Application of the BenTaher-Rachidi method in numerical sequences

Renata Vieira
Federal University of Ceara
A. Mister Hull, s/n, Pici
60455-760 Fortaleza, Brazil
e-mail: re.passosm@gmail.com

Milena Mangueira, *Francisco Regis Alves*
Federal Institute of Education, Science and Technology of Ceara
Av. Treze de Maio, 2081
60040-531 Fortaleza, Brazil
e-mail: fregis@gmx.fr

Paula Catarino
University of Trás-os-Montes and Alto Douro
Quinta de Prados
5000-801 Vila Real, Portugal
e-mail: pcatarino23@gmail.com

Resumo: Este artigo apresenta a aplicação do estudo sobre o método de BenTaher-Rachidi para a resolução de sequências numéricas lineares e recorrentes de ordem superior. Assim, obtém-se a fórmula de Binet, pelo método de BenTaher-Rachidi, nas sequências de Lucas, Pell, Leonardo, Mersenne, Oresme, Jacobsthal, Padovan, Perrin e Narayana.

Abstract This article presents the application of the study on the BenTaher-Rachidi method for the solving of linear and recurrent numerical sequences of higher order. Thus, Binet's formula is obtained, using the BenTaher-Rachidi method, in the sequences of Lucas, Pell, Leonardo, Mersenne, Oresme, Jacobsthal, Padovan, Perrin and Narayana.

palavras-chave: fórmula de Binet; método BenTaher-Rachidi; método tradicional; sequências numéricas.

keywords: Binet's formula; BenTaher-Rachidi method; traditional method; numeric sequences.

1 Introduction

Sequences have been extensively studied in mathematical literature over the years due to their wide applicability. A linear recursive sequence is defined as one that has an infinite number of terms, generated by a linear recurrence, called the recurrence formula, which allows the calculation of its immediate predecessor terms. However, this recurrence is not the only way to define linear recursive sequences, and it is still necessary to know its initial elements.

You can extract several properties and theorems from the recurrence of a sequence. There are several methods of solving a recurrence taking into account its characteristics: linear, non-linear, homogeneous and non-homogeneous order [8]. It is possible to obtain the terms of a sequence without the need to apply the recurrence formula, that is, through the generating matrix or other mechanisms, such as Binet's formula.

Usually, when using Binet's formula, the solving of a linear system is necessary, however [15] present a technique that consists of finding the necessary terms without using a linear system for cases in which there is a matrix of Vandermonde. With that, [8] presented a comparison between the traditional method of solving a recurrence with the BenTaher-Rachidi method.

Continuing the work of [8], we will introduce the traditional and BenTaher-Rachidi's methods of solving a recurrence, and we will also present linear sequences that fulfilled a recurrence, whose characteristic polynomial has simple roots, namely: Lucas sequence, Pell, Leonardo, Mersenne, Jacobsthal, Padovan, Perrin and Narayana. The BenTaher-Rachidi method will also be applied to these sequences in order to obtain their respective Binet's formulas using this alternative method presented.

2 Methods of resolving a recurrence

Primarily, Binet's formula was described in terms of another formula, introducing the notion of Binet's factorial formula. However, its resolution is still linked to the resolution of linear systems, which can be solved using the Vandermonde system. Thus, we have that each numerical sequence of the linear and recurrent type presents its respective characteristic polynomial. Therefore, it is necessary to know their respective roots to perform the calculation of the Binet's formula. Generally, for this calculation using the traditional method, one has to solve the Vandermonde system or invert the associated matrices, making the calculation difficult. Therefore, another method, called the BenTaher-Rachidi method, is then studied, facilitating this mathematical calculation [15].

2.1 Traditional method

In [8], we can find two methods of solving a recurrence. The first, called **traditional method**, consists of using the formula,

$$V_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j} n^j \right) \lambda_i^n, \tag{1}$$

for $n \geq 0$, which $\beta_{i,j}$ are coefficients determined through a linear system of r equations that are used as boundary condition the coefficients $(V_j)_{0\leq j\leq r-1}$, and s is the number of distinct roots of the polynomial characteristic of the given recurrence.

As a way of exemplifying, the resolution for the Fibonacci sequence, presenting its recurrence formula $F_n = F_{n-1} + F_{n-2}, n \geq 2$, with $F_0 = 0, F_1 = 1$ characteristic polynomial $\lambda^2 - \lambda - 1 = 0$, whose roots are $\lambda_1 = \frac{1-\sqrt{5}}{2}, \lambda_2 = \frac{1+\sqrt{5}}{2}$. Using the formula presented in Equation 1, we have that: $F_n = \beta_{1,0}\lambda_1^n + \beta_{2,0}\lambda_2^n$. For this, it is necessary to calculate the values of the coefficients $\beta_{1,0}$ and $\beta_{2,0}$. Therefore, it is possible to integrate the data of the polynomial with the recurrence formula, to then assemble a system of equations, such as

$$\begin{cases} \beta_{1,0} + \beta_{2,0} = F_0 \\ \lambda_1\beta_{1,0} + \lambda_2\beta_{2,0} = F_1 \end{cases}$$

For that, it is necessary that the system of equations be solved, which presents difficulties for the calculation of sequences of higher order than the second order.

Thus, solving the determined system, we have that

$$\beta_{1,0} = -\frac{1}{\lambda_2 - \lambda_1} = -\frac{1}{\sqrt{5}} \qquad \text{and} \qquad \beta_{2,0} = \frac{1}{\lambda_2 - \lambda_1} = \frac{1}{\sqrt{5}}$$

Finally, the Binet's formula of the Fibonacci sequence is given by

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

In [15], it is possible to establish some new expressions of the Binet's formula for the sequences without using the resolution-based approach of the linear system, therefore approaching a method based only on combinatorial expression of sequences.

