Bessel Potentials and Lions-Calderón Spaces

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Resumo: O principal objetivo deste trabalho é apresentar um gradiente fracionário, utilizado por Tien-Tsan Shieh e Daniel E. Spector em 2015 para estudar uma nova classe de equações com derivadas parciais. Esse gradiente fracionário permite dar uma caracterização para os espaços de Lions-Calderón (que são também conhecidos na literatura como espaços de potenciais de Bessel) idêntica à caracterização usual dos espaços de Sobolev. Para além desta interessante caracterização, apresentamos ainda dois resultados, um sobre existência e unicidade de soluções fracas para uma equação com derivadas parciais fracionárias e uma caracterização do dual dos espaços de Lions-Calderón em termos de derivadas parciais fracionárias, que correspondem a uma generalização dos correspondentes resultados no caso clássico (i.e. $s = 1$) para o caso fracionário (i.e. $s \in (0, 1)$).

Abstract The main goal of this short survey is to present a fractional gradient, used by Tien-Tsan Shieh and Daniel E. Spector in 2015, to study a new class of partial differential equations. This fractional gradient allows us to provide a characterization of the Lions-Calderón spaces (also known in the literature as Bessel potential spaces) similar to the usual characterization of the Sobolev spaces. In addition to this characterization, we also present two results, one about existence and uniqueness of weak solutions to a fractional partial differential equation and a characterization of the dual space of the Lions-Calderón spaces with the help of fractional partial derivatives, which correspond to a generalization of the corresponding results in the classical case (i.e., $s = 1$) to the fractional case (i.e. $s \in (0, 1)$).

palavras-chave: Espaços de Lions-Calderón; Potenciais de Bessel; Gradiente fracionário de Riesz distribucional.

keywords: Lions-Calderón spaces; Bessel potentials; Distributional Riesz fractional gradient.

1 Bessel Potentials

In Harmonic Analysis and as well as in potential theory two important potentials are studied, the Riesz potential that is related to the powers of the Laplacian, i.e., $(-\Delta)^{-s/2}$ for $0 < s < N$, and the Bessel potential, which was studied primarily by Aronszajn and Smith in [2] and [3], and is related to the powers of the Helmholtz operator, i.e., $(I - \Delta)^{-s/2}$. Without further ado we are going to introduce the two potentials here, although the Riesz potential will only be used explicitly in the second section.

Definition 1.1. *Let $0 < s < N$ and $x \in \mathbb{R}^N$. The Riesz potential of order s, $\mathcal{I}_s$, is defined by*

$$\mathcal{I}_s f(x) = (I_s * f)(x) = \frac{1}{\gamma(N,s)} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-s}}\, dy$$

where

$$I_s(x) = \frac{1}{\gamma(N,s)|x|^{N-s}}$$

is called the Riesz kernel, and

$$\gamma(N,s) = \frac{\pi^{N/2} 2^s \Gamma(s/2)}{\Gamma((N-s)/2)}.$$

We point out without giving a proof (but the interested reader can see it in [14]), that one has the identity $\mathscr{F} I_s(\xi) = (2\pi|\xi|)^{-s}$ and so, for $\varphi \in \mathscr{S}(\mathbb{R}^N)$ (or $\varphi \in \mathscr{S}'(\mathbb{R}^N)$), one has $\mathscr{F}(\mathcal{I}_s\varphi)(\xi) = (2\pi|\xi|)^{-s}\mathscr{F}\varphi(\xi)$.[1]

Definition 1.2. *Let $s \in \mathbb{R}$ and $x \in \mathbb{R}^N$. We define the Bessel kernel G_s of order s as being*

$$G_s(x) = \mathscr{F}^{-1}\left((1 + 4\pi^2|\xi|^2)^{-s/2}\right)(x),$$

and consequently, for $u \in \mathscr{S}'(\mathbb{R}^N)$ we define the Bessel potential $\mathscr{J}_s$ of order s of u as

$$\mathscr{J}_s u = G_s * u = \mathscr{F}^{-1}\left((1 + 4\pi^2|\xi|^2)^{-s/2}\mathscr{F}u(\xi)\right).$$

[1] $\mathscr{S}(\mathbb{R}^N)$ will always denote the Schwartz space whose elements are usually called rapidly decreasing functions, and consequently, its dual space, often called the space of tempered distributions, will be denoted by $\mathscr{S}'(\mathbb{R}^N)$. The interest of this spaces concerns mainly with the fact that they are useful when one wants to apply the Fourier transform $\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}^N} \varphi(x)e^{-2\pi i x\cdot\xi}\, dx$. One should also note that some coefficients that appear when one takes the Fourier transform of a function might differ depending on the definition of this transform.

We note that the map $\mathscr{J}_s : \mathscr{S}(\mathbb{R}^N) \to \mathscr{S}(\mathbb{R}^N)$ is in fact onto because if $\psi \in \mathscr{S}(\mathbb{R}^N)$, then $\mathscr{F}\psi \in \mathscr{S}(\mathbb{R}^N)$ and consequentely $\xi \mapsto \mathscr{F}\varphi(\xi) = (1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}\psi(\xi) \in \mathscr{S}(\mathbb{R}^N)$, which implies both that $\varphi \in \mathscr{S}(\mathbb{R}^N)$ and that $\psi = \mathscr{J}_s\varphi$.

Now we are going to present some properties about the Bessel potential on Lebesgue spaces. For that to make sense we must notice that when we take the Fourier transform of a function in L^p, with $p \neq 1$ or 2, what is actually happening is that we are taking it on the tempered distribution that is associated to that function.

Theorem 1.1 (Integral characterization of the Bessel kernel; see [12])**.** *Assume that $s > 0$. Then,*

(1) $G_s(x) = \frac{1}{(4\pi)^{N/2}\Gamma(s/2)} \int_0^\infty e^{-\frac{t}{4\pi}} e^{-\frac{|x|^2\pi}{t}} t^{\frac{s-N}{2}} \frac{dt}{t}$*; and*

(2) $G_s(x) \in L^1(\mathbb{R}^N)$.

Proof.

