Is there switching without suspended horseshoes?

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Resumo: Comutação em redes heteroclínicas está associada à existência de ferraduras suspensas. Trajetórias que exibem comutação encontram-se dentro destes conjuntos transitivos. Revisitando o cenário de Shilnikov-Holmes, nesta Nota, descrevemos uma classe redes atractoras que exibem comutação heteroclínica, com sensibilidade relativamente às condições iniciais e sem ferraduras suspensas na sua vizinhança.

Abstract: Infinite switching behaviour near networks is associated with the existence of suspended horseshoes. Trajectories that realize switching lie within these transitive sets. Revisiting the Shilnikov-Holmes scenario, in this Note, we describe a class of attracting networks exhibiting forward switching and sensitive dependence on initial conditions, without suspended horseshoes in its neighbourhood.

palavras-chave: Rede Heteroclínica, Comutação, Sombreamento, Conjunto Atractor.

keywords: Heteroclinic network, switching, shadowing, attracting set.

1 Introduction

In dynamical systems, a heteroclinic cycle is a set of finitely many invariant saddles and trajectories connecting them. A connected union of finitely many heteroclinic cycles is a heteroclinic network. These objects are associated with intermittent dynamics and used to model stop-and-go behaviour in various applications, including neuroscience, geophysics, game theory and

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populations dynamics. The study of homo/heteroclinic cycles and networks is well-established as an interesting subject in the dynamical systems community.

Complex behaviour near a homo/heteroclinic network is often connected to the occurence of *switching*. There are different types of switching, leading to increasingly complex behaviour near the network:

- *switching at a node* [1] characterised by existence of initial conditions, near an incoming connection to that node, whose trajectory follows any of the possible outgoing connections at the same node. Incoming connection does not predetermine the outgoing choice at the node.
- *switching along a heteroclinic connection* [2, 7], which extends the notion of switching at a node to initial conditions whose trajectories follow a prescribed homo/heteroclinic connection.
- *infinite switching* [3, 14, 12, 19], which ensures that any sequence of connections in the network is a possible path near the network. This is different from *random switching* in which trajectories shadow the network in a non-controllable way [17].

The absence of *switching along a connection* prevents *infinite switching* and, therefore, chaotic behaviour near the network. The term *switching* has also been used to describe simpler dynamics where there is one change in the choices observed in trajectories. This is the case described in [13]. In this case, the network consists of two cycles and trajectories are allowed to change from a neighbourhood of one cycle to a neighbourhood of the other cycle. This change is referred to as switching, although it is a very weak example of this phenomenon. In [6], the expression *railroad switching* is used in relation to switching at a node. Complex behaviour near a network can also arise from the presence of noise-induced switching, see [4]. We do not address the presence of noise in this article.

The authors of [17] find a form of complicated switching (possibly not infinite) leading to regular and irregular cycling near a network. There are several examples in the literature where the existence of infinite switching leads to chaotic behaviour near the network, see [3, 5, 15, 18, 20]. All the networks considered by these authors have at least one invariant saddle at which the linearized vector field has non-real eigenvalues. Infinite switching seems to be related with the existence of non-uniformly hyperbolic suspended horseshoes in its neighbourhood. See for example the works [12, 15, 21]. In these articles, the authors proved the existence of infinitely many initial conditions that realize a given forward infinite path. The associated solutions lie on the sequence of suspended horseshoes that accumulate on the network. The natural question is:

(Q1) are there homo/heteroclinic networks exhibiting infinite switching and without suspended horseshoes around it?

The main goal of this Note is to answer this question. We will exhibit a class of vector fields whose flow has an attracting network (in the sense of Lyapunov) exhibiting infinite switching, high sensitivity with respect to initial conditions and without suspended horseshoes around it. Switching behaviour is due to the combination of the strong spiralling distortion of the two-dimensional invariant manifold around the cycle with the existence of at least two outcoming directions.

The example is based on the most famous and rich examples in the dynamical systems theory: the Shilnikov model of a homoclinic cycle to a saddle-focus with negative saddle value [23, 24, 28]. This particular case has also been studied by Holmes [10], this is why we call *Shilnikov-Holmes* scenario.

2 Preliminaries

Let M be a compact three-dimensional manifold possibly without boundary and let $\mathcal{X}^r(M)$ be the Banach space of C^r vector fields on M endowed with the C^r Whitney topology with $r \geq 2$. Consider a vector field $f : M \to TM$ defining a system

$$\dot{x} = f(x), \qquad x(0) = x_0 \in M \tag{1}$$

and denote by $\varphi(t, x_0)$, with $t \in \mathbf{R}$, the associated flow (with initial condition x_0).