In this work, it is possible to find variations in the resolutions of the Binet's formula, be they: for the simple roots of the equation, for roots with multiplicities and for sequences of orders greater than or equal to two. Nevertheless, applications of these new expressions of Binet's formula are presented to solve the usual linear systems of Vandermonde equations. Furthermore, explicit formulas are obtained for the inverse entries of their associated matrices. Illustrative examples and a comparison is made with two current methods and some numerical aspects of the results that have been presented.

2.2 BenTaher-Rachidi method

The second method discussed is the **BenTaher-Rachidi method**, which is a technique presented in [15]. This method consists of finding the coefficients $\beta_{i,0}$ of the linear system (2), without the need to use a linear system for cases in which A is a Vandermonde matrix.

$$\begin{cases} \beta_{1,0} + \beta_{2,0} + \cdots + \beta_{r,0} = V_0 \\ \lambda_1\beta_{1,0} + \lambda_2\beta_{2,0} + \cdots + \lambda_r\beta_{r,0} = V_1 \\ \qquad\qquad \vdots \\ \lambda_1^{r-1}\beta_{1,0} + \lambda_2^{r-1}\beta_{2,0} + \cdots + \lambda_r^{r-1}\beta_{r,0} = V_{r-1} \end{cases} \tag{2}$$

This linear system can be written as $Ax = b$, where A is a Vandermonde matrix, x is the unknown vector $\beta_{i,0}$ and b is the vector of conditions recurrence contour.

Furthermore, it is worth noting that a Vandermonde matrix is defined by a square matrix in which each column (or row) is a geometric progression where the first term is 1.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ & & \vdots & \\ \lambda_1^m & \lambda_2^m & \cdots & \lambda_m^m \end{bmatrix}.$$

Thus, the solution of a homogeneous linear recurrence whose characteristic polynomial has only simple roots is given by the equation,

$$V_n = \sum_{i=1}^{r} \frac{1}{p'(\lambda_i)} \left(\sum_{p=0}^{r-1} \frac{A_p}{\lambda_i^{p+1}} \right) \lambda_i^n, \tag{3}$$

for $n \geq r$, where $A_p = a_{r-1}V_p + \cdots + a_pV_{r-1}$.

It is noteworthy that this BenTaher-Rachidi method does not require the resolution of a linear system. Thus, in the article [8], this method is applied to solve the polynomial characteristic of the Fibonacci sequence, thus finding the Binet's formula of this sequence. Furthermore, a comparison was made with the traditional method to show that it is possible to arrive at the same solution regardless of the method used.

With this, next, we will solve the polynomial characteristic of other linear sequences to present a new obtainment of the Binet's formula of these sequences.

3 Application of the BenTaher-Rachidi method

In this section, the BenTaher-Rachidi method will be applied to the numerical sequences of linear and recurring character, establishing a new alternative for the calculation of the Binet formula. Emphasizing that the respective application for the Fibonacci sequence has already been carried out in the work of [8], then the Lucas, Pell, Leonardo, Mersenne, Oresme, Jacobsthal, Padovan, Perrin and Narayana sequences are addressed.

3.1 Lucas Sequence

Lucas's sequence was developed by French mathematician Édouard Anatole Lucas (1842-1891), in which he made some mathematical contributions such as the well-known Tower of Hanoi [2]. And yet, the mathematician performed tests for prime numbers based on linear and recurring sequences, thus establishing a relationship of the twelfth prime number of Mersenne, a 39-digit number that remained the largest prime number for many years, and being the highest prime number found without the aid of computational and technological resources [14].

Lucas studied the Fibonacci sequence and in one of his generalizations, created the Lucas sequence, where he changed only the two initial values to 2 and 1, remaining with the same recurrence. The Lucas numbers form a second order sequence, linear and recurring, having its recurrence formula $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$, with $L_0 = 2$ and $L_1 = 1$. Its characteristic polynomial is identical to that of Fibonacci, $x^2 - x - 1$, having the same roots and presenting the same relationship with the gold number. Thus, Binet's formula for the Lucas sequence is given by, [13],

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{4}$$

Using the characteristic polynomial $p(\lambda) = \lambda^2 - \lambda - 1$, deriving p we get the polynomial $p'(\lambda) = 2\lambda - 1$ and we can calculate $p'(\lambda_1) = -\sqrt{5}$ and $p'(\lambda_2) = \sqrt{5}$.

Using the boundary conditions, we can calculate $A_0 = a_1 L_0 + a_0 L_1 = 3$ and $A_1 = a_1 L_1 = 1$, where $a_0 = 1$ and $a_1 = 1$, given by the coefficients of the recurrence relation. With that, using the formula (3), recurrence is given by

$$L_n = \frac{1}{p'(\lambda_1)}.\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}.\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}}\right)\lambda_2^n$$

By replacing the previously obtained values, it is possible to obtain

$$\begin{aligned} L_n &= \frac{1}{-\sqrt{5}}\left(\frac{3}{\left[\frac{1-\sqrt{5}}{2}\right]^1} + \frac{1}{\left[\frac{1-\sqrt{5}}{2}\right]^2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &\quad + \frac{1}{\sqrt{5}}\left(\frac{3}{\left[\frac{1+\sqrt{5}}{2}\right]^1} + \frac{1}{\left[\frac{1+\sqrt{5}}{2}\right]^2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\ &= \frac{1}{-\sqrt{5}}\left(\frac{3.2(1-\sqrt{5})+4}{(1-\sqrt{5})^2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{3.2(1+\sqrt{5})+4}{(1+\sqrt{5})^2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\ &= \left(\frac{6\sqrt{5}-10}{6\sqrt{5}-10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{10+6\sqrt{5}}{10+6\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\ &= \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n \end{aligned}$$

resulting in the same answer obtained in Equation (4).

3.2 Pell Sequence

Pell's sequence carries this name in honor of the English mathematician John Pell (1611-1685), known for being extremely reserved, which made him recognized as one ot the most enigmatic mathematicians of the 17th century [9]. In [2], Pell acquired credit for the development of the study of Pell's equations, or Diophantine equation, described by $x^2 - Ay^2 = 1$, with x and y numbers integers and A not squared whole.