(1) Using a well-known property of the Γ-functions, we have for $a, s > 0$ the following equality holds

$$a^{-s/2} = \frac{1}{\Gamma(s/2)} \int_0^{+\infty} e^{-ta} t^{s/2} \frac{dt}{t}. \tag{1}$$

Setting $a = \frac{(1+4\pi^2|\xi|^2)}{4\pi}$ we obtain

$$(4\pi)^{s/2}(1+4\pi^2|\xi|^2)^{-s/2} = \frac{1}{\Gamma(s/2)} \int_0^{+\infty} e^{-\frac{t}{4\pi}} e^{-\pi t|\xi|^2} t^{s/2} \frac{dt}{t}. \tag{2}$$

By taking the inverse Fourier transform of (2), and applying the Tonelli theorem and the inverse Fourier transform of the Gaussian function, we get

$$\begin{aligned} G_s(x) &= \frac{1}{(4\pi)^{s/2}\Gamma(s/2)} \mathscr{F}^{-1} \left(\int_0^{+\infty} e^{-\frac{t}{4\pi}} e^{-\pi t|\xi|^2} t^{s/2} \frac{dt}{t} \right)(x) \\ &= \frac{1}{(4\pi)^{s/2}\Gamma(s/2)} \int_0^{+\infty} e^{-\frac{t}{4\pi}} \mathscr{F}^{-1}\left(e^{-\pi t|\xi|^2}\right)(x) t^{s/2} \frac{dt}{t} \\ &= \frac{1}{(4\pi)^{N/2}\Gamma(s/2)} \int_0^{\infty} e^{-\frac{t}{4\pi}} e^{-\frac{|x|^2\pi}{t}} t^{\frac{s-N}{2}} \frac{dt}{t}. \end{aligned}$$

(2) Using the integral characterization obtained in the previous item, the fact that $\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}$ and the identity (1), we obtain the following chain of equalities

$$\int_{\mathbb{R}^N} G_s(x)\,dx = \frac{1}{(4\pi)^{s/2}\Gamma(s/2)} \int_{\mathbb{R}^N} \int_0^{+\infty} e^{-\frac{t}{4\pi}} e^{-\frac{\pi|x|^2}{t}} t^{\frac{s-N}{2}} \frac{dt}{t}\,dx$$
$$= \frac{1}{(4\pi)^{s/2}\Gamma(s/2)} \int_0^{+\infty} e^{-\frac{t}{4\pi}} t^{\frac{s-N}{2}} \left(\int_{\mathbb{R}^N} e^{-\frac{\pi|x|^2}{t}}\,dx\right) \frac{dt}{t}$$
$$= \frac{1}{(4\pi)^{s}\Gamma(s/2)} \int_0^{+\infty} e^{-\frac{t}{4\pi}} t^{\frac{s-N}{2}} t^{N/2} \frac{dt}{t} = 1.$$

□

As a consequence of this theorem when $s > 0$ we have that $\mathscr{J}_s : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a continuous map, since by Young's inequality for convolution we have

$$\|\mathscr{J}_s u\|_{L^p(\mathbb{R}^N)} = \|G_s * u\|_{L^p(\mathbb{R}^N)} \leq \|G_s\|_{L^1(\mathbb{R}^N)}\|u\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)}.$$

Lemma 1.1 (see [14]). *For $s \in \mathbb{R}$, $\mathscr{J}_s : L^p(\mathbb{R}^N) \to \mathscr{S}'(\mathbb{R}^N)$ is injective.*

Proof. Let $g_1, g_2 \in L^p(\mathbb{R}^N)$ such that $\mathscr{J}_s(g_1) = \mathscr{J}_s(g_2)$ and consider $\varphi \in \mathscr{S}(\mathbb{R}^N)$ a rapidly decreasing function. Applying Fubini twice we get that

$$\int_{\mathbb{R}^N} \mathscr{J}_s(g_1)(x)\varphi(x)dx = \int_{\mathbb{R}^N\times\mathbb{R}^N} G_s(x-y)g_1(y)\varphi(x)\,dxdy$$
$$= \int_{\mathbb{R}^N} g_1(y)\,\mathscr{J}_s(\varphi)(y)dy.$$

After doing the same to g_2 we obtain that $\int_{\mathbb{R}^N} (g_1 - g_2)\mathscr{J}_s(\varphi)\,dx = 0$ for all $\varphi \in \mathscr{S}(\mathbb{R}^{\mathbb{N}})$. We have already pointed out that $\mathscr{J}_s : \mathscr{S}(\mathbb{R}^N) \to \mathscr{S}(\mathbb{R}^N)$ is surjective so $\int_{\mathbb{R}^N} (g_1 - g_2)\psi\,dx = 0$ for all $\psi \in \mathscr{S}(\mathbb{R}^N)$, and by density this means that $g_1 = g_2$ a.e.. □

For the next property that relates the Riesz potential with the Bessel potential we need the following lemma.

Lemma 1.2 (Wiener's Theorem; see [12]). *If $\varphi_1 \in L^1(\mathbb{R}^N)$ and $\mathscr{F}\varphi_1 + 1 \neq 0$ everywhere, then there exists $\varphi_2 \in L^1(\mathbb{R}^N)$ such that $(\mathscr{F}\varphi_1(\xi) + 1)^{-1} = \mathscr{F}\varphi_2(\xi) + 1$ for all $\xi \in \mathbb{R}^N$.*

Proof. See [12] pages 249-251. □

Theorem 1.2 (Relation between the Riesz and Bessel potentials; see [12] and [14])**.** *Let $s > 0$.*

(1) There exists a finite measure μ_s on $\mathbb{R}^N$ so that its Fourier transform[2] *is given by*

$$\mathscr{F}\mu_s(\xi) = \frac{(2\pi|\xi|)^s}{(1+4\pi^2|\xi|^2)^{s/2}}.$$

(2) There exists a finite signed measure ν_s on $\mathbb{R}^N$ so that

$$(1+4\pi^2|x|^2)^{s/2} = \mathscr{F}\nu_s(x)\,(1+(2\pi|x|)^s)\,.$$

Proof.

(1) The idea is to use the Taylor expansion $(1-t)^{s/2} = 1 + \sum_{m=1}^{\infty} A_{m,s}t^m$ where $A_{m,s} = \frac{(-s/2)(1-(s/2))\cdots(m-1-(s/2))}{m!}$, valid for all $|t| < 1$. Note that for $m > (s/2)+1$, $A_{m,s}$ has always the same sign and $(1-t)^{s/2}$ remains bounded as $t \to 1$, which imply that $\sum_{m=1}^{\infty} |A_{m,s}| < +\infty$. So, if we set $t = (1+4\pi^2|\xi|^2)^{-1}$ we obtain

$$\left(\frac{4\pi^2|\xi|^2}{1+4\pi^2|\xi|^2}\right)^{s/2} = 1 + \sum_{m=1}^{\infty} A_{m,s}(1+4\pi^2|\xi|^2)^{-m}.$$

By defining the signed measure $\mu_s = \delta_0 + \left(\sum_{m=1}^{\infty} A_{m,s}G_{2m}\right)dx$ we obtain the desired result.

