# On the area-preserving Domain-Straightening Theorem

*Mário Bessa* and *Pedro Morais*
Universidade da Beira Interior
Rua Marquês d'Ávila e Bolama
6201-001 Covilhã, Portugal
e-mail: bessa@ubi.pt,pmorais@ubi.pt

**Resumo:** O clássico teorema da retificação do domínio no plano garante que, sob determinadas condições de não degeneracidade da derivada da função $f\colon \mathbb{R}^2 \to \mathbb{R}$ num ponto $p$ existem abertos $U$ e $V$ de $\mathbb{R}^2$, $p \in V$ e um difeomorfismo $h\colon U \to V$ tal que $f \circ h$ tem todas as suas curvas de nível em $h^{-1}(V)$ contidas em retas. Provamos que tal deformação $h$ pode ser feita preservando a área.

**Abstract:** The classical Domain-Straightening Theorem on the plane says that under a non-degeneracy condition on the derivative of a map $f\colon \mathbb{R}^2 \to \mathbb{R}$ in a point $p$ there exists a neighborhood $V \subset \mathbb{R}^2$ of $p$, $U \subset \mathbb{R}^2$ and a diffeomorphism $h\colon U \to V$ such that $f \circ h$ has all level curves in $h^{-1}(V)$ defined by straight lines. We prove that such deformation $h$ can be made area-preserving.

**palavras-chave:** Teorema da retificação do domínio; fluxos que preservam a área; teoremas de caixa de fluxo.

**keywords:** Domain-Straightening Theorem; area-preserving flows; flowbox theorems.

## 1 Area-preserving flowbox theorem

There is a lemma in Celestial Mechanics which says that there are no perfect coordinate systems. Indeed, the problem that we have at hand can be made extremely simple just by picking the right coordinates. It is no coincidence that this observation came accross in this particular area which is rich on imposing constraints related with the invariance of volume forms, symplectic forms, contact forms, et cetera. Yet, we may wonder why there is the need for coordinate systems preserve some invariants? Actually, when working with perturbations of flows/vector fields it is convenient to have good coordinates to perform perturbations explicitly, furthermore, once we perturb maintaining the invariant (e.g. volume form) we would like to 're-turn' to the initial scenario and so we are keenly interested that these change of coordinates keep the geometric invariant unchanged, otherwise they are completely useless. Let us consider a simple and enlightening example related with a perturbation of an orbit of a planar flow $F^t$ associated to a divergence-free vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ (i.e. $(F^t)' = F$ and $\nabla \cdot F = 0$). From Liouville's formula we know that

$$\det D_q F^t = \exp\left(\int_0^t \nabla \cdot F(F^s(q))\, ds\right), \tag{1}$$

for all $q \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Hence $\nabla \cdot F = 0$ implies that $\det D_q F^t = 1$ and so $F^t$ preserves the area. When perturbing the $F^t$ orbit of a point we begin by perturbing $F$ by $\hat{F}$ however we must keep $\nabla \cdot \hat{F} = 0$ otherwise, by (1), $\hat{F}^t$ may no longer preserve the area. This can be a hard task but if we, by some manner or means, could begin with a trivial divergence-free vector field like $F = (1, 0)$, even defined in a neighborhood of the points we want to perturb, then we fairly simplify our problem.

When trivializing coordinates are considered the examples are immense and just to mention some we have: Darboux charts on contact and also symplectic forms [9, 1], the generalization of Darboux charts (Carathéodory-Jacobi-Lie theorem) [11], the volume-preserving Moser charts [14]; the flow-box theorems on $\mathbb{R}$-actions (i.e. flows) proved in [1] for the general case, proved in [3] for the volume-preserving case followed by [2, 7] and proved in [15, 1] for Hamiltonian flows followed by [4, 6]; flowbox-like theorems on $\mathbb{R}^n$-actions [11] followed by the symplectic [13] and also the volume-preserving counterpart [5]. The examples continue on and on as the quadratic charts provided by Morse and Morse-Palais lemmas [10] show. Here we will be interested in conservative coordinates and in particular in the area-preserving Domain-Straightening Theorem. A proof of this theorem aka *canonical form for a submersion* and without any kind of area-preserving constraits can be found in lots of places e.g. [12, § 8]. In brief terms it says that if $f\colon \mathbb{R}^2 \to \mathbb{R}$ has a nonzero derivative at $p$, then $f$ can be 'straightened out' that is to say that around $p$ its level curves can be transformed into lines under a suitable change of coordinates. Here the novelty is that we perform the straightened in a conservative fashion (see Theorem 2) and so we begin by proving a flowbox theorem for area-preserving flows. Since we are dealing with flows in $\mathbb{R}^2$ the proof can be simplified avoiding the use of Dacorogna-Moser PDE (see [8]) as it was used e.g. in [3, 5]. Given a diffeomorphism $\varphi$ let $\varphi_* \frac{\partial}{\partial x}$ stand for the push-forward of $\frac{\partial}{\partial x} := (1, 0)$ by $\varphi$.