Pell's sequence was already known in Greek antiquity around 100 years after Christ, as part of an ancient algorithm to create successive approximations to $\sqrt{2}$, known as Theon's ladder. This sequence has the recurrence formula defined by $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$ and its initial values are $P_0 = 0$ and $P_1 = 1$.

This sequence has a characteristic polynomial defined by $x^2 - 2x - 1 = 0$ where one root is positive, known as the silver number (2.41), and its other root is a negative number [2]. This silver number represents the convergence relationship between the neighboring terms of the sequence. From the polynomial characteristic of the Pell sequence, it is possible to obtain its Binet's formula, by the traditional method, as was done in [11]. With that, we have that its Binet's formula is presented as

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \tag{5}$$

Now, using the BenTaher-Rachidi method and using the characteristic polynomial $p(\lambda) = \lambda^2 - 2\lambda - 1$, deriving p we get the polynomial $p'(\lambda) = 2\lambda - 2$ and we can calculate $p'(\lambda_1) = -2\sqrt{2}$ and $p'(\lambda_2) = 2\sqrt{2}$.

Using the boundary conditions, we can calculate $A_0 = a_1 P_0 + a_0 P_1 = 2$ and $A_1 = a_1 P_1 = 1$, where $a_0 = 2$ and $a_1 = 1$, given by the coefficients of the recurrence relation. Thus, using the formula (3), the recurrence is given by

$$P_n = \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}}\right)\lambda_2^n$$

By replacing the previously obtained values, it is possible to obtain:

$$\begin{aligned}
P_n &= \frac{1}{-2\sqrt{2}}\left(\frac{2}{(1-\sqrt{2})^1} + \frac{1}{(1-\sqrt{2})^2}\right)(1-\sqrt{2})^n \\
&\qquad + \frac{1}{2\sqrt{2}}\left(\frac{2}{(1+\sqrt{2})^1} + \frac{1}{(1+\sqrt{2})^2}\right)(1+\sqrt{2})^n \\
&= \frac{1}{-2\sqrt{2}}\left(\frac{2-2\sqrt{2}+1}{(1-\sqrt{2})^2}\right)(1-\sqrt{2})^n + \frac{1}{2\sqrt{2}}\left(\frac{2+2\sqrt{2}+1}{(1+\sqrt{2})^2}\right)(1+\sqrt{2})^n \\
&= \frac{1}{-2\sqrt{2}}\left(\frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right)(1-\sqrt{2})^n + \frac{1}{2\sqrt{2}}\left(\frac{3+2\sqrt{2}}{3+2\sqrt{2}}\right)(1+\sqrt{2})^n \\
&= \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n - (1-\sqrt{2})^n]
\end{aligned}$$

resulting in the same answer obtained in Equation (5).

3.3 Leonardo Sequence

Historically, little is known about Leonardo's sequence. The authors of [2] believe that these numbers were studied by Leonardo de Pisa, known as Leonardo Fibonacci, and, therefore, has not been proven in any work in the literature, due to the lack of research related to that sequence. This sequence is very similar to the Fibonacci sequence, including a relationship between Leonardo's numbers and Fibonacci numbers. This relationship is defined by [7] as $Le_n = 2F_{n+1} - 1$.

Leonardo's sequence was initially presented by [7], in which there are two recurrences for this sequence, namely: $Le_n = Le_{n-1} + Le_{n-2} + 1$ and $Le_n = 2Le_{n-1} - Le_{n-3}$, for $n \geq 2$, being $Le_0 = Le_1 = 1$. Its characteristic polynomial is given by $x^3 - 2x^2 + 1 = 0$, in which there are three real roots, one equal to 1 and the other two equal to the roots of the characteristic Fibonacci equation, $x_2 = \dfrac{1+\sqrt{5}}{2}$ and $x_3 = \dfrac{1-\sqrt{5}}{2}$ [2, 17]. It is worth mentioning that these Leonardo numbers have their convergence relation between the neighboring terms of the sequence as being the gold number (1.61), as well as the result of one of its real roots. As for their Binet's formula, [7] define it using the relationship $Le_n = 2F_{n+1} - 1$ and the Binet's formula of the Fibonacci sequence. With that, we have that the Binet formula for Leonardo's sequence is given by

$$Le_n = 2\left(\frac{x_2^{n+1} - x_3^{n+1}}{x_2 - x_3}\right) - 1, \tag{6}$$

on what $x_2 = \dfrac{1+\sqrt{5}}{2}$ and $x_3 = \dfrac{1-\sqrt{5}}{2}$ are the roots of the polynomial characteristic of the sequence.

Now, using the BenTaher-Rachidi method and using the characteristic polynomial

$$p(\lambda) = \lambda^3 - 2\lambda^2 + 1,$$

deriving p we get the polynomial $p'(\lambda) = 3\lambda^2 - 4\lambda$ and we can calculate

$$p'(\lambda_1) = -1, \quad p'(\lambda_2) = \frac{5-\sqrt{5}}{2} \qquad \text{and} \qquad p'(\lambda_3) = \frac{5+\sqrt{5}}{2}.$$

Using the boundary conditions, we can calculate

$$A_0 = a_2Le_0 + a_1Le_1 + a_0Le_2 = 5 \qquad \text{and} \qquad A_1 = a_2Le_1 + a_1Le_2 = -1,$$

where $a_0 = 2$, $a_1 = 0$ and $a_2 = -1$, as given by the coefficients of the recurrence relation. Thus, using the formula (3), the recurrence is given by,

$$Le_n = \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}}+\frac{A_1}{\lambda_1^{1+1}}+\frac{A_2}{\lambda_1^{2+1}}\right)\lambda_1^n$$
$$+\frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}}+\frac{A_1}{\lambda_2^{1+1}}+\frac{A_2}{\lambda_2^{2+1}}\right)\lambda_2^n+\frac{1}{p'(\lambda_3)}\left(\frac{A_0}{\lambda_3^{0+1}}+\frac{A_1}{\lambda_3^{1+1}}+\frac{A_2}{\lambda_3^{2+1}}\right)\lambda_3^n$$

