselves. In fact, one can proceed via:

(a) explicit formulae for uniform random walks;

(b) rigorous numerical integration;

(c) asymptotic analysis of the weighted integrals in (22).

Path (a) is quite elegant, path (b) is very robust, and path (c) gathers elements from both. By construction, paths (a), (b) are able to provide a solution to a finite number of exponents only. On the other hand, path (c) relies on asymptotics, and as such it naturally misses a few initial cases. Therefore each of the paths is useful on its own, and the three of them intertwine nicely together.

The integrals in (22) are related to the *cube slicing problem*, addressed in Question 1. To see why this is the case, consider the unit cube $Q_d := [-\frac{1}{2}, \frac{1}{2}]^d \subset \mathbb{R}^d$, and let $H \subset \mathbb{R}^d$ be a linear subspace of codimension 1. Then the volume of the $(d-1)$-dimensional section $H \cap Q_d$ is at least 1, and at most $\sqrt{2}$. The lower bound is best possible, and attained if and only if H is parallel to a face of Q_d. The upper bound is also best possible, and attained if and only if H contains a $(d-2)$-dimensional face of Q_d. These results were obtained by K. Ball [1], as a consequence of the key inequality

$$\frac{1}{\pi}\int_{-\infty}^{\infty} |\mathrm{sinc}(r)|^p\,\mathrm{d}r \leq \sqrt{\frac{2}{\pi}},$$

which holds for every $p \geq 2$, with equality if and only $p = 2$. Even though many partial results are known, the complete answer to Question 1 for generic values of d, k remains a topic of current research interest; see [13] and the references therein.

[7]The sinc function is defined as $\mathrm{sinc}(r) := \frac{\sin r}{r}$.

8.2 Higher dimensions

The sharp form of inequality (2) for $q = 4$ is unknown if $d \geq 8$. Without this starting point, our bootstrapping approach to proving Theorem 1 seems condemned from the very start. On the other hand, the whole proof strategy can be made to work in dimensions $d \in \{4, 5, 6, 7\}$, but at least three new difficulties arise.

Firstly, the results in §2–§5 can all be adapted to the higher dimensional case, even though the discussion in §5 (in particular, the proof of Lemma 1) requires some care. In fact, a complete answer to Question 3 is known, and reveals a curious difference that occurs in the *four-dimensional* case: The minimal codimension of a proper subalgebra of $\mathfrak{so}(d)$ equals $d - 1$ if $d \geq 3, d \neq 4$, but equals 2 if $d = 4$; see [12]. This stems from the fact that the group $\mathrm{SO}(4)/\{\pm I\}$ is *not* simple, whereas all other groups $\mathrm{SO}(d)$, $d \neq 4$, are simple (after modding out by $\{\pm I\}$ is d if even). In turn, this relates back to the existence of quaternions, and partly accounts for some exotic aspects of the geometry of 4-manifolds.

Secondly, the computations from §6 rely on a solution to Question 2, which for general values of n was obtained recently, but only *under the additional assumption that d is odd*; see [2, 9]. This can be partly explained by the formula which generalizes (14) to all dimensions $d \geq 2$:

$$(\sigma_{d-1} * \sigma_{d-1})(\xi) = \frac{\sigma_{d-2}(\mathbb{S}^{d-2})}{2^{d-3}} \frac{1}{|\xi|} (4 - |\xi|^2)_+^{\frac{d-3}{2}}, \tag{23}$$

together with the realization that the right-hand side of (23) defines a polynomial expression in the variables $|\xi|, |\xi|^{-1}$ if and only if d is odd. For a generalization of (15) to dimensions $d \in \{3, 5, 7, 9\}$, see Figure 2. To the best of our knowledge, a complete answer to Question 2 in even dimensions remains a fascinating, largely open problem, which via the theory of hypergeometric functions and modular forms exhibits some deep connections to number theory; see [3] and the references therein.

Thirdly, the higher-dimensional generalization of (21) boils down to the inequality[8]

$$\Phi_{d,q}(\mathbf{1}) \leq \frac{1}{\sigma_{d-1}(\mathbb{S}^{d-1})} \frac{q + 2d - \delta_{d,4}}{q+1} \Phi_{d,q+2}(\mathbf{1}), \tag{24}$$

which needs to be checked for each of the relevant values of d, q. An explicit formula for the Fourier transform $\widehat{\sigma}_{d-1}$ is known in all dimensions $d \geq 2$, but

[8] The Kronecker delta satisfies $\delta_{d,4} = 1$ if $d = 4$, and $\delta_{d,4} = 0$ if $d \neq 4$. The introduction of $\delta_{d,4}$ is justified by the distinct behaviour of $\mathfrak{so}(4)$ discussed above.