**Theorem 1 (Area-preserving flowbox theorem)** *Let be given an open set $A \subset \mathbb{R}^2$, a $C^r$ $(r \geq 1)$ divergence-free vector field $F\colon A \to \mathbb{R}^2$ and $p \in A$ such that $F(p) \neq \vec{0}$. Then there exist open sets $U$ and $V$ and a $C^r$ area-preserving diffeomorphism $\varphi\colon U \to V$ such that $p \in V$ and $\varphi_* \frac{\partial}{\partial x} = F$ for all $q \in U$.*

**Proof:**
No loss of generality comes from assuming $p = \vec{0}$ and $F(p) = \frac{\partial}{\partial x}$. Let $F^t = (F_1^t, F_2^t)$ be the flow associated to $F = (F_1, F_2)$. Clearly, there exists a vertical transversal section $\{0\}\times] - \epsilon, \epsilon[$ through $p$ for some $\epsilon > 0$. Let us consider now, for a small $\hat{\epsilon} > 0$, a diffeomorphism $\hat{\phi}\colon ] - \hat{\epsilon}, \hat{\epsilon}[\to] - \epsilon, \epsilon[$ such that $\hat{\phi}(0) = 0$. Consider a small neighborhood $U$ of $\vec{0}$ on which makes sense the following definition: given $(\hat{x}, \hat{y}) \in U$ let $\hat{\varphi}(\hat{x}, \hat{y}) = F^{\hat{x}}(0, \hat{\phi}(\hat{y}))$. Taking $\hat{x} = 0$ we have $\frac{\partial}{\partial \hat{y}} F_1^{\hat{x}}(0, \hat{\phi}(\hat{y})) = 0$ and also $\frac{\partial}{\partial \hat{y}} F_2^{\hat{x}}(0, \hat{\phi}(\hat{y})) = \frac{\partial}{\partial \hat{y}} \hat{\phi}(\hat{y}) = \hat{\phi}'(\hat{y})$. Thus computing the derivatives when $\hat{x} = 0$ and taking into account that the first column is the time-derivative of a flow, say the vector field, we obtain:

$$D_{(0,\hat{y})}\hat{\varphi} = \begin{pmatrix} F_1(F^0(0, \hat{\phi}(\hat{y}))) & 0 \\ F_2(F^0(0, \hat{\phi}(\hat{y}))) & \hat{\phi}'(\hat{y}) \end{pmatrix} \tag{2}$$

and

$$\det D_{(0,\hat{y})}\hat{\varphi} = \hat{\phi}'(\hat{y}) F_1(0, \hat{\phi}(\hat{y})). \tag{3}$$

Since we would like to have $\det D\hat{\varphi}(0, \hat{y}) = 1$ we need to solve the first order differential equation, with initial condition $\hat{\phi}(0) = 0$, given by

$$\hat{\phi}'(\hat{y}) = \frac{1}{F_1(0, \hat{\phi}(\hat{y}))}, \tag{4}$$

and reescaling the domain a little bit if necessary. By transversality and smoothness assumptions, (4) has a unique solution $\phi$. Finally, for $(\hat{x}, \hat{y}) \in U$, we define the diffeomorphism $\varphi\colon U \to V$ by

$$\varphi(\hat{x}, \hat{y}) = F^{\hat{x}}(0, \phi(\hat{y})). \tag{5}$$

Using the chain rule we get:

$$D_{(\hat{x},\hat{y})}\varphi = D_{(\hat{x},\hat{y})} F^{\hat{x}}(0, \phi(\hat{y})) = D_{(\hat{x},\hat{y})} F^{\hat{x}} \varphi(0, \hat{y}) = D_{\varphi(0,\hat{y})} F^{\hat{x}} \cdot D_{(0,\hat{y})}\varphi. \tag{6}$$

Using the multiplicative property of the determinant, the fact that $F$ is divergence-free, (2) equipped with the solution $\phi$ of (4), and Liouville's formula

$$\det D_q F^{\hat{x}} = \exp\left(\int_0^{\hat{x}} \nabla \cdot F(F^t(q))\, dt\right) \tag{7}$$

we obtain:

$$\det D_{(\hat{x},\hat{y})} F^{\hat{x}}(0,\phi(\hat{y})) = \det D_{\varphi(0,\hat{y})} F^{\hat{x}} \cdot \det D_{(0,\hat{y})}\varphi = 1.$$