By replacing the previously obtained values, it is possible to obtain

$$\begin{aligned}
Le_n &= \frac{1}{-1}\left(\frac{5}{1^1}+\frac{(-1)}{1^2}+\frac{(-3)}{1^3}\right)1^n \\
&\quad +\frac{1}{\frac{5-\sqrt{5}}{2}}.\left(\frac{5}{(\frac{1+\sqrt{5}}{2})^1}+\frac{(-1)}{(\frac{1+\sqrt{5}}{2})^2}+\frac{(-3)}{(\frac{1+\sqrt{5}}{2})^3}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \\
&\quad \frac{1}{\frac{5+\sqrt{5}}{2}}.\left(\frac{5}{(\frac{1-\sqrt{5}}{2})^1}+\frac{(-1)}{(\frac{1-\sqrt{5}}{2})^2}+\frac{(-3)}{(\frac{1-\sqrt{5}}{2})^3}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\
&= -1+\left(\frac{1}{\left(\frac{5-\sqrt{5}}{2}\right)}\right)\left(\frac{5\left(\frac{1+\sqrt{5}}{2}\right)^2-\left(\frac{1+\sqrt{5}}{2}\right)-3}{\left(\frac{1-\sqrt{5}}{2}\right)^3}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\
&+ \left(\frac{1}{\left(\frac{5+\sqrt{5}}{2}\right)}\right)\left(\frac{5\left(\frac{1-\sqrt{5}}{2}\right)^2-\left(\frac{1-\sqrt{5}}{2}\right)-3}{\left(\frac{1-\sqrt{5}}{2}\right)^3}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\
&= -1+\left(\frac{2}{(5-\sqrt{5})}\right)\left(\frac{5\left(\frac{6+2\sqrt{5}}{4}\right)-\frac{1}{2}-\frac{\sqrt{5}}{2}-3}{\left(\frac{6-2\sqrt{5}}{4}\right)\left(\frac{1-\sqrt{5}}{2}\right)}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\
&+ \left(\frac{2}{(5+\sqrt{5})}\right)\left(\frac{5\left(\frac{6-2\sqrt{5}}{4}\right)-\frac{1}{2}+\frac{\sqrt{5}}{2}-3}{\left(\frac{6-2\sqrt{5}}{4}\right)\left(\frac{1-\sqrt{5}}{2}\right)}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n
\end{aligned}$$

$$= \quad -1+\left(\frac{2}{\left(5-\sqrt{5}\right)}\right)\left(\frac{\frac{15}{2}+\frac{5\sqrt{5}}{2}-\frac{1}{2}-\frac{\sqrt{5}}{2}-\frac{6}{2}}{\left(\frac{16+8\sqrt{5}}{8}\right)}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n}$$
$$+ \quad \left(\frac{2}{\left(5+\sqrt{5}\right)}\right)\left(\frac{\frac{15}{2}-\frac{5\sqrt{5}}{2}-\frac{1}{2}+\frac{\sqrt{5}}{2}-\frac{6}{2}}{\left(\frac{16+8\sqrt{5}}{8}\right)}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n}$$

By doing algebraic manipulations, it is possible to write the equation as

$$\begin{aligned} Le_n &= -1+\frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\ &= \frac{2}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]-1 \end{aligned}$$

resulting in the same answer obtained in Equation (6).

3.4 Mersenne Sequence

The Mersenne numbers make up the Mersenne sequence, such numbers honoring the Frenchman Marin Mersenne (1588-1648). Marin Mersenne was a Franciscan who offered his home for meetings with contemporary scientists, such as Descartes, Galileo, Fermat, Pascal and Torricelli with an interest in discussing and studying mathematics and scientific subjects [2]. Mersenne contributed to number theory, specifically Mersenne's prime numbers, which are all natural numbers in the form $M_n = 2^n - 1$ where n is a natural number.

The Mersenne sequence has as its recurrence formula $M_n = 3M_{n-1} - 2M_{n-2}$, for $n \geq 2$, being $M_0 = 0$ and $M_1 = 1$ their initial values. And yet, this sequence has a second degree polynomial, $x^2 - 3x + 2 = 0$, where they have two real roots, one equal to 2 and the other equal to 1. We have that the Binet's formula of the Mersenne sequence is presented by [4], where it is defined as

$$M_n = 2^n - 1, n \geq 0. \tag{7}$$

Using the characteristic polynomial $p(\lambda) = \lambda^2 - 3\lambda + 2$, deriving p we get the polynomial $p'(\lambda) = 2\lambda - 3$ and we can calculate $p'(\lambda_1) = -1$ and $p'(\lambda_2) = 1$.

Using the boundary conditions, we can calculate $A_0 = a_1V_0 + a_0V_1 = a_1M_0 + a_0M_1 = 3$ and $A_1 = a_1V_1 = a_1M_1 = -2$, where $a_0 = 3$ and $a_1 = -2$, given by the coefficients of the recurrence relation. Thus, using the formula

(3), the recurrence is given by,

$$M_n = \frac{1}{p'(\lambda_1)}.\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}.\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}}\right)\lambda_2^n$$

By replacing the previously obtained values, it is possible to obtain:

$$\begin{aligned} M_n &= \frac{1}{-1}.\left(\frac{3}{1^1} + \frac{-2}{1^2}\right)1^n + \frac{1}{1}.\left(\frac{3}{2^1} + \frac{(-2)}{2^2}\right)2^n \\ &= -1(3-2) + 2^n\left(\frac{3}{2} - \frac{1}{2}\right) \\ &= 2^n - 1 \end{aligned}$$

resulting in the same answer obtained in Equation (7).

3.5 Oresme Sequence

The Oresme sequence was created by the German philosopher Nicole Oresme (1320 - 1382), being a linear and recurrent second order sequence [3]. This sequence exposes a graphical representation of qualities and speeds, it is also believed that Oresme used primitive ideas, known today as the improper integral, to perform the sum of the infinite series, obtaining a value of 2.