(2) This proof is just a consequence of Lemma 1.2 and so we just need to check its hypothesis. Consider $\varphi_1(x) = G_s(x) + \sum_{m=1}^{\infty} A_{m,s}G_{2m}(x)$. The first thing we need to check is the integrability of φ_1, in fact we note that using the monotone convergence theorem we obtain that $\varphi_1 \in L^1(\mathbb{R}^N)$ because $\|G_j\|_{L^1(\mathbb{R}^N)} = 1$ for all $j > 0$ and the series $\sum_{m=1}^{\infty} A_{m,s}$ converges absolutely; the second and last thing that we need to check is that $\mathscr{F}\varphi_1(\xi) + 1$ is nowhere vanishing. Indeed $\mathscr{F}\varphi_1(\xi) = \mathscr{F}\mu_s(\xi) - \mathscr{F}\delta_0(\xi) + \mathscr{F}G_s(\xi)$ which implies that

$$\mathscr{F}\varphi_1(\xi) + 1 = \frac{(2\pi|\xi|)^s + 1}{(1+4\pi^2|\xi|^2)^{s/2}} > 0 \quad \forall \xi \in \mathbb{R}^N.$$

Then, we can apply Lemma 1.2 in order to conclude that there exists a function $\varphi_2 \in L^1(\mathbb{R}^N)$ such that $(1+4\pi^2|\xi|^2)^{s/2} = (1+(2\pi|\xi|^s))(\mathscr{F}\varphi_2(\xi)+1)$ which gives the desired result with $\nu_s = \delta_0 + \varphi_2(x)dx$.

[2]the Fourier transform of a Radon measure μ is given by $\mathscr{F}\mu(y) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot y}\,d\mu(x)$

□

Remark. *Like we pointed out in the beginning of this section, the Riesz and Bessel potentials are related with some real powers of the Laplacian and the Helmholtz operators, respectively. Although the Riesz potential is only defined for* $0 < s < N$*, the Laplacian operator can be defined for all* s *as* $u \mapsto (-\Delta)^s u := \mathscr{F}^{-1}(|2\pi\xi|^{2s}\mathscr{F}u)$*. As a consequence, the previous theorem establishes, in the first point, a boundedness result for the formal quotient operator*

$$\frac{(-\Delta)^{s/2}}{(I-\Delta)^{s/2}}, \quad s > 0 \tag{3}$$

on every $L^p(\mathbb{R}^N)$ *with* $1 \leq p \leq +\infty$*; while the second point allows us to write the Fourier transform of the positive powers of the Helmholtz operator as the product of the Fourier transform of a measure with the sum of 1 (the identity of* $\mathbb{R}^N$*) and the Fourier transform of the Laplacian with the same power, which is interesting because the Helmholtz operator is the sum of the identity operator* I *with the Laplacian.*

2 Lions-Calderón Spaces

In this section we will deal with a space that we called Lions-Calderón space. This space was introduced by Aronzajn and Smith in [2] and in [3], in the Hilbertian case, $p = 2$, and later the ideas were generalized by A. P. Calderón in [4], J. L. Lions in [10] and J. L. Lions and E. Magenes in [11] to the non-Hilbertian case, $p \neq 2$. It is important to have in mind that A. P. Calderón's work has its foundations and is more directed towards Harmonic Analysis, while J. L. Lions' work has its foundations in the theory of complex interpolation in Functional Analysis (in fact, these spaces appear only as an example in [10]) and is later used in [11] to study the regularity of the non-homogeneous Dirichlet problem for an elliptic partial differential equation of order $2m$. Since from the beginning these spaces were more used in the theory of Harmonic Analysis rather than in the theory of partial differential equations (although, in recent years some interest has begun to emerge in the theory of partial differential equations), the notations and nomenclature used nowadays are generally the ones that came from Harmonic Analysis, which as we will explain later has some drawbacks. In order to make a historical appreciation and establish links between these two perspectives, we will introduce these spaces in the way that both authors did[3].

Calderón presented this space in the following way:"Let s be a real number and $1 \leq p \leq +\infty$. We define $L^p_s(\mathbb{R}^N)$ to be the image of $L^p(\mathbb{R}^N)$ under $\mathscr{J}_s$. If $f \in L^p_s(\mathbb{R}^N)$ then $f = \mathscr{J}_s g$ for some $g \in L^p(\mathbb{R}^N)$. This g is unique; we define the norm $\|f\|_{p,s}$ of $f \in L^p_s(\mathbb{R}^N)$ by $\|f\|_{p,s} = \|g\|_p$." In the last years some authors (for example, [13] and [14]) adopted the notation $L^{p,s}(\mathbb{R}^N)$ and called this space, Bessel potential space.[4]

In contrast, Lions presented it in the following way: "We will indicate with $H^{s,p}(\mathbb{R}^N)$, $1 < p < +\infty$, s real, the (Banach) space of tempered distributions u such that $\mathscr{F}^{-1}((1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}u) \in L^p(\mathbb{R}^N)$ with the norm $\|u\|_{H^{s,p}(\mathbb{R}^N)} = \|\mathscr{F}^{-1}((1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}u)\|_{L^p(\mathbb{R}^N)}$."[5]

At first sight $L^p_s(\mathbb{R}^N)$ and $H^{s,p}(\mathbb{R}^N)$ may be different spaces, however we note that they are exactly the same, since

$$\begin{aligned} u \in H^{s,p}(\mathbb{R}^N) &\Leftrightarrow \exists f \in L^p(\mathbb{R}^N) : f = \mathscr{F}^{-1}\left((1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}u\right) \\ &\Leftrightarrow \exists f \in L^p(\mathbb{R}^N) : u = \mathscr{F}^{-1}\left((1+4\pi^2|\xi|^2)^{-s/2}\mathscr{F}f\right) \\ &\Leftrightarrow u \in L^p_s(\mathbb{R}^N). \end{aligned}$$