We are left to see that $F$ is the push-forward of $\frac{\partial}{\partial x}$ by $\varphi$. This means that $F(q) = D\varphi \frac{\partial}{\partial x}(\varphi^{-1}(q))$. Recall that $\frac{\partial}{\partial x} := (1,0)$. Let $(x,y) = \varphi(\hat{x},\hat{y})$. Noticing that $D_q F^t \cdot F(q) = F(F^t(q))$ we obtain,

$$\begin{aligned} \varphi_* \frac{\partial}{\partial x}(x,y) &= D_{(\hat{x},\hat{y})}\varphi \frac{\partial}{\partial x}\, \varphi^{-1}(x,y) = D_{(\hat{x},\hat{y})}\varphi \frac{\partial}{\partial x}(\hat{x},\hat{y}) \\ &\overset{(6)}{=} D_{\varphi(0,\hat{y})} F^{\hat{x}} \cdot D_{(0,\hat{y})}\varphi \cdot (1,0) = D_{\varphi(0,\hat{y})} F^{\hat{x}}\, F(\varphi(0,\hat{y})) \\ &= F(F^{\hat{x}}(\varphi(0,\hat{y}))) = F(F^{\hat{x}}(0,\phi(\hat{y}))) \overset{(5)}{=} F(\varphi(\hat{x},\hat{y})). \ \square \end{aligned}$$

# 2 Area-preserving Domain-Straightening Theorem

Formaly, the Domain-Straightening Theorem says that given an open set $A \subset \mathbb{R}^2$, a $C^r$ $(r \geq 2)$ function $f\colon A \to \mathbb{R}$ and $p \in A$ such that $f(p) = 0$ and $\nabla f(p) \neq \vec{0}$, then there exist open sets $U$ and $V$ and a $C^r$ diffeomorphism $h\colon U \to V$ such that $p \in V$ and $f \circ h(x,y) = y$ for all $(x,y) \in U$. By making use of Theorem 1 and a somehow subtly trivial property of the vector perpendicular to the gradient and with the same norm (see (8) bellow) we will obtain:

**Theorem 2 (Area-preserving Domain-Straightening Theorem)** *Let be given an open set $A \subset \mathbb{R}^2$, a $C^r$ $(r \geq 2)$ function $f\colon A \to \mathbb{R}$ and $p \in A$. If $f(p) = 0$ and $\nabla f(p) \neq \vec{0}$, then there exist open sets $U$ and $V$ and an area-preserving $C^r$ diffeomorphism $h\colon U \to V$ such that $p \in V$ and $f \circ h(x,y) = y$ for all $(x,y) \in U$.*

**Proof:**
We assume that $p = \vec{0}$. Take a neighborhood $V \subset \mathbb{R}^2$ of $p$ such that $\nabla f(q) \neq \vec{0}$ for all $q \in V$. Let $\nabla f(q) = \left(\frac{\partial f}{\partial x}(q), \frac{\partial f}{\partial y}(q)\right)$ and we consider the vector field tangent to the level curves of $f$ defined as $\nabla^{\perp} f(q) = \left(\frac{\partial f}{\partial y}(q), -\frac{\partial f}{\partial x}(q)\right)$ for $q \in V$. By Clairaut-Schwarz theorem we have

$$\operatorname{div}(\nabla^{\perp} f(q)) = \nabla \cdot \nabla^{\perp} f(q) = 0 \tag{8}$$

and so, by (7), the flow formed by the integral curves of $\nabla^{\perp} f$ is area-preserving. Take $h = \varphi$ given by Theorem 1. We are left to check that

$f \circ h(x,y) = y$ for all $(x,y) \in U$. We take partial derivatives and in one hand we get

$$D_{(x,y)}(f \circ \varphi) \cdot (1,0) = D_{\varphi(x,y)} f \cdot D_{(x,y)} \varphi \cdot (1,0) = \nabla f(\varphi(x,y)) \cdot \nabla^{\perp} f(\varphi(x,y)) = 0.$$

On the other hand we have

$$\begin{aligned} D_{(x,y)}(f \circ \varphi) \cdot (0,1) &= D_{\varphi(x,y)} f \cdot D_{(x,y)} \varphi \cdot (0,1) = \nabla f(\varphi(x,y)) \cdot (0, \phi'(y)) \\ &= \frac{\partial f}{\partial y}(\varphi(x,y)) \phi'(y) = 1, \end{aligned}$$

by (4) and taking in consideration that the first component of the vector field $\nabla^{\perp} f$ is $\frac{\partial f}{\partial y}(\varphi(x,y))$. As $\varphi(0,0) = F^0(0,0) = \vec{0}$ and $p = \vec{0}$ we get $f \circ \varphi(x,y) = y$ for all $(x,y) \in U$. $\square$

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