The Oresme sequence is defined for every $n \geqslant 2$ by

$$O_{n+2} = O_{n+1} + \left(-\frac{1}{4}\right)O_n,$$

and the initial conditions $O_0 = 0, O_1 = O_2 = \frac{1}{2}$. Its characteristic polynomial is given by $x^2 - x + \frac{1}{4}$, where $\lambda = \frac{1}{2}$ being its real root of multiplicity 2. Using the traditional method via the Binet formula, we have

$$O_n = \alpha_0\left(\frac{1}{2}\right)^n + \alpha_1 n\left(\frac{1}{2}\right)^n,$$

for $n \geqslant 0$. Given the initial conditions, we obtain $\alpha_0 = 0$ and $\alpha_1 = 1$. Thus, we get $O_n = n\left(\frac{1}{2}\right)^n$.

Now we will apply the Bentaher-Rachidi method in the general setting to the Oresme sequence. That is, as matter of fact, we need to appeal the general sitting of this method, when the associated characteristic polynomial of the linear recursive sequence owns distincts roots of multiplicity $\geqslant 1$, introducing the Stirling numbers of the first kind. Indeed, applying the Theorem 2.9 [15] in this case, we have

$$O_n = A_0 V_{n-2} + A_1 V_{n-3},$$

for every $n \geqslant 2$, where $V_n = (c_0 + c_1 n)\left(\frac{1}{2}\right)^n$, with $c_0 = 1$ and $c_1 = S_{1,1} = 1$ ($S_{1,1}$ are the first kind Stirling numbers), $A_0 = -\frac{1}{4}V_0 + V_1$ and $A_1 = -\frac{1}{4}V_1$. Thereby, a direct calculation yields

$$O_n = \frac{1}{2}\left(\frac{1}{2}\right)^n (1 + n - 2) - \frac{1}{8}\left(\frac{1}{2}\right)^{n-3} (1 + n - 3) = \frac{n}{2^n}$$

3.6 Jacobsthal Sequence

The Jacobsthal sequence was defined by the German mathematician Ernest Erich Jacobsthal (1882-1965), this sequence has a great similarity with the Fibonacci sequence and presents several applications of which we can exemplify the use of these numbers in the area of computing in directives to change the program execution flow [2].

Jacobsthal sequence is defined by recurrence $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ and being $J_0 = 0$ and $J_1 = 1$ their initial conditions. This sequence carries many mathematical properties, highlighting its characteristic polynomial $x^2 - x - 2 = 0$, having two real roots, $x_1 = -1$ and $x_2 = 2$ [2], where the root equal to 2 also represents the convergence relationship between neighboring terms of the sequence. Due to the characteristic polynomial, we have the Binet's formula for the Jacobsthal sequence is given by [6], being defined as

$$J_n = \frac{2^n - (-1)^n}{3} \tag{8}$$

From the characteristic polynomial $p(\lambda) = \lambda^2 - \lambda - 2$, deriving p we get the polynomial $p'(\lambda) = 2\lambda - 1$ and we can calculate $p'(\lambda_1) = -3$ and $p'(\lambda_2) = 3$. Using the boundary conditions, we can calculate $A_0 = a_1 J_0 + a_0 J_1 = 1$ and $A_1 = a_1 J_1 = 2$, where $a_0 = 1$ and $a_1 = 2$, given by the coefficients of the recurrence relation. Thus, using the formula (3), the recurrence is given by,

$$J_n = \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}}\right)\lambda_2^n$$

By replacing the previously obtained values, it is possible to obtain

$$\begin{aligned} J_n &= \frac{1}{-3}\left(\frac{1}{-1^1} + \frac{2}{(-1)^2}\right)(-1)^n + \frac{1}{3}\left(\frac{1}{2^1} + \frac{(2)}{2^2}\right)2^n \\ &= -\frac{1}{3}(-1+2)(-1)^n + \frac{1}{3}\left(\frac{1}{2} + \frac{1}{2}\right)2^n \\ &= -\frac{1}{3}(-1)^n + \frac{1}{3}2^n \\ &= \frac{1}{3}[2^n - (-1)^n] \end{aligned}$$

resulting in the same answer presented in Equation (8).

3.7 Padovan Sequence

This sequence was created by the Italian architect Richard Padovan (1935-), it is considered as a cousin of the Fibonacci sequence [1], the first being a linear, recurring, third order and integer sequence. And yet, in the work of [16, 18] there is an emphasis on the mathematical historical process of this sequence, the Dutchman Hans Van Der Laan (1904 - 1991), who stands out after the Second World War, used the early Christian abbey basilica as an example to train architects in rebuilding churches [19]. The process of rebuilding the churches had been carried out by Lan and his brother, eventually discovering a new standard of measurement given by an irrational number, a number known as a plastic number or radiant number, and was first studied by Gérard Cordonnier.

Padovan sequence is defined by recurrence $Pa_n = Pa_{n-2} + Pa_{n-3}$, for $n \geq 3$ and being $Pa_0 = Pa_1 = Pa_2 = 1$ it is initial terms, still presenting its respective characteristic polynomial $x^3 - x - 1 = 0$, having three roots

$$x_1 = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{27}}} \approx 1,32$$

$$x_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} \approx -0,66 + 0,56i$$

$$x_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} \approx -0,66 - 0,56i.$$

With that, we will use the notations x_1, x_2, x_3 to facilitate the calculations, since this sequence has complex roots and with higher algebraic values. The relationship between the value of 1.32, which is presented as the real solution of the characteristic polynomial, and the convergence relationship between the neighboring terms of the sequence is also emphasized, thus creating a similarity.