Having said that both spaces are equal, we point out that both notations have some problems. The first is related to the fact that there are more mathematical relevant spaces than letters in the latin alphabet, for example $L^{s,p}$ is already used to denote the Lorentz space (see [7] and [15]), and $H^{s,p}$ is used for example to denote Nikol'skii spaces (see [1]). In order to overcome this problem, for the rest of this work we will use the notation $\Lambda^{s,p}(\mathbb{R}^N)$ to denote these spaces[6] with the norm $\|u\|_{\Lambda^{s,p}(\mathbb{R}^N)} = \|\mathscr{F}^{-1}((1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}u)\|_{L^p(\mathbb{R}^N)} = \|\mathscr{J}_{-s}u\|_{L^p(\mathbb{R}^N)}$. Another important aspect that we touch here is the nomenclature of these spaces. There has been a solid tradition in functional analysis and in the theory of partial differential equations to name the spaces in honor to the authors that introduced them (we can point out several examples such as the Lebesgue spaces, Sobolev spaces, Besov spaces and Triebel-Lizorkin spaces), and so,

[3]With minor alterations to make it compatible with our notation, for example in [4] the author uses E_n to denote the n-dimensional euclidean space $\mathbb{R}^n$ and u as the real number that we have been denoting by s.

[4]It is also interesting to note that some authors with formation in Harmonic Analysis also call this spaces generalized Sobolev spaces as in [8].

[5]Translated freely from the french.

[6]to our knowledge this notation does not coincide with the notation of any other space

in order to continue this long standing tradition we propose the name Lions-Calderón spaces for $\Lambda^{s,p}$.

After this introduction, we are going to present now some properties about Lions-Calderón spaces.

Lemma 2.1 (see [9]). *Let $s \in \mathbb{R}$ and $p \in [1, +\infty]$. Then $\Lambda^{s,p}(\mathbb{R}^N)$ is a Banach space.*

Proof. First we observe that $\Lambda^{s,p}(\mathbb{R}^N)$ with $s \in \mathbb{R}$ and $p \in [1, +\infty]$ is a vector space because $\mathscr{J}_s$ is linear; and is a normed vector space thanks to the properties of the norms in $L^p(\mathbb{R}^N)$ and injectivity of the Bessel potential. However we still need to prove that $\Lambda^{s,p}(\mathbb{R}^N)$ is complete. For that consider a Cauchy sequence $\{f_m\}_{m\in\mathbb{N}} \subset \Lambda^{s,p}(\mathbb{R}^N)$, by the definition of the norm $\|\cdot\|_{\Lambda^{s,p}(\mathbb{R}^N)}$, $\{\mathscr{J}_{-s} f_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^N)$, and so there is a $g \in L^p(\mathbb{R}^N)$ such that $\|\mathscr{J}_{-s} f_m - g\|_{L^p(\mathbb{R}^N)} \to 0$ as $m \to +\infty$. And again using the properties of the norm $\|\cdot\|_{\Lambda^{s,p}(\mathbb{R}^N)}$ we see that $\|f_m - \mathscr{J}_s g\|_{\Lambda^{s,p}(\mathbb{R}^N)} = \|\mathscr{J}_{-s} f_m - g\|_{L^p(\mathbb{R}^N)}$ and so $\{f_m\}_{m\in\mathbb{N}}$ is convergent in $\Lambda^{s,p}(\mathbb{R}^N)$. □

Lemma 2.2 (see [14]). *Let $s \in \mathbb{R}$ and $1 \leq p \leq +\infty$. Then $\Lambda^{s+\varepsilon,p}(\mathbb{R}^N) \subset \Lambda^{s,p}(\mathbb{R}^N)$ with $\|f\|_{\Lambda^{s,p}(\mathbb{R}^N)} \leq \|f\|_{\Lambda^{s+\varepsilon,p}(\mathbb{R}^N)}$ for all $\varepsilon > 0$.*

Proof. Let $f \in \Lambda^{s+\varepsilon,p}(\mathbb{R}^N)$. Then

$$\begin{aligned}\|f\|_{\Lambda^{s,p}(\mathbb{R}^N)} &= \|\mathscr{J}_{-s} f\|_{L^p(\mathbb{R}^N)} = \|\mathscr{J}_{\varepsilon}(\mathscr{J}_{-s-\varepsilon} f)\|_{L^p(\mathbb{R}^N)} \\ &\leq \|\mathscr{J}_{-s-\varepsilon} f\|_{L^p(\mathbb{R}^N)} = \|f\|_{\Lambda^{s+\varepsilon,p}(\mathbb{R}^N)}.\end{aligned}$$

□

Lemma 2.3 (see [9]). *Let $s \in \mathbb{R}$ and $p \in [1, +\infty)$, then $\mathscr{S}(\mathbb{R}^N)$ is dense in $\Lambda^{s,p}(\mathbb{R}^N)$.*

Proof. Let $f \in \Lambda^{s,p}(\mathbb{R}^N)$. Since $\mathscr{S}(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for $p \in [1, +\infty)$, this means that for every $\varepsilon > 0$ there exists a $g \in \mathscr{S}(\mathbb{R}^N)$ such that

$$\|f - \mathscr{J}_s g\|_{\Lambda^{s,p}(\mathbb{R}^N)} = \|\mathscr{J}_{-s} f - g\|_{L^p(\mathbb{R}^N)} < \varepsilon.$$

At the same, using the fact $\mathscr{J}_s : \mathscr{S}(\mathbb{R}^N) \to \mathscr{S}(\mathbb{R}^N)$, we obtain that $\mathscr{J}_s g \in \mathscr{S}(\mathbb{R}^N)$ and then $\mathscr{S}(\mathbb{R}^N)$ is dense in $\Lambda^{s,p}(\mathbb{R}^N)$. □

In fact we have a stronger result, but first we need to state the following one.

Theorem 2.1 (see [4],[9] and [14]). *Suppose k is a positive integer and $1 < p < +\infty$. Then $W^{k,p}(\mathbb{R}^N) = \Lambda^{k,p}(\mathbb{R}^N)$ and the two norms are equivalent.*

The idea of the proof is the following: using a similar argument as in the proof of Theorem 2.4 below with $s = 1$ we are able to prove that $u \in \Lambda^{1,p}(\mathbb{R}^N)$ if and only if u and $\frac{\partial u}{\partial x_j}$ (taken in the sense of distributions),where $j = 1, ..., N$, are elements of $\Lambda^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$, and then the case with $k \geq 1$ follows immediately. For the complete proof we suggest [14].