2.1 Heteroclinic Cycle and Network

Throughout the present article, we use the following definitions:

Definition 1 *Let $n \in \mathbf{N}$. A* heteroclinic cycle *is a finite collection of equilibria $\{O_1, \ldots, O_n\}$ of (1) together with a set of trajectories $\{\gamma_1, \ldots, \gamma_n\}$ where γ_j is a solution of (1) such that for $j \in \{1, \ldots, n\}$ we have:*

$$\lim_{t \to -\infty} \gamma_j = O_j \qquad \textit{and} \qquad \lim_{t \to +\infty} \gamma_j = O_{j+1} \tag{2}$$

and $O_{n+1} \equiv O_1$. When $n = 1$, γ_1 will be called a homoclinic connection *and the set $\{O_1, \gamma_1\}$ is a* homoclinic cycle *associated to O.*

A homoclinic cycle associated to O is the union of the equilibrium O and a trajectory biasymptotic to O in forward and backward times. In this paper, we will be focused on a type of equilibrium of (1) such that its spectrum (*i.e.* the eigenvalues of *df* computed at the equlibrium) consists of one pair of non-real complex numbers with negative real part and one positive real eigenvalue. This is what we call a *saddle-focus*.

Definition 2 (Field [8]) *Let $n \in \mathbf{N}$ and $\mathcal{A} = \{O_i, 1 \leq i \leq n\}$ be a finite ordered set of saddle-foci. We say that a flow-invariant set $\Gamma \subset M$ is a* heteroclinic network *associated to $\mathcal{A}$ if there is a finite set of ordered nonempty sets $\mathcal{A}_j$ of $\mathcal{A}$ such that:*

1. *each subset $\mathcal{A}_j$ determines a heteroclinic cycle Γ_j and $\Gamma = \bigcup_j \Gamma_j$;*
2. *for $1 \leq l, p \leq n$, if $W^u(O_l) \cap W^s(O_p) \neq \emptyset$, then there exists $1 \leq j \leq n$ such that $W^u(O_l) \cap W^s(O_p) \subset \Gamma_j$;*
3. *$\bigcup_j \mathcal{A}_j = \mathcal{A}$.*

Roughly speaking, a *heteroclinic network* associated to $\mathcal{A}$ consists of the set of equilibria and a finite union of heteroclinic connections associated to the saddles in $\mathcal{A}$. A homoclinic *figure eight* is the union of an equilibrium and two homoclinic connections associated to it.

2.2 Switching

Let Γ be a heteroclinic network associated to $\mathcal{A} = \{O_1, \dots, O_n\}$, a set of n saddle-foci.

Definition 3 *If $k \in \mathbf{N}$, a finite path of order k on Γ is a sequence $(\gamma_1, \dots, \gamma_k)$ of k heteroclinic connections in Γ such that $\gamma_j \subset W^u(O_{j \pmod n}) \cap W^s(O_{j+1 \pmod n})$ for all $j \in \{1, \dots, k\}$. We use the notation σ^k for this type of finite path. For an infinite path, take $k \in \mathbf{N}$.*

Let N_Γ be a neighbourhood of the network Γ and let $V_{O_j} \subset N_\Gamma$ be a neighbourhood of O_j, $j \in \{1, \dots, n\}$. For each heteroclinic connection γ_i in Γ, consider a point $p_i \in \gamma_i$ and a neighbourhood $V_i \subset N_\Gamma$ of p_i. The collection of these neighbourhoods should be pairwise disjoint.

Definition 4 *Given neighbourhoods as above, we say that the trajectory of a point q follows a finite path σ^k, if there exist two monotonically increasing sequences of times $(t_j)_{j \in \{1,\dots,k+1\}}$ and $(z_j)_{j \in \{1,\dots,k\}}$ such that for all $j \in \{1, \dots, k\}$, we have $t_j < z_j < t_{j+1}$ and:*

(i) $\varphi(t,q) \subset N_\Gamma$ *for all* $t \in]t_1, t_{k+1}[$*;*

(ii) $\varphi(t_j,q) \in V_{O_j}$ *for all* $j \in \{1,\ldots,k+1\}$ *and* $\varphi(z_j,q) \in V_{\sigma(j)}$ *for all* $j \in \{1,\ldots,k\}$*;*

(iii) for all $j = 1,\ldots,k-1$ *there exists a proper subinterval* $I \subset]z_j, z_{j+1}[$ *such that, given* $t \in]z_j, z_{j+1}[$*,* $\varphi(t,q) \in V_{O_j}$ *if and only if* $t \in I$*.*

The notion of a trajectory following an infinite path can be stated similarly.

Along the paper, when we refer to points that follow a path, we mean that their trajectories do it. Based in [3, §2], we define:

Definition 5 *There is:*

(i) finite switching *near* Γ *if for each finite path and for each neighbourhood* N_Γ *there is a trajectory in* N_Γ *that follows it and*

(ii) infinite switching *(or simply* switching*) near* Γ *by requiring that for each infinite path and for each neighbourhood* N_Γ *there is a trajectory in* N_Γ *that follows it.*

An infinite path on Γ can be considered as a pseudo-orbit of (1) with infinitely many discontinuities. Switching near Γ means that any pseudo-orbit in Γ can be realized. In [14], using connectivity matrices, the authors gave an equivalent definition of switching, emphasising the possibility of coding all trajectories that remain in a given neighbourhood of the network in both finite and infinite times.

3 The main results

Our object of study is the dynamics around a special type of networks, for which we give a rigorous description here.