Thus, we have Padovan's Binet's formula, as being:

$$Pa_n = \frac{(x_2-1)(x_3-1)}{(x_1-x_2)(x_1-x_3)}x_1^n + \frac{(x_1-1)(x_3-1)}{(x_2-x_1)(x_2-x_3)}x_2^n + \frac{(x_1-1)(x_2-1)}{(x_3-x_1)(x_3-x_2)}x_3^n \quad (9)$$

Using the BenTaher-Rachidi method, we have that from the polynomial

$$p(\lambda) = \lambda^3 - \lambda - 1,$$

the derivative is then calculated, resulting in $p'(\lambda) = 2\lambda^2 - 1$. Therefore, according to the formula of the method and the sequence coefficients, we have that

$$A_0 = a_2Pa_0 + a_1Pa_1 + a_0Pa_2 = 2$$
$$A_1 = a_2Pa_1 + a_1Pa_2 = 2$$
$$A_2 = a_2Pa_2 = 1.$$

Therefore, by considering Formula 3, we get:

$$\begin{aligned} Pa_n &= \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}} + \frac{A_2}{\lambda_1^{2+1}}\right)\lambda_1^n \\ &+\frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}} + \frac{A_2}{\lambda_2^{2+1}}\right)\lambda_2^n + \frac{1}{p'(\lambda_3)}\left(\frac{A_0}{\lambda_3^{0+1}} + \frac{A_1}{\lambda_3^{1+1}} + \frac{A_2}{\lambda_3^{2+1}}\right)\lambda_3^n \end{aligned}$$

Performing the replacement of previously calculated values and using Girard relations $x_1x_2x_3 = 1$,$x_1 + x_2 + x_3 = 0$ and $x_1x_2 + x_1x_3 + x_2x_3 = -1$, we have that

$$\begin{aligned} Pa_n &= \frac{1}{3x_1^2-1}\left(\frac{2}{x_1} + \frac{2}{x_1^2} + \frac{1}{x_1^3}\right)(x_1)^n + \frac{1}{3x_2^2-1}\left(\frac{2}{x_2} + \frac{2}{x_2^2} + \frac{1}{x_2^3}\right)(x_2)^n \\ &\quad + \frac{1}{3x_3^2-1}\left(\frac{2}{x_3} + \frac{2}{x_3^2} + \frac{1}{x_3^3}\right)(x_3)^n \\ &= \left(\frac{2x_1^2+2x_1+1}{3x_1^5-x_1^3}\right)(x_1)^n + \left(\frac{2x_2^2+2x_2+1}{3x_2^5-x_2^3}\right)(x_2)^n \\ &\quad + \left(\frac{2x_3^2+2x_3+1}{3x_3^5-x_3^3}\right)(x_3)^n \\ &= \left[\frac{2x_1^2+2x_1+x_1x_2x_3}{x_1(3x_1^4-x_1^2)}\right](x_1)^n + \left[\frac{2x_2^2+2x_2+x_1x_2x_3}{x_2(3x_2^4-x_2^2)}\right](x_2)^n \\ &\quad + \left[\frac{2x_3^2+2x_3+x_1x_2x_3}{x_3(3x_3^4-x_3^2)}\right](x_3)^n \\ &= \left[\frac{2x_1+x_1x_2x_3-x_1x_2-x_1x_3}{x_1(3x_1^3-x_1)}\right](x_1)^n + \left[\frac{2x_2+x_1x_2x_3-x_1x_2-x_2x_3}{x_2(3x_2^3-x_2)}\right](x_2)^n \\ &\quad + \left[\frac{2x_3+x_1x_2x_3-x_1x_3-x_2x_3}{x_3(3x_3^3-x_3)}\right](x_3)^n \\ &= \left[\frac{2+x_2x_3+x_1}{x_1(3x_1^2-1)}\right](x_1)^n + \left[\frac{2+x_2x_3+x_2}{x_2(3x_2^2-1)}\right](x_2)^n + \left[\frac{2+x_1x_2+x_3}{x_3(3x_3^2-1)}\right](x_3)^n \\ &= \left(\frac{x_2x_3-x_2-x_3+1}{3x_1^2-1}\right)(x_1)^n + \left(\frac{x_1x_3-x_1-x_3+1}{3x_2^2-1}\right)(x_2)^n \\ &\quad + \left(\frac{x_1x_2-x_1-x_2+1}{3x_3^2-1}\right)(x_3)^n \end{aligned}$$

that is

$$Pa_n = \left[\frac{(x_2-1)(x_3-1)}{2x_1^2+x_1^2-1}\right](x_1)^n + \left[\frac{(x_1-1)(x_3-1)}{2x_2^2+x_2^2-1}\right](x_2)^n + \left[\frac{(x_1-1)(x_2-1)}{2x_3^2+x_3^2-1}\right](x_3)^n$$

$$= \left[\frac{(x_2-1)(x_3-1)}{x_1^2-x_1x_2-x_1x_3+x_2x_3}\right](x_1)^n + \left[\frac{(x_1-1)(x_3-1)}{x_2^2-x_2x_3-x_1x_2+x_1x_3}\right](x_2)^n + \left[\frac{(x_1-1)(x_2-1)}{x_3^2-x_1x_3-x_3x_3+x_1x_2}\right](x_3)^n$$

$$= \frac{(x_2-1)(x_3-1)}{(x_1-x_2)(x_1-x_3)}(x_1)^n + \frac{(x_1-1)(x_3-1)}{(x_2-x_1)(x_2-x_3)}(x_2)^n + \frac{(x_1-1)(x_2-1)}{(x_3-x_1)(x_3-x_2)}(x_3)^n$$

obtaining the formula presented in the Equation (9).

3.8 Perrin Sequence

The Perrin sequence was developed by French engineer Olivier Raoul Perrin (1841-1910), who, in his spare time liked to produce scientific works, specifically for the area of mathematics. It is believed that in 1876 this sequence was mentioned implicitly by Édouard Lucas, known for creating the Lucas sequence and Lucas numbers. One can find applicability of this sequence in graph theory, and it has recently been used to discover the coordinates of taxis in urban networks in a confidential way [10].