Theorem 2.2 (Density; cf. [11]). *Let $s \geq 0$ and $p \in (1, +\infty)$, then $C_c^\infty(\mathbb{R}^N)$ is dense in $\Lambda^{s,p}(\mathbb{R}^N)$.*

Proof. Consider a function $f \in \Lambda^{s,p}(\mathbb{R}^N)$ and let $\varepsilon > 0$ arbitrary. Consider, by Lemma 2.3, a function $g \in \mathscr{S}(\mathbb{R}^N)$ such that $\|f-g\|_{\Lambda^{s,p}(\mathbb{R}^N)} \leq \varepsilon$. Then let $k > s$ be a positive integer and let $h \in C_c^\infty(\mathbb{R}^N)$ such that $\|h-g\|_{W^{k,p}(\mathbb{R}^N)} \leq \varepsilon$ (this being possible because $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}(\mathbb{R}^N)$). Since the norms of $W^{k,p}(\mathbb{R}^N)$ and $\Lambda^{k,p}(\mathbb{R}^N)$ are equivalent by Theorem 2.1, and $\|g - h\|_{\Lambda^{s,p}(\mathbb{R}^N)} \leq \|g - h\|_{\Lambda^{k,p}(\mathbb{R}^N)}$ by Lemma 2.2, then $\|g - h\|_{\Lambda^{s,p}(\mathbb{R}^N)} < C\varepsilon$, where C is a positive constant that does not depend on g or h, allowing us to conclude that $\|f - h\|_{\Lambda^{s,p}(\mathbb{R}^N)} \leq (1+C)\varepsilon$ and consequently that $C_c^\infty(\mathbb{R}^N)$ is dense in $\Lambda^{s,p}(\mathbb{R}^N)$. □

Lemma 2.4. *If $s \in \mathbb{R}$, $p \in (1, +\infty)$ and $p' = \frac{p}{p-1}$, then $(\Lambda^{s,p}(\mathbb{R}^N))' \cong \Lambda^{-s,p'}(\mathbb{R}^N)$.*

Proof. We just need to prove that if $\varphi \in \left(\Lambda^{s,p}(\mathbb{R}^N)\right)'$ then $\varphi \in \Lambda^{-s,p'}(\mathbb{R}^N)$, taken the appropriate isomorphisms. Suppose that $\varphi \in \left(\Lambda^{s,p}(\mathbb{R}^N)\right)' \subset \mathscr{S}'(\mathbb{R}^N)$ and so

$$\|\varphi\|_{(\Lambda^{s,p}(\mathbb{R}^N))'} = \sup_{\substack{f \in \Lambda^{s,p}(\mathbb{R}^N) \\ f \neq 0}} \frac{|\langle \varphi, f \rangle|}{\|f\|_{\Lambda^{s,p}(\mathbb{R}^N)}} = \sup_{\substack{g \in L^p(\mathbb{R}^N) \\ g \neq 0}} \frac{|\langle \varphi, \mathscr{J}_s g \rangle|}{\|g\|_{\Lambda^p(\mathbb{R}^N)}}$$

$$= \sup_{\substack{g \in L^p(\mathbb{R}^N) \\ g \neq 0}} \frac{|\langle \mathscr{J}_s \varphi, g \rangle|}{\|g\|_{\Lambda^p(\mathbb{R}^N)}} = \|\mathscr{J}_s \varphi\|_{L^{p'}(\mathbb{R}^N)} = \|\varphi\|_{\Lambda^{-s,p'}(\mathbb{R}^N)}.$$

□

Remark. *From now on we will use an abuse of notation and write $\Lambda^{-s,p'}(\mathbb{R}^N)$ instead of $\left(\Lambda^{s,p}(\mathbb{R}^N)\right)'$.*

Let us now introduce the fractional Sobolev space, as it is done in [1] or in [6]. The reason why we introduce this space is because later on this section we will establish a relationship between the Lions-Calderón spaces and a space of functions in $L^p(\mathbb{R}^N)$ with a notion of fractional derivatives also in $L^p(\mathbb{R}^N)$.

Definition 2.1. *Let $s \in (0,1)$ and $p \in [1,+\infty)$. We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ as*

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{s+\frac{N}{p}}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

with the intrinsic natural norm[7]

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \right)^{1/p}.$$

It is important to notice that the space $W^{s,p}(\mathbb{R}^N)$ can be seen as an intermediate Banach space between $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$, because of the way they are defined in terms of trace spaces (see for example [1]) and because the Gagliardo seminorm

$$[u]_{W^{s,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \right)^{1/p}$$

can be seen as a generalization of the Hölder continuity to L^p (see for example [14]).

Now we state some other interesting properties about the Lions-Calderón spaces without proof.

Theorem 2.3 (see [4] and [11])**.** *Let $1 \le p < +\infty$.*

(1) If $0 < s-t < N/p$ and $1 < p \le q \le \frac{Np}{N-(s-t)p}$, then $\Lambda^{s,p}(\mathbb{R}^N) \subset \Lambda^{t,q}(\mathbb{R}^N)$ continuously;

(2) If $0 \le \mu \le s - \frac{N}{p} < 1$, then $\Lambda^{s,p}(\mathbb{R}^N) \subset C^{0,\mu}(\mathbb{R}^N)$ continuously;

[7] the fact that we say that this norm is intrinsic comes from the fact that the norm depends only on the immediate properties of the element involved, unlike an equivalent norm to this called the "trace norm" (see for example [1]).

(3) For any real $s \in (0,1)$, we have $\Lambda^{s,2}(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$, where the norms in the two spaces are equivalent. In particular, for any $u \in W^{s,2}(\mathbb{R}^N)$

$$[u]^2_{W^{s,2}(\mathbb{R}^N)} = 2C \int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} |\mathscr{F}u(\xi)|^2 \, d\xi,$$

where C is a constant depending only on N and s;

(4) If $1 < p < +\infty$ and $\varepsilon > 0$, then for every s we have $\Lambda^{s+\varepsilon,p}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N) \subset \Lambda^{s-\varepsilon,p}(\mathbb{R}^N)$, where both inclusions are continuous.

Remark. *In light of the item (3) of the previous theorem, we will denote the space $\Lambda^{s,2}(\mathbb{R}^N)$ when $s \in (0,1)$ as $H^s(\mathbb{R}^N)$, since this terminology is well established in the community of partial differential equations and functional analysis for the space $W^{s,2}(\mathbb{R}^N)$. In fact we go further and denote $\Lambda^{s,2}(\mathbb{R}^N)$ by $H^s(\mathbb{R}^N)$ for all $s \in \mathbb{R}$.*

In an interesting article [13] the authors considered the notion of distributional Riesz fractional gradient (for short fractional gradient) that generalizes the idea of derivatives of integer order to derivatives of fractional order. This notion of derivative turns out to be quite adequate to the theory of Calculus of Variations (see for example [5]).