3.1 The first result

We consider a family of vector fields in $\mathcal{X}^r(M)$, $r \geq 2$, with a flow given by the unique solution $\varphi(t,x) \in M$ of (1) satisfying the following hypotheses:

(H1) The point O is a saddle-focus where the eigenvalues of $\mathrm{d}f|_O$ are $-C \pm \alpha i$ and E, where $C, E, \alpha \in \mathbf{R}^+$.

(H2) There are two trajectories γ_1 and γ_2 biasymptotic to O.

(H3) $C > E$.

Hypotheses **(H1)** and **(H2)** imply that $\Gamma = \{O\} \cup \gamma_1 \cup \gamma_2$ is a homoclinic *figure eight* (see Figure 1(a)). Our main result says that although Γ is attracting (due to **(H3)**), the approach to the network is chaotic.

Theorem 1 *For a vector field $f : M \to TM$ whose flow satisfies **(H1)**–**(H3)**, the following conditions hold:*

(a) *there are no suspended horseshoes in the neighbourhood of Γ;*

(b) *the network Γ is asymptotically stable, in the sense that all trajectories starting in a small open neighbourhood of Γ are attracted to the network;*

(c) *there is infinite switching near Γ, realized by infinitely many initial conditions.*

It is clear that **(b)** implies **(a)**. The proof of **(b)** may be found in [9, 25]. The proof of **(c)** is addressed in §6 of the present article. In particular, in a small neighbourhood of Γ, N_Γ , if g is C^2-close to f, the set of non-wandering trajectories of f in N_Γ consists of O and one or two attracting limit cycles. Using Theorem 1, we conclude that the transient dynamics should visit the ghost of the homoclinic cycles (in any prescribed order) before falling on the basins of attractions of the periodic solutions. Details in [28].

There are several papers in the literature considering the case where the inequality **(H3)** fails, all of them dealing with the complexity of solutions in a neighbourhood of Γ, namely a sequence of suspended non-uniformly horseshoes accumulating on the network. Finitely many of these horseshoes survive under the addition of generic perturbing terms. The natural context to construct examples where hypotheses **(H1)**–**(H2)** hold is in the symmetric setting. We address the reader to [20, 29], where the authors claim the existence of a countable set of parameter values for which a homoclinic *figure eight* may be observed.

3.2 The second result

In this section, we generalize Theorem 1 by considering networks as those depicted in Figure 1(b). More specifically, we consider a family of vector fields in $\mathcal{X}^r(M)$, $r \geq 2$, with a flow given by the unique solution $\varphi(t, x) \in M$ of (1) satisfying the following hypotheses. For $n \in \mathbf{N}$ and $j \in \{1, \ldots, n\}$:

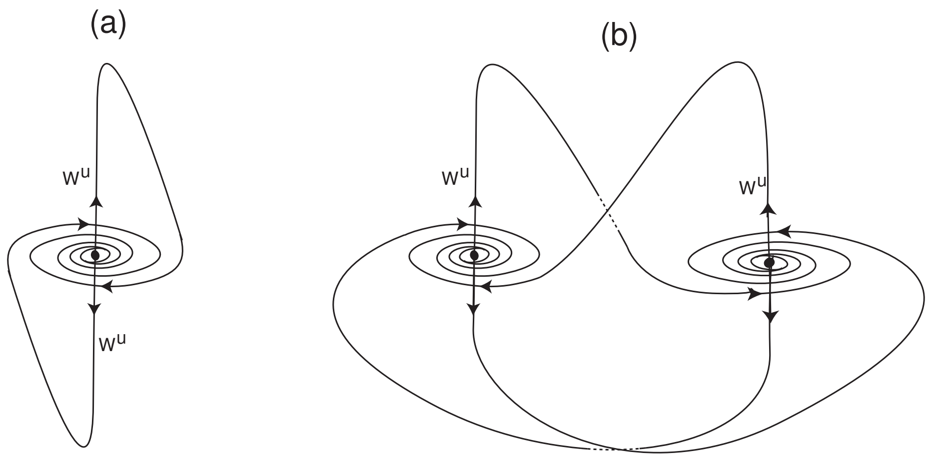

Figure 1: Schemes of the networks. (a) Γ: homoclinic *figure eight*. (b): Γ^g: heteroclinic network $n = 2$.

(G1) The point O_j is a saddle-focus equilibrium where the eigenvalues of $\mathrm{d}f|_{O_j}$ are $-C_j \pm \alpha_j i$ and E_j, where $C_j, E_j, \alpha_j \in \mathbf{R}^+$.

(G2) There are two trajectories γ_1^j and γ_2^j such that

$$\lim_{t\to+\infty} \gamma_1^j = \lim_{t\to+\infty} \gamma_2^j = O_j \quad \text{and} \quad \lim_{t\to-\infty} \gamma_1^j = \lim_{t\to-\infty} \gamma_2^j = O_{(j-1)} \pmod{n}.$$

(G3) $C_j > E_j$.