This sequence has a great similarity with the Padovan sequence, presenting the same recurrence relation, differing only the initial terms, and even a characteristic polynomial. Thus, its recurrence is defined as $Pe_n = Pe_{n-2} + Pe_{n-3}$, for $n \geq 3$, being $Pe_0 = 3$, $Pe_1 = 0$ and $Pe_2 = 2$ its initial terms, as this sequence has the same recurrence as the Padovan sequence, the same characteristic polynomial can be presented $x^3 - x - 1 = 0$, having the same roots seen previously. Therefore, Perrin's Binet's formula is given by

$$Pe_n = \frac{(3x_2x_3+2)}{(x_1-x_2)(x_1-x_3)}x_1^n + \frac{(3x_1x_3+2)}{(x_2-x_1)(x_2-x_3)}x_2^n + \frac{(3x_1x_2+2)}{(x_3-x_1)(x_3-x_2)}x_3^n$$

Thus, this sequence changes only its initial values to $Pe_0 = 3, Pe_1 = 0$ and $Pe_2 = 2$, resulting in $A_0 = a_2Pe_0 + a_1Pe_1 + a_0Pe_2 = 3$, $A_1 = a_2Pe_1 + a_1Pe_2 = 2$ and $A_2 = a_2Pe_2 = 2$.

Using the BenTaher-Rachidi method, we have that from the polynomial $p(\lambda) = \lambda^3 - \lambda - 1$, the derivative is then calculated, resulting in $p'(\lambda) =$

$2\lambda^2 - 1$. Therefore, according to the formula of the method and the sequence coefficients, we have that

$$A_0 = a_2Pa_0 + a_1Pa_1 + a_0Pa_2 = 2$$
$$A_1 = a_2Pa_1 + a_1Pa_2 = 2 \text{ and}$$
$$A_2 = a_2Pa_2 = 1.$$

That done, we have that, from the Formula (3), we get

$$\begin{aligned} Pe_n &= \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}} + \frac{A_2}{\lambda_1^{2+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}} + \frac{A_2}{\lambda_2^{2+1}}\right)\lambda_2^n \\ &\quad + \frac{1}{p'(\lambda_3)}\left(\frac{A_0}{\lambda_3^{0+1}} + \frac{A_1}{\lambda_3^{1+1}} + \frac{A_2}{\lambda_3^{2+1}}\right)\lambda_3^n \end{aligned}$$

Using Girard relations,

$$x_1x_2x_3 = 1, \quad x_1 + x_2 + x_3 = 0, \quad \text{and} \quad x_1x_2 + x_1x_3 + x_2x_3 = -1$$

and the operations previously presented, we have that

$$\begin{aligned} Pe_n &= \frac{1}{3x_1^2 - 1}\left(\frac{3}{x_1} + \frac{2}{x_1^2} + \frac{2}{x_1^3}\right)(x_1)^n + \frac{1}{3x_2^2 - 1}\left(\frac{3}{x_2} + \frac{2}{x_2^2} + \frac{2}{x_2^3}\right)(x_2)^n \\ &\quad + \frac{1}{3x_3^2 - 1}\left(\frac{3}{x_3} + \frac{2}{x_3^2} + \frac{2}{x_3^3}\right)(x_3)^n \\ &= \left(\frac{3x_1^2 + 2x_1 + 2}{3x_1^5 - x_1^3}\right)(x_1)^n + \left(\frac{3x_2^2 + 2x_2 + 2}{3x_2^5 - x_2^3}\right)(x_2)^n + \left(\frac{3x_3^2 + 2x_3 + 2}{3x_3^5 - x_3^3}\right)(x_3)^n \\ &= \left[\frac{3x_1^2 + 2x_1 + 2}{x_1(3x_1^4 - x_1^2)}\right](x_1)^n + \left[\frac{3x_2^2 + 2x_2 + 2}{x_2(3x_2^4 - x_2^2)}\right](x_2)^n + \left[\frac{3x_3^2 + 2x_3 + 2}{x_3(3x_3^4 - x_3^2)}\right](x_3)^n \\ &= \left[\frac{3x_1 - 2x_1x_3 - 2x_1x_2}{x_1(3x_1^3 - x_1)}\right](x_1)^n + \left[\frac{3x_2 - 2x_1x_2 - 2x_2x_3}{x_2(3x_2^3 - x_2)}\right](x_2)^n + \\ &\quad \left[\frac{3x_3 - 2x_1x_3 - 2x_2x_3}{x_3(3x_3^3 - x_3)}\right](x_3)^n \\ &= \left[\frac{3 - 2x_3 - 2x_2}{x_1(3x_1^2 - 1)}\right](x_1)^n + \left[\frac{3 + x_1 - 2x_3}{x_2(3x_2^2 - 1)}\right](x_2)^n + \left[\frac{3 - 2x_1 - 2x_2}{x_3(3x_3^2 - 1)}\right](x_3)^n \\ &= \left(\frac{3x_2x_3 + 2}{3x_1^2 - 1}\right)(x_1)^n + \left(\frac{3x_1x_2 + 2}{3x_2^2 - 1}\right)(x_2)^n + \left(\frac{3x_1x_2 + 2}{3x_3^2 - 1}\right)(x_3)^n \\ &= \left(\frac{3x_2x_3 + 2}{2x_1x_1 + x_1^2 - 1}\right)(x_1)^n + \left(\frac{3x_1x_2 + 2}{2x_2x_2 + x_2^2 - 1}\right)(x_2)^n + \left(\frac{3x_1x_2 + 2}{2x_3x_3^3 + x_3^2 - 1}\right)(x_3)^n \end{aligned}$$

that is

$$Pe_n = \left(\frac{3x_2x_3+2}{x_1^2-x_1x_2-x_1x_3+x_2x_3}\right)(x_1)^n + \left(\frac{3x_1x_2+2}{x_2^2-x_1x_2-x_2x_3+x_1x_3}\right)(x_2)^n + \left(\frac{3x_1x_2+2}{x_3^2-x_1x_3-x_2x_3+x_1x_2}\right)(x_3)^n$$

$$= \left(\frac{3x_2x_3+2}{(x_1-x_2)(x_1-x_3)}\right)(x_1)^n + \left(\frac{3x_1x_2+2}{(x_2-x_1)(x_2-x_3)}\right)(x_2)^n + \left(\frac{3x_1x_2+2}{(x_3-x_1)(x_3-x_2)}\right)(x_3)^n$$

However, there is the formula presented in Equation (10).