Definition 2.2. *Let $s \in (0,1)$ and consider $u \in L^p(\mathbb{R}^N)$ with $p \in (1,+\infty)$ such that $\mathcal{I}_{1-s}u$ is well-defined. The distributional Riesz fractional gradient is given by*

$$(D^s u)_j = \frac{\partial^s u}{\partial x_j^s}, \quad j = 1, ..., N,$$

where $\frac{\partial^s u}{\partial x_j^s}$ is taken in the distributional sense with

$$\left\langle \frac{\partial^s u}{\partial x_j^s}, v \right\rangle = -\left\langle \mathcal{I}_{1-s}u, \frac{\partial v}{\partial x_j} \right\rangle = -\int_{\mathbb{R}^N} (\mathcal{I}_{1-s}u) \frac{\partial v}{\partial x_j} \, dx, \quad \forall v \in C_c^\infty(\mathbb{R}^N).$$

With this notion we can define the space of fractionallly differentiable functions $X^{s,p}(\mathbb{R}^N) = \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_{X^{s,p}(\mathbb{R}^N)}}$ where $1 < p < +\infty$, $s \in (0,1)$ and

$$\|u\|_{X^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \|D^s u\|^p_{L^p(\mathbb{R}^N)} \right)^{1/p}, \quad u \in C_c^\infty(\mathbb{R}^N).$$

The next theorem asserts that the Lions-Calderón space is essentially the same as the space of fractionally differentiable functions, but before stating and proving this theorem we need to introduce the Riesz transform[8] and some of its properties without proof.

The vector-valued Riesz transform for $f \in L^p(\mathbb{R}^N)$ with $1 \leq p < +\infty$ is given by

$$\mathcal{R}(f)(x) = (\mathcal{R}_j(f)(x))_j = \left(c_n \,\text{p.v.} \int_{\mathbb{R}^N} \frac{y_j}{|y|^{N+1}} f(x-y)\,dy\right)_j, \quad j = 1, ..., N,$$

with $c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$. Two important properties of these transforms are:

(1) $\mathscr{F}(R_j f)(\xi) = -i\frac{\xi_j}{|\xi|}\mathscr{F} f$;

(2) if $f \in L^p(\mathbb{R}^N)$ with $1 < p < +\infty$, then $\|\mathcal{R}_j f\|_{L^p(\mathbb{R}^N)} \leq C_p \|f\|_{L^p(\mathbb{R}^N)}$.

Theorem 2.4 (see [13])**.** *If $p \in (1, +\infty)$ and $s \in (0,1)$, then $X^{s,p}(\mathbb{R}^N) = \Lambda^{s,p}(\mathbb{R}^N)$.*

Proof. We start by proving that $\Lambda^{s,p}(\mathbb{R}^N) \subset X^{s,p}(\mathbb{R}^N)$. For that, let us consider that $u \in \Lambda^{s,p}(\mathbb{R}^N)$, which means that there exists a function $f \in L^p(\mathbb{R}^N)$ such that $u = G_s * f$. Then, as it was pointed out before, since $s > 0$, $\|u\|_{L^p(\mathbb{R}^N)} \leq \|f\|_{L^p(\mathbb{R}^N)}$, and so $u \in L^p(\mathbb{R}^N)$. The only thing left to be proved in this part is that $D^s u \in L^p(\mathbb{R}^N)$. For that, assume firstly that $f \in C_c^\infty(\mathbb{R}^N)$ and then we will argue by density to prove the desired result. Since $f \in C_c^\infty(\mathbb{R}^N)$ and $u \in L^p(\mathbb{R}^N)$, then for all $v \in C_c^\infty(\mathbb{R}^N)$

$$\begin{aligned}\left\langle \frac{\partial^s u}{\partial x_j^s}, v \right\rangle &= -\left\langle I_{1-s} * u, \frac{\partial v}{\partial x_j} \right\rangle = -\left\langle I_{1-s} * (G_s * f), \frac{\partial v}{\partial x_j} \right\rangle \\ &= -\left\langle G_s * (I_{1-s} * f), \frac{\partial v}{\partial x_j} \right\rangle = -\left\langle G_s * \frac{\partial^s f}{\partial x_j^s}, v \right\rangle.\end{aligned}$$

In this sense the Fourier transform of $\frac{\partial^s u}{\partial x_j^s}$ is given by

$$\begin{aligned}&\left\langle \mathscr{F}\frac{\partial^s u}{\partial x_j^s}, v \right\rangle = -\left\langle \mathscr{F}\left(G_s * \frac{\partial^s f}{\partial x_j^s}\right), v \right\rangle = -\left\langle G_s * \frac{\partial^s f}{\partial x_j^s}, \mathscr{F} v \right\rangle \\ &= -\left\langle G_s * (I_{1-s} * f), \frac{\partial \mathscr{F} v}{\partial x_j} \right\rangle = -\left\langle G_s * (I_{1-s} * f), \mathscr{F}(-2\pi i \xi_j v) \right\rangle \\ &= -\left\langle \mathscr{F} G_s \mathscr{F} I_{1-s} \mathscr{F} f, -2\pi i \xi_j v \right\rangle = \left\langle (1 + 4\pi^2|\xi|^2)^{-s/2}\left((2\pi)^s i \xi_j |\xi|^{-1+s}\right)\mathscr{F} f, v \right\rangle \\ &= \left\langle -i\frac{\xi_j}{|\xi|}\frac{(2\pi|\xi|)^s}{(1+4\pi^2|\xi|^2)^{s/2}}\mathscr{F} f, v \right\rangle.\end{aligned}$$

[8]The Riesz transform is one important example of the theory of singular integrals studied extensively in [15].

Note that everything is well-defined since $I_{1-s} \in \mathscr{S}(\mathbb{R}^N)'$ and $f \in \mathscr{S}(\mathbb{R}^N)$.