Hypotheses **(G1)** and **(G2)** imply that $\Gamma^g = \bigcup_j \{O_j\} \cup \bigcup_j [\gamma_1^j \cup \gamma_2^j]$ is a heteroclinic network. Theorem 1 may be generalized in the following way:

Theorem 2 *For a vector field $f : M \to TM$ whose flow satisfies **(G1)**–**(G3)**, the following conditions hold:*

(a) *there are no suspended horseshoes in the neighbourhood of Γ^g;*

(b) *the network Γ^g is asymptotically stable, in the sense that all trajectories starting in a small open neighbourhood of Γ^g are attracted to the network;*

(c) *there is infinite switching near Γ^g, realized by infinitely many initial conditions.*

Theorem 2 is clearly a generalization of Theorem 1 ($n = 1$). An explicit example of a network satisfying **(G1)**–**(G3)** is described by Tigan *et al* [26, 27] in the context of the Lu and T systems, a class of Lorenz systems having the nonlinear terms of order two. Of course the inequality (4) of [27, Th 3.1] cannot be verified; otherwise the flow would have a dense trajectory. The symmetry plays an important goal to ensure the existence of two trajectories between the saddle-foci (see [26, Th. 2.2]). The proof of Theorem 2 runs along the same lines to that of Theorem 1. Switching is related with the sensitive dependence on initial conditions.

Structure of the article:

In §4 we linearize the vector field around the saddle-focus, obtaining an isolating block around it; this section is concerned with introducing the notation for the proof of switching. In §5, we obtain a geometrical description of the way the flow transforms a segment of initial conditions across the stable manifold of O. This curve is wrapped around the isolating block and accumulates on the unstable manifold of O (λ-Lemma for flows) and, in particular, on the next connection. The local stable manifold of O crosses infinitely many times the previous curve. The geometric setting is explored in §6 to obtain intervals of the segment that are mapped by the flow into curves next to O in a position similar to the first one. This allows to establish the recurrence neeeded for infinite switching. For any infinite sequence of homoclinic connections, say:

$$\gamma_2 \to \gamma_1 \to \gamma_1 \to \gamma_1 \to \gamma_2 \to \gamma_2 \to \dots$$

without using perturbation theory, we find infinitely many trajectories that visits the neighbourhoods of these connections in the same sequence. Throughout this note, we have endeavoured to make a self contained exposition bringing together all topics related to the proofs. We have stated short lemmas and we have drawn illustrative figures to make the paper easily readable.

4 Local Maps

The behaviour of the vector field f in the neighbourhood of the network Γ is given, up to topological equivalence, by the linear part of f in the neighbourhood of O and by the transition map between two discs transversal to the flow in those neighbourhoods. In this section, we choose coordinates in the neighbourhood of O in order to put f in the canonical form and we assume that the transition map is linear. The main point is the application of Samovol's Theorem [22] to C^1–linearize the flow around O, and to introduce cylindrical coordinates around the equilibrium. There are no C^1-resonances here. We use neighbourhoods with boundary transverse to the linearized flow.

4.1 C^1–linearization

Since O is hyperbolic, by Samovol's Theorem [22], the vector field f is C^1–conjugate to its linear part in a ε-small open neighbouhood around O, $\varepsilon > 0$. We choose cylindrical coordinates (ρ, θ, z) near O so that the linearized vector field can be written as:

$$\begin{cases} \dot{\rho} = -C\rho \\ \dot{\theta} = \alpha \\ \dot{z} = Ez \end{cases} . \tag{3}$$

After a linear rescaling of the local variables, we consider a cylindrical neighbourhood of O of radius 1 and height 2 that we denote by V – see Figure 2(a). Their boundaries consist of three components: the cylinder wall parametrized by $x \in \mathbf{R} \pmod{2\pi}$ and $|y| \leq 1$ with the usual cover $(x, y) \mapsto (1, x, y) = (\rho, \theta, z)$ and two disks (top and bottom). We take polar coverings of these disks $(r, \phi) \mapsto (r, \phi, j) = (\rho, \theta, z)$ where $j \in \{-1, +1\}$, $0 \leq r \leq 1$ and $\phi \in \mathbf{R} \pmod{2\pi}$. By convention, the intersection points of Γ with the wall of the cylinder has 0 and π angular coordinate. The set of points in the cylinder wall with positive (resp. negative) second coordinate is denoted by Σ^{in}_{+}. (resp Σ^{in}_{-}). The similar holds for Σ^{out}.

As depicted in Figure 2, the cylinder wall of V is denoted by Σ^{in}. Note that $W^s_{loc}(O)$ corresponds to the circle $y = 0$. The top and the bottom of the cylinder are simply denoted by Σ^{out}. The boundary of V can be written as the disjoint union:

$$\partial V = \Sigma^{in} \dot{\cup} \Sigma^{out} \dot{\cup} \Omega,$$

where Ω is the part of ∂V where the flow is not transverse. It follows by the above construction that:

Lemma 3 *Let $j \in \{-, +\}$. Solutions starting:*

1. *at Σ^{in} go inside the cylinder V in positive time;*

2. *at Σ^{out} go outside the cylinder V in positive time;*

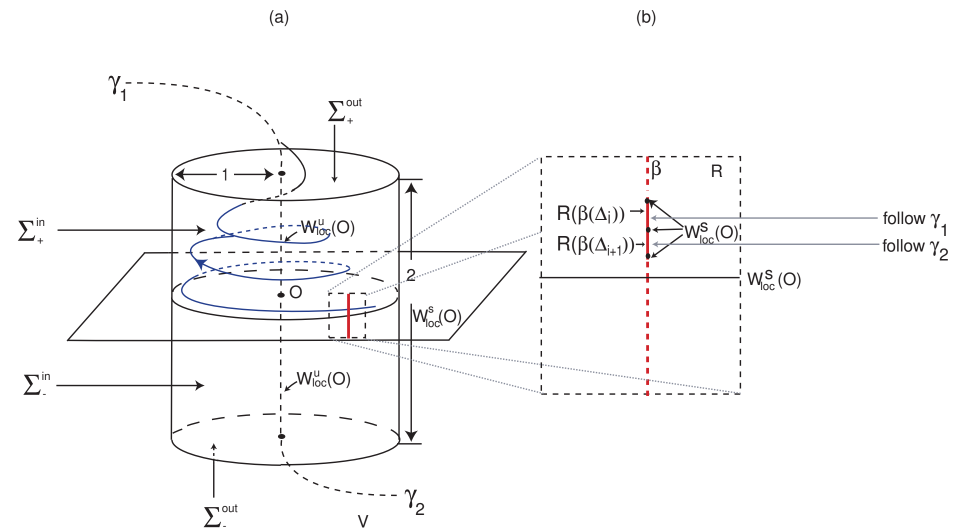

Figure 2: Cylindrical neighbourhood of the saddle-focus O. For $i, j \in \{-,+\}$, on a segment β there are infinitely many subsegments that are mapped by R into Σ_i^{in}, each one containing a point mapped into $W^s_{loc}(O)$. The small sub-segments contain smaller ones that are mapped by R^2 into Σ_j^{in} and the process may be continued forming a nested sequence.

3. at $\Sigma_j^{in}\backslash W^s(O)$ leave the cylindrical neighbourhood V at Σ_j^{out}.

If $(x, y) \in \Sigma^{in}\backslash W^s_{loc}(O)$, let $T(x, y)$ be the time of flight through V of the trajectory whose initial condition is (x, y). It only depends on $y \neq 0$ and is given explicitly by

$$T(x,y) \approx \frac{1}{E} \ln\left(\frac{1}{|y|}\right) = -\frac{1}{E}\ln|y|. \tag{4}$$

In particular $\lim_{y\to 0} T(x, y) = +\infty$. Now, we obtain the expression of the local map that sends points in the boundary where the flow goes in, into points in the boundary where the flows goes out. The local map $\Phi_O : \Sigma^{in} \to \Sigma^{out}$ near O is given by

$$\Phi_O(x,y) = \left(|y|^\delta, -\tfrac{\alpha}{E}\ln|y| + x\right) = (r,\phi) \tag{5}$$

where $\delta = \frac{C}{E} > 1$ is the *saddle index* of O. Observe that if $x_0 \in \mathbf{R}$ is fixed, then

$$\lim_{y\to 0} |\Phi_O(x_0, y)| = (0, +\infty).$$

In [11], the author obtains precise asymptotic expansions for the local map $\Phi_O : \Sigma^{in} \to \Sigma^{out}$. In the present article, we omit high order terms because they are not needed to our purposes.

4.2 Transition and First Return Maps

Let $j \in \{+, -\}$. By the *Tubular Flow Theorem* [16], solutions starting near $\Sigma_j^{out} \cap W^u_{loc}(O)$ follow one of the connections in Γ. We may then define the transition map $\Psi : \Sigma^{out} \to \Sigma^{in}$ by flow box fashion and the return map to Σ^{in} by:

$$R = \Psi \circ \Phi_O : \Sigma^{in} \backslash W^s_{loc}(O) \to \Sigma^{in}.$$

Hereafter, we concentrate our attention on initial conditions that do not escape from N_Γ; otherwise take a smaller subset in Σ^{in} where the returm map is well defined. The explicit expression for the return map is highly nonlinear since the distortion near the hyperbolic saddle-foci is tremendous.

The analytic expression of $R : \Sigma^{in}_+ \to \Sigma^{in}_+$ may be written as

$$(x, y) \mapsto \left(|y|^\delta \cos\left(-\frac{\alpha}{E} \ln |y| + x \right), |y|^\delta \sin\left(-\frac{\alpha}{E} \ln |y| + x \right) \right),$$

implying that the first coordinate of $\Phi_O \circ R^n$ is of order $|y|^{\delta(n+1)}$, $n \in \mathbf{N}$. Using (**H3**), it follows that $\delta > 1$ and then Γ is attracting. In particular, there are no dense solutions near Γ.

5 Local Geometry

The coordinates and notations of §4 will be used to study the geometry of the local dynamics near the saddle-focus. This is the main goal of the present section but first we introduce the concept of a *segment* on Σ^{in}.