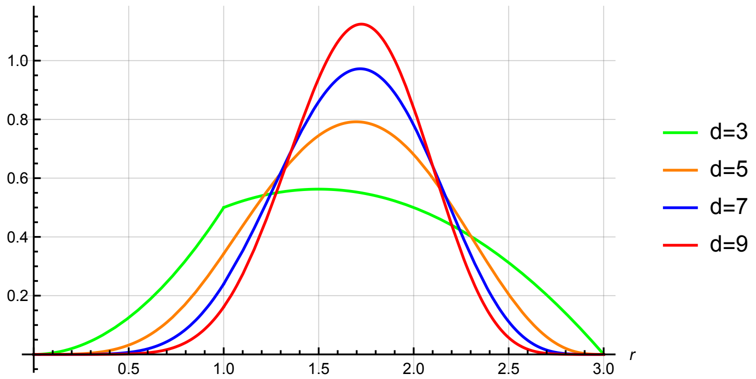

Figura 2: Plot of the function $r \mapsto r^{d-1}(\sigma_{d-1} * \sigma_{d-1} * \sigma_{d-1})(r)$ for $0 \leq r \leq 3$, when $d \in \{3, 5, 7, 9\}$. Multiplication by r^{d-1} distorts the picture but clarifies the behavior, because the surface area of a sphere of radius r in $\mathbb{R}^d$ is proportional to r^{d-1}; in particular, one-dimensional integrals of the plotted function are proportional to integrals of the function in $\mathbb{R}^d$.

it involves the Bessel function $J_{(d-2)/2}$, which is *not* an elementary function whenever d is even; see [17, Ch. VIII, §3]. In fact, setting $\nu = (d-2)/2$, we have that

$$\widehat{\sigma}_{d-1}(x) = (2\pi)^{\frac{d}{2}} |x|^{-\nu} J_\nu(|x|),$$

and consequently (24) holds if and only if

$$\int_0^\infty |J_\nu(r)|^q \, r^{d-1-q\nu} \, \mathrm{d}r \leq \frac{((\frac{d}{2})!)^2}{2^{2-d}} \frac{q+2d-\delta_{d,4}}{q+1} \int_0^\infty |J_\nu(r)|^{q+2} \, r^{d-1-(q+2)\nu} \, \mathrm{d}r. \tag{25}$$

A careful combination of the paths (a), (b), (c) outlined in §8.1 above can be used to verify inequality (25), and therefore (24), in the appropriate range of exponents and dimensions. Details can be consulted in [16, §7].

8.3 $\mathbb{C}$-valued maximizers

It is natural to ask about general complex-valued maximizers of $\Phi_{d,q}$, for $d \geq 2$ and even $q \geq 2\frac{d+1}{d-1}$. In [16, Theorem 1.2], we show that in this case any $\mathbb{C}$-valued maximizer of $\Phi_{d,q}$ is of the form $ce^{i\xi\cdot\omega}F(\omega)$, for some $\xi \in \mathbb{R}^d$, some $c \in \mathbb{C} \setminus \{0\}$, and some nonnegative, antipodally symmetric maximizer F of $\Phi_{d,q}$. Given the discussion in §8.1 and §8.2, all $\mathbb{C}$-valued maximizers of $\Phi_{d,q}$ are then given by $ce^{i\xi\cdot\omega}$, for some $\xi \in \mathbb{R}^d$ and $c \in \mathbb{C} \setminus \{0\}$, provided $d \in \{3, 4, 5, 6, 7\}$ and $q \geq 4$ is an even integer.

8.4 Open problems

We collect some of the outstanding problems which have been mentioned throughout the present note, and add a few others to the list.

1. Do constant functions maximize $\Phi_{2,6}$? If this is indeed the case, then [16, Theorem 1.1] implies that constant functions maximize $\Phi_{2,q}$ as well, for every even integer $q \geq 6$.

2. Are non-zero solutions of the Euler–Lagrange equation which generalizes (6) to arbitrary dimensions $d \geq 2$ and exponents $q \geq 2\frac{d+1}{d-1}$,

$$(|\widehat{f\sigma}_{d-1}|^{q-2}\widehat{f\sigma}_{d-1})^{\vee}|_{\mathbb{S}^{d-1}} = \lambda f,$$

   necessarily C^∞-smooth even when q is *not* an even integer?

3. Do maximizers of $\Phi_{d,q}$ exist at the endpoint $q = 2\frac{d+1}{d-1}$ if $d \geq 4$? See [8, Theorem 1.1] for a conditional result along these lines.

4. Assuming the answer to the question in (3) to be affirmative, do constant functions maximize $\Phi_{d,q}$ if $q = 2\frac{d+1}{d-1}$, in all dimensions $d \geq 4$? Conversely, are all real-valued maximizers of $\Phi_{d,2\frac{d+1}{d-1}}$ constant?

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