But by Theorem 1.2 and the Fourier transform of the Riesz transform we obtain that $\frac{\partial^s u}{\partial x_j^s} = \mu_s * \mathcal{R}_j f$. With this information and using Young's inequality for convolution (applied to Radon measures) and the boundedness properties of the Riesz transform we conclude that

$$\|D^s u\|_{L^p(\mathbb{R}^N)} \leq \|\mu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} \|\mathcal{R} f\|_{L^p(\mathbb{R}^N)} \leq C \|\mu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} \|f\|_{L^p(\mathbb{R}^N)},$$

where $\|\mu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} = \sup_{\varphi \in C_c(\mathbb{R}^N), \|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \varphi \, d\mu_s$. Now, using the fact that $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, we are able to conclude the inclusion that we wanted to prove.

Now we prove the converse inclusion, $X^{s,p}(\mathbb{R}^N) \subset \Lambda^{s,p}(\mathbb{R}^N)$. We start by considering the function $f := \nu_s * \left(u + \sum_{j=1}^N \mathcal{R}_j \frac{\partial^s u}{\partial x_j^s} \right)$, where ν_s is the same measure that appears in the item (2) of the Theorem 1.2 and $u \in X^{s,p}(\mathbb{R}^N)$. Observe that $f \in L^p(\mathbb{R}^N)$ because

$$\|f\|_{L^p(\mathbb{R}^N)} \leq \|\nu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} \left\| u + \sum_{j=1}^N \mathcal{R}_j \frac{\partial^s u}{\partial x_j^s} \right\|_{L^p(\mathbb{R}^N)} \leq C_{N,p} \|\nu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} \|u\|_{X^{s,p}(\mathbb{R}^N)}.$$

Assuming now that $u \in \mathscr{S}(\mathbb{R}^N) \cap X^{s,p}(\mathbb{R}^N)$ (which is dense in $X^{s,p}(\mathbb{R}^N)$ because $C_c^\infty(\mathbb{R}^N) \subset \mathscr{S}(\mathbb{R}^N)$). Then

$$\begin{aligned}
\mathscr{F} G_s \, \mathscr{F} f = & \mathscr{F} G_s \, \mathscr{F} \nu_s \left(\mathscr{F} u + \sum_{j=1}^N \frac{-i\xi_j}{|\xi|} ((2\pi)^s i \xi_j |\xi|^{-1+s}) \mathscr{F} u \right) \\
= & \mathscr{F} G_s \, \mathscr{F} \nu_s \left(\mathscr{F} u + (2\pi|\xi|)^s \mathscr{F} u \right) = \mathscr{F} G_s \left(1 + 4\pi^2 |\xi|^2\right)^{s/2} \mathscr{F} u \\
= & \mathscr{F} (G_s * G_{-s}) \, \mathscr{F} u = \mathscr{F} u,
\end{aligned}$$

and therefore $u = G_s * f$. Then by density, we get the desired result for $u \in X^{s,p}(\mathbb{R}^N)$. □

From the proof of this theorem we can notice two things:

1) in conjugation with the Theorem 2.1 we observe that for $s \in (0,1]$ and $t > 0$, $u \in \Lambda^{t+s,p}(\mathbb{R}^N)$ if and only if $u \in \Lambda^{t,p}(\mathbb{R}^N)$ and $\frac{\partial^s u}{\partial x_j^s} \in \Lambda^{t,p}(\mathbb{R}^N)$, for all $j = 1, \dots, N$;

2) the norms $\|\cdot\|_{\Lambda^{s,p}(\mathbb{R}^N)}$ and $\|\cdot\|_{X^{s,p}(\mathbb{R}^N)}$ are equivalent, and so $X^{s,p}(\mathbb{R}^N)$ is a Banach space for every $s \in (0,1)$ and $1 < p < \infty$. Moreover $X^{s,2}(\mathbb{R}^N)$ is a Hilbert space for the inner product

$$(u, v)_{X^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} uv + D^s u \cdot D^s v \, dx.$$

3 Development and Applications

In this section we present a development of the theory of Lions-Calderón spaces that is related to the classic theory of Sobolev spaces, and an application to the theory of partial differential equations.

We start by introducing the notion of s-divergence as considered in [5, 16].

Definition 3.1. *Let $s \in (0,1)$. We define the s-divergence, which we will denote by div^s, of a smooth vector field $\varphi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ as*

$$\operatorname{div}^s \varphi(x) = \frac{1}{\gamma(N,s)} \int_{\mathbb{R}^N} \frac{\operatorname{div}\varphi(y)}{|x-y|^{N+s-1}}\, dy,$$

for all $x \in \mathbb{R}^N$.

It interesting to note that this notion of s-divergence is closely related to the notion of the Riesz fractional gradient, in the sense of the following lemma.

Lemma 3.1 (see [5])**.** *Let $s \in (0,1)$. Then for all $f \in C_c^\infty(\mathbb{R}^N)$ and $\varphi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} f \operatorname{div}^s \varphi\, dx = -\int_{\mathbb{R}^N} \varphi \cdot D^s f\, dx. \tag{4}$$

Proof. Using integration by parts, the Lebesgue's dominated convergence theorem and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} f \operatorname{div}^s \varphi\, dx &= \frac{1}{\gamma(N,s)} \int_{\mathbb{R}^N} f(x) \lim_{\varepsilon\to 0} \int_{\{|y|>\varepsilon\}} \frac{\operatorname{div}_y \varphi(x+y)}{|y|^{N+s-1}}\, dy\, dx \\
&= \frac{N+s-1}{\gamma(N,s)} \int_{\mathbb{R}^N} f(x) \lim_{\varepsilon\to 0} \int_{\{|y|>\varepsilon\}} \frac{y\cdot\varphi(x+y)}{|y|^{N+s+1}}\, dy\, dx \\
&= \frac{N+s-1}{\gamma(N,s)} \int_{\mathbb{R}^N} \lim_{\varepsilon\to 0} \int_{\{|x-y|>\varepsilon\}} f(x) \frac{(y-x)\cdot\varphi(y)}{|y-x|^{N+s+1}}\, dy\, dx \\
&= \frac{N+s-1}{\gamma(N,s)} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\{|x-y|>\varepsilon\}} f(x) \frac{(y-x)\cdot\varphi(y)}{|y-x|^{N+s+1}}\, dy\, dx \\
&= -\frac{N+s-1}{\gamma(N,s)} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\{|x-y|>\varepsilon\}} \varphi(y) \cdot \frac{(x-y) f(x)}{|x-y|^{N+s+1}}\, dx\, dy \\
&= -\frac{N+s-1}{\gamma(N,s)} \int_{\mathbb{R}^N} \varphi(y) \cdot \lim_{\varepsilon\to 0} \int_{\{|x|>\varepsilon\}} \frac{x f(x+y)}{|x|^{N+s+1}}\, dx\, dy \\
&= -\frac{N+s-1}{\gamma(N,s)} \int_{\mathbb{R}^N} \varphi(y) \cdot \lim_{\varepsilon\to 0} \int_{\{|x|>\varepsilon\}} \frac{D_x f(x+y)}{|x|^{N+s-1}}\, dx\, dy = -\int_{\mathbb{R}^N} \varphi\cdot D^s f\, dy.
\end{aligned}$$