Definition 6 *Let $j \in \{-, +\}$. A* segment *β on Σ^{in}_j is a smooth regular parametrized curve $\beta : [0, 1] \to \Sigma^{in}_j$ that meets $W^s_{loc}(O)$ at the point $\beta(0)$ and such that, writing $\beta(s) = (x(s), y(s))$, both x and y are monotonic and bounded functions of s.*

The definition of *segment* may be relaxed: the components do not need to be monotonic for all $s \in [0, 1]$. We use the assumption of monotonicity to simplify the arguments.

Definition 7 *Let $a \in \mathbf{R}$, D be a disc centered at $p \in \mathbf{R}^2$ and ℓ a line passing through p.*

1. *A* spiral *on D around the point p is a smooth curve*

$$\alpha : [a, +\infty[\to D,$$

satisfying $\lim_{s\to+\infty} \alpha(s) = p$ *and such that if* $\alpha(s) = (r(s), \phi(s))$ *is its expression in polar coordinates around p then:*

   (a) the map r is bounded by two monotonically decreasing maps converging to zero as $s \to +\infty$;

   (b) the map ϕ is monotonic for some unbounded subinterval of $[a, +\infty[$ and

   (c) $\lim_{s\to+\infty} |\phi(s)| = +\infty$.

2. *A* double spiral *on D around the point p is the union of two spirals accumulating on p and a curve connecting the other end points.*

3. *Given a* spiral α *on D around the point p, a* half circle *on D bounded by ℓ is a connected component of $\alpha\backslash\ell$.*

The next result characterizes the local dynamics near the saddle-focus.

Lemma 4 *Let $j \in \{+, -\}$. A* segment *β on Σ_j^{in} is mapped by Φ_O into a spiral on Σ_j^{out} accumulating on the point defined by $\Sigma_j^{out} \cap W_{loc}^u(O)$.*

Proof: The proof will be done for $j = +$; the other case is analogous. Let β be a segment on Σ_+^{in}. Write $\beta(s) = (x^\star(s), y^\star(s)) \in \Sigma_+^{in}$, where:

- $s \in [0, 1]$,
- $y^\star$ is an increasing map as function of s and
- $\lim_{s\to 0^+} x^\star(s) = \lim_{s\to 0^+} y^\star(s) = 0$.

The function Φ_O maps the segment $\beta \subset \Sigma_+^{in}$ into the curve defined by:

$$\Phi_O(\beta(s)) = \Phi_O[x^\star(s), y^\star(s)] = \left[|y^\star(s)|^\delta, -\frac{\alpha}{E}\ln|y^\star(s)| + x^\star(s)\right] = (r^\star(s), \phi^\star(s)).$$

The map $\Phi_O \circ \beta$ is a spiral on Σ_+^{out} accumulating on the point defined by $\Sigma_+^{out} \cap W_{loc}^u(O)$ because $r(s)$ and $\phi(s)$ are monotonic (see Remark 1) and

$$\lim_{s\to 0^+} |y^\star(s)|^\delta = 0 \quad \text{and} \quad \lim_{s\to 0^+} \left|-\frac{\alpha}{E}\ln|y^\star(s)| + x^\star(s)\right| = +\infty.$$

$\square$

Remark 1 *Let $j \in \{+,-\}$. The coordinates $(r,\phi) \in \Sigma_j^{out}$ may be chosen so as to make the map $\phi^\star$ increasing or decreasing, according to our convenience. From now on, we omit the dependence of $x^\star, y^\star$ on s to simplify the notation.*

6 Proof of Theorem 1(c)

In this section we put together the information about the return map. First observe that if $A \subset M$, we denote by $\overline{A}$ and $int(A)$ its topological closure and topological interior, respectively. In what follows we remove the point $\beta(0)$ from the $graph(\beta)$ because the return map R is not defined at this point. From now on, let us fix N_Γ, a small neighbourhood of Γ. Figure 2(b) illustrates the main idea of the proof.

6.1 The return map

The next result shows that there are infinitely many points in $graph(\beta) \subset \Sigma_+^{in}$ which are mapped under R into $W^s_{loc}(O)$ and that separate segments of initial conditions that follow the different connections γ_1 and γ_2.

Lemma 5 *Let $\mathcal{R}$ be a rectangle in Σ^{in} centered at one point of $\Gamma \cap \Sigma^{in}$. For any segment $\beta : (0,1] \to \mathcal{R} \cap \Sigma_+^{in}$, there is a family of intervals of the type $\Delta_i = [a_i, a_{i+1}]$ such that for all $i \in \mathbf{N}$, we have:*

1. *$R \circ \beta(a_i) \in W^s_{loc}(O)$;*
2. *$R \circ \beta([a_i, a_{i+1}])$ is one half-circle in Σ_+^{in} bounded by $W^s_{loc}(O)$;*
3. *$R \circ \beta([a_{i+1}, a_{i+2}])$ is one half-circle in Σ_-^{in} bounded by $W^s_{loc}(O)$.*