3.9 Narayana Sequence

The Narayana sequence was introduced by the Indian mathematician Narayana Pandita (1340 - 1400) and, similarly to the Fibonacci sequence, it is derived from a problem that presents the numbers of Narayana is that of the herd of cows and calves that was proposed by Narayana in the 14th century, in which: *"A cow gives birth to a calf every year. In turn, the calf gives birth to another calf when it is three years old. What is the number of progenies produced for twenty years by a cow?"* [12]. When answering this problem, one finds the terms that make up the Narayana sequence, which are $1, 1, 1, 2, 3, 4, 6, 9, 13 \cdots$.

The Narayana sequence is a third order numerical sequence, presenting its recurrence formula $N_n = N_{n-1} + N_{n-3}$, for $n \geq 3$ and with the initial values $N_0 = N_1 = N_2 = 1$. Its respective characteristic polynomial is given by the equation $x^3 - x^2 - 1 = 0$, with three roots $\alpha \approx 1,465$, $\beta = 0,108(0,866i - 0,5) + 1,02(-0,866i - 0,5) + 0,3$ and $\gamma = 1,023(0,866i - 0,5) + 0,108(-0,866i - 0,5) + 0,3$. Next, Narayana's Binet's formula given by

$$N_n = \left(\frac{\alpha}{3\alpha-2}\right)(\alpha)^n + \left(\frac{\beta}{3\beta-2}\right)(\beta)^n + \left(\frac{\gamma}{3\gamma-2}\right)(\gamma)^n \qquad (10)$$

Applying the BenTaher-Rachidi method, we have that from the polynomial $p(\lambda) = \lambda^3 - \lambda^2 - 1$, the derivative is calculated, resulting in $p'(\lambda) = 3\lambda^2 - 2\lambda$. Thus, from the formula of the method and the sequence coefficients, we have that: $A_0 = a_2N_0 + a_1N_1 + a_0N_2 = 2$, $A_1 = a_2N_1 + a_1N_2 = 1$ and $A_2 = a_2N_2 = 1$. From the Formula 3, Girard's relations $\alpha\beta\gamma = 1$,$\alpha + \beta + \gamma = 1$

and $\alpha\beta + \alpha\gamma + \beta\gamma = 0$ and the operations previously presented, we have that

$$
\begin{aligned}
N_n &= \frac{1}{3\alpha^2 - 2\alpha}\left(\frac{2}{\alpha} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3}\right)(\alpha)^n + \frac{1}{3\beta^2 - 2\beta}\left(\frac{2}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3}\right)(\beta)^n \\
&\qquad + \frac{1}{3\gamma^2 - 2\gamma}\left(\frac{2}{\gamma} + \frac{1}{\gamma^2} + \frac{1}{\gamma^3}\right)(\gamma)^n \\
&= \left(\frac{2\alpha^2 + \alpha + 1}{3\alpha^5 - 2\alpha^4}\right)(\alpha)^n + \left(\frac{2\beta^2 + \beta + 1}{3\beta^5 - 2\beta^4}\right)(\beta)^n + \left(\frac{2\gamma^2 + \gamma + 1}{3\gamma^5 - 2\gamma^4}\right)(\gamma)^n \\
&= \left[\frac{2\alpha + \alpha\beta\gamma - \alpha\beta - \alpha\gamma}{\alpha(3\alpha^3 - 2\alpha^2)}\right](\alpha)^n + \left[\frac{2\beta + \alpha\beta\gamma - \alpha\beta - \beta\gamma}{\beta(3\beta^3 - \beta^2)}\right](\beta)^n \\
&\qquad + \left[\frac{2\gamma + \alpha\beta\gamma - \alpha\gamma - \beta\gamma}{\gamma(3\gamma^3 - \gamma^2)}\right](\gamma)^n \\
&= \left[\frac{2 + \beta\gamma - \beta - \gamma}{\alpha(3\alpha^2 - 2\alpha)}\right](\alpha)^n + \left[\frac{2 + \alpha\gamma - \alpha - \gamma}{\beta(3\beta^2 - 2\beta)}\right](\beta)^n + \left[\frac{2 + \alpha\beta - \alpha - \beta}{\gamma(3\gamma^2 - 2\gamma)}\right](\gamma)^n \\
&= \left[\frac{-\alpha\beta - \alpha\gamma + \alpha}{\alpha(3\alpha - 2)}\right](\alpha)^n + \left[\frac{-\alpha\beta - \beta\gamma + \beta}{\beta(3\beta - 2)}\right](\beta)^n + \left[\frac{-\alpha\gamma - \beta\gamma + \gamma}{\gamma(3\gamma - 2)}\right](\gamma)^n \\
&= \left(\frac{\alpha}{3\alpha - 2}\right)(\alpha)^n + \left(\frac{\beta}{3\beta - 2}\right)(\beta)^n + \left(\frac{\gamma}{3\gamma - 2}\right)(\gamma)^n
\end{aligned}
$$

Finally, there is the formula presented in Equation (10).

4 Conclusion

Arising from the junction of the resolution of the Binet's formula with the Vandermonde system, this worked presented the application of the resolution through the BenTaher-Rachidi method. Thus, based on the work of [8, 15], it was possible to discuss this new way of obtaining the Binet's formula of the Lucas, Pell, Leonardo, Mersenne, Oreseme, Jacobsthal, Padovan, Perrin and Narayana sequences.

Thus, this method makes the resolution simpler, despite presenting the calculation of the derivative of a function, presenting itself as an alternative way of solving the Binet's formula. For future work, research is projected from different perspectives, such as its application in the area of computing, applied science and others.

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