□

This allow us to say that if $\varphi \in L^{p'}(\mathbb{R}^N;\mathbb{R}^N)$ for $p' \in (1,+\infty)$, the s-divergence of φ exists in the sense of distributions because on the one hand we have that for all $f \in C_c^\infty(\mathbb{R}^N) \subset \Lambda^{s,p}(\mathbb{R}^N)$, $D^s f \in L^p(\mathbb{R}^N)$ by Theorem 2.4 and so, by the Hölder inequality, the right hand side of (4) exists, while on the other hand by density of $C_c^\infty(\mathbb{R}^N)$ in $L^{p'}(\mathbb{R}^N)$ and by linearity of the right hand side of (4), we conclude that $\operatorname{div}^s \varphi \in \Lambda^{-s,p'}(\mathbb{R}^N)$.

Theorem 3.1. *Assume that $L \in \Lambda^{-s,p'}(\mathbb{R}^N)$, with $s \in (0,1)$, $p \in (1,+\infty)$ and $p' = p/(p-1)$. Then there are $v_0, v_1, ..., v_N \in L^{p'}(\mathbb{R}^N)$ such that*

$$\langle L, u\rangle = \int_{\mathbb{R}^N} uv_0 + \sum_{j=1}^{N} (D^s u)_j v_j \, dx, \quad \forall u \in \Lambda^{s,p}(\mathbb{R}^N).$$

Proof. Let $P : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N, \mathbb{R}^{N+1})$ defined as $P(u) = [u, D^s u]$. Note that

$$\|Pu\|_{L^p(\mathbb{R}^N,\mathbb{R}^{N+1})} = \|u\|_{X^{s,p}(\mathbb{R}^N)}$$

which means that P is an isometry of $X^{s,p}(\mathbb{R}^N)$ onto a subspace $W \subset L^p(\mathbb{R}^N, \mathbb{R}^{N+1})$. Then we start by defining the linear functional L^* on W as $\langle L^*, Pu\rangle = \langle L, u\rangle$, which by the isometric isomorphism between $X^{s,p}(\mathbb{R}^N)$ and W we obtain that $\|L^*\|_{W'} = \|L\|_{(X^{s,p}(\mathbb{R}^N))'}$. Now, using Hahn-Banach theorem, there exists an extension $\tilde{L}$ of L^* to all $L^p(\mathbb{R}^N, \mathbb{R}^{N+1})$ such that $\|L^*\|_{W'} = \|\tilde{L}\|_{(L^p(\mathbb{R}^N,\mathbb{R}^{N+1}))'}$. Knowing this, when we apply the Riesz representation theorem, we obtain that there exist $v_0, ..., v_N \in L^{p'}(\mathbb{R}^N)$ such that

$$\left\langle \tilde{L}, w \right\rangle = \int_{\mathbb{R}^N} \sum_{j=0}^{N} w_j v_j \, dx, \quad \forall w = (w_j)_{j=0}^N \in L^p(\mathbb{R}^N;\mathbb{R}^{N+1}),$$

and thus, for $u \in \Lambda^{s,p}(\mathbb{R}^N) = X^{s,p}(\mathbb{R}^N)$

$$\langle L, u\rangle = \langle L^*, Pu\rangle = \left\langle \tilde{L}, Pu \right\rangle = \int_{\mathbb{R}^N} uv_0 + \sum_{j=1}^{N} (D^s u)_j v_j \, dx.$$

□

Remark. *This result, together with the fact that* $\operatorname{div}^s : L^{p'}(\mathbb{R}^N;\mathbb{R}^N) \to \Lambda^{-s,p'}(\mathbb{R}^N)$*, allow us to state that the element $L \in \Lambda^{-s,p'}(\mathbb{R}^N)$ considered in the previous theorem can be identified with the distribution* $v_0 - \operatorname{div}^s v$

where $v = [v_1, ..., v_n] \in L^{p'}(\mathbb{R}^N, \mathbb{R}^N)$*. We also remark that this result is similar to the one that we find in the classical case* $s = 1$*, consisting, up to our knowledge, a new result even for the Hilbertian case* $p = 2$ *with* $0 < s < 1$*, where the Lions-Calderón spaces correspond to the classical fractional Sobolev spaces as pointed in the item (3) of Theorem 2.3, and also a new characterization for their dual spaces for negative* s*.*

Now we present an application of Lions-Calderón spaces to the theory of partial differential equations that is based on a result stated and proved in [13] and is just a simple consequence of the Lax-Milgram theorem.

Theorem 3.2 (cf. [13])**.** *Let* $0 < s < 1$*. Suppose that* $f \in H^{-s}(\mathbb{R}^N)$ *and consider the functions* $A = [a^{ij}]_{N\times N}$ *where for each* $i, j = 1, ..., N$ *we have* $a^{ij} : \mathbb{R}^N \to \mathbb{R}$*, and* $b : \mathbb{R}^N \to \mathbb{R}$ *that are bounded and measurable such that*

$$A(x)y \cdot y \geq \lambda_1 |y|^2 \text{ and } b(x) \geq \lambda_2$$

for some $\lambda_1, \lambda_2 > 0$ *and for almost all* $x \in \mathbb{R}^N$ *and all* $y \in \mathbb{R}^N$*. Then there exists a unique* $u \in H^s(\mathbb{R}^N)$ *such that*

$$\int_{\mathbb{R}^N} A(x)D^s u \cdot D^s v + b(x)uv \, dx = \langle f, v\rangle, \quad \forall v \in H^s(\mathbb{R}^N).$$

Proof. Consider the bilinear form $B : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$B[u, v] = \int_{\mathbb{R}^N} A(x)D^s u \cdot D^s v + b(x)uv \, dx.$$

In order to apply the Lax-Milgram theorem we just need to check that B is continuous and coercive. For the continuity, when we apply the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} B[u, v] &\leq C \left