Proof: We concentrate our attention on initial conditions that do not escape from N_Γ; recall that Γ is an attractor. By Lemma 4, the image of $\beta \subset \mathcal{R}$ under Φ_O is a spiral accumulating on $\Sigma_+^{out} \cap W^u_{loc}(O)$. In its turn, this spiral is mapped by Ψ into another spiral in Σ^{in} accumulating on one point of $\Gamma \cap \Sigma^{in}$. The curve $W^s_{loc}(O)$ cuts transversely this spiral into infinitely many points. Let $s = a_i$ the points for which $R \circ \beta(a_i) \in W^s_{loc}(O)$. By construction, it is easy to see that if $s \in (a_i, a_{i+1})$ either $R \circ \beta(s) \in \Sigma_+^{in}$ or $R \circ \beta(s) \in \Sigma_-^{in}$. Suppose, without loss of generality, that the first case holds. Then, by continuity of R, for $s \in (a_{i+1}, a_{i+2})$ we get $R \circ \beta(s) \in \Sigma_-^{in}$. $\square$

Lemma 6 *Let $\mathcal{R}$ be a rectangle in Σ^{in} centered at one point of $\Gamma \cap \Sigma^{in}$ and let Δ_i be as in Lemma 5. Then for sufficiently large $i \in \mathbf{N}$, there are intervals of the type $\Delta_{i,j} = [b_{i,j}, b_{i,j+1}]$ such that for all $j \in \mathbf{N}$, we have:*

1. $R^2 \circ \beta(b_{i,j}) \in W^s_{loc}(O)$.
2. $[b_{i,j}, b_{i,j+1}] \subset \Delta_i$
3. $R^2 \circ \beta([b_{i,j}, b_{i,j+1}])$ *is one half-circle in* Σ^{in}_+ *bounded by* $W^s_{loc}(O)$;
4. $R^2 \circ \beta([b_{i,j+1}, b_{i,j+2}])$ *is one half-circle in* Σ^{in}_- *bounded by* $W^s_{loc}(O)$.

Proof: Our starting point is the half-circle $R \circ \beta(int(\Delta_i)) \subset \Sigma^{in}_+$ bounded by $W^s_{loc}(O)$, $i \in \mathbf{N}$, whose existence has been proved in Lemma 5. Since each half-circle can be seen as two connected segments, using Lemma 4 the set $\Phi_O \circ R \circ \beta(int(\Delta_i)) \subset \Sigma^{out}_+$ is a double spiral accumulating on $W^u_{loc}(O) \cap \Sigma^{out}_+$, which is mapped under Ψ into a double spiral accumulating on $\Gamma \cap \Sigma^{in}$. The line $W^s_{loc}(O)$ intersects this double spiral infinitely many times. Let $s = \beta_{i,j}$ the sequence of points such that $R^2 \circ \beta(b_{i,j}) \in W^s_{loc}(O)$. For a given $i \in \mathbf{N}$, the arguments used before may be used to conclude that, for each i there exists a sequence $(b_{i,j})_j$ such that $R^2 \circ \beta([b_{i,j}, b_{i,j+1}])$ is one half-circle in Σ^{in}_+ bounded by $W^s_{loc}(O)$ and $R^2 \circ \beta([b_{i,j+1}, b_{i,j+2}])$ is another half-circle in Σ^{in}_- bounded by $W^s_{loc}(O)$. $\square$

6.2 Finite switching

In the previous sections we have proved that any segment cutting transversely the stable manifold of O contains subsegments Δ_i that are mapped into new segments cutting transversely the stable manifold of O. Starting with a segment β on Σ^{in}_+, we may obtain, recursively, nested compact subsets containing initial conditions that follow any prescribed sequence of connections. For $k \in \mathbf{N}$, let $\sigma : \{1, \dots, k\} \to \{1, 2\}$ be an arbitrary map.

Definition 8 *Let $k, l \in \mathbf{N}$. We say that the path $\sigma^k = \left(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(k)}\right)$ of order k on Γ is inside the path $\sigma^{k+l} = \left(\gamma_{\omega(1)}, \dots, \gamma_{\omega(k+l)}\right)$ of order $k+l$ if $\sigma(i) = \omega(i)$ for all $i \in \{1, \dots, k\}$. We denote this relation by $\sigma^k \prec \sigma^{k+l}$.*

Proposition 7 *There is finite switching near the network Γ defined by a vector field satisfying **(H1)**–**(H2)**.*

Proof: Given a path of order $k \in \mathbf{N}$, $\sigma^k = \left(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(k)}\right) \in \{\gamma_1, \gamma_2\}^k$, we want to find trajectories that follow it. Let us fix a segment Δ_i given by Lemma 5 and set that all initial conditions in $\Delta_i \setminus W^s_{loc}(O)$ follow the connection $\gamma_{\sigma(1)}$. Take any closed subset $\mathcal{A}_i$ of Δ_i. By construction, all initial conditions starting in $\mathcal{A}_i$ follow the connection $\gamma_{\sigma(1)}$. The set $R \circ \beta(\Delta_i)$ is one half-circle on Σ^{in}_+ cutting transversely $W^s_{loc}(O) \cap \Sigma^{in}$ infinitely many times. By Lemma 6, one can obtain again sequences of points in $\Delta_{i,j}$, where a similar result to that in Lemma 5 can be stated for R^2 instead of R. Take a closed subset $\mathcal{A}_{i,j}$ of $\Delta_{i,j}$. By construction, all initial conditions starting in $\mathcal{A}_{i,j}$ follow the path $(\gamma_{\sigma(1)}, \gamma_{\sigma(2)})$. A recursive argument allows the construction of a compact set $\mathcal{A}_{\sigma(1),\sigma(2),\ldots,\sigma(k)}$ of initial conditions whose trajectories follow σ^k. $\square$

6.3 Proof of Theorem 1(c)

We first need to introduce some extra terminology. Given a path of order $k \in \mathbf{N}$,

$$\sigma^k = \left(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(k)}\right) \in \{\gamma_1, \gamma_2\}^k,$$

we denote by $\mathcal{Q}(\sigma^k)$ the compact set $\mathcal{A}_{\sigma(1)\sigma(2)\ldots\sigma(k)}$ obtained in the proof of Proposition 7 and we say that $\mathcal{Q}(\sigma^k)$ is an admissible set with respect to σ^k. Recall that all points in $\mathcal{Q}(\sigma^k)$ correspond to infinitely many solutions following σ^k (with positive Lebesgue measure).

Remark 2 *Let $l \in \mathbf{N}$. By the construction in the proof of Proposition 7, if $\sigma^k \prec \sigma^{k+l}$ one can get admissible sets such that $\mathcal{Q}(\sigma^k) \supset \mathcal{Q}(\sigma^{k+l})$.*

Proof: Fix an infinite path $\sigma^\infty = (\gamma_{\sigma(j)})_{j \in \mathbf{N}}$, with $\sigma : \mathbf{N} \to \{1, 2\}$. For each $k \in \mathbf{N}$ define the finite path $\sigma^k = (\gamma_{\sigma(j)})_{j \in \{1,\ldots,k\}}$. Taking into account Remark 2 it follows that there exists an infinite sequence of admissible sets $\{\mathcal{Q}(\sigma^k)\}_{k \in \mathbf{N}}$ such that $\mathcal{Q}(\sigma^k) \supset \mathcal{Q}(\sigma^{k+1})$ for all $k \in \mathbf{N}$. Since the sequence of compact sets $\{\mathcal{Q}(\sigma^k)\}_{k \in \mathbf{N}}$ is nested, it follows that

$$\mathcal{B} = \bigcap_{k=1}^{\infty} \mathcal{Q}(\sigma^k) \neq \emptyset.$$

Any initial condition in $\mathcal{B}$ gives a trajectory which follows σ^∞. The different solutions are distinguished by the number of revolutions around the isolating block of O [21]. By construction we find trajectories realising the required switching arbitrarily close to Γ. $\square$

The stable manifold of O continued along Γ in negative time has a helicoid form. When it crosses the section Σ_+^{out} it has a folded shape. Infinite switching between γ_1 and γ_2 is controlled by the complicated shape of $\Psi^{-1} \circ R^{-n}(W^s_{loc}(O) \cap \Sigma_+^{in})$, $n \in \mathbf{N}$, where well defined. See Figure 3.

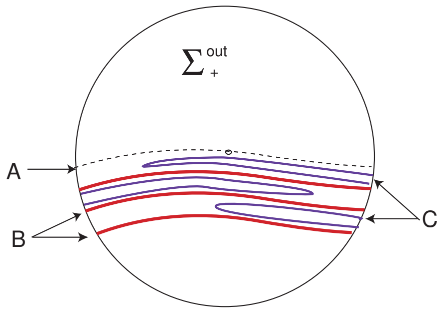

Figure 3: The structure of $W^s(O) \cap \Sigma_+^{out}$ under R^{-n}, $n \in \mathbf{N}$. $A = \Psi^{-1}(W^s_{loc}(O) \cap \Sigma^{in})$. $B = \Psi^{-1} \circ R^{-1}(W^s_{loc}(O) \cap \Sigma_+^{in})$. $C = \Psi^{-1} \circ R^{-2}(W^s_{loc}(O) \cap \Sigma_+^{in})$.

7 Concluding remark

In the literature there are attracting networks Γ^g satisfying hypotheses **(G1)**–**(G3)** with a very complex numerical behaviour. Although the ω-limite set of Γ^g is simple (the statistical limit set is the set of equilibria), the dynamics of the solutions in its neighbourhood seems to be chaotic. Lebesgue almost all solutions seem to exhibit sensitive dependence on initial solutions jumping between the heteroclinic connections, in a non-controllable way. This phenomenon is due to the spiralling of the saddles forced by non-real eigenvalues of df at the equilibria. Hypothesis **(G3)** prevents the emergence of suspended horseshoes near Γ^g.

Generalizing Holmes' work [10], this article contributes for the understanding of a particular type of attracting networks exhibiting complex dynamics in its neighbourhood. This class of examples should serve as a warning to all those doing numerics near heteroclinic networks who deduce the dynamics near a network is chaotic merely because associated time series have no regular patterns.

Going back to the title of this article, based on Theorems 1 and 2, the answer to the question **(Q1)** is *yes*. At this point, a related question arises: *for three-dimensional flows, are there other classes of heteroclinic* *networks (beside those described in Theorem 3.2) where infinite switching holds without suspended horseshoes emerging in its neighbourhood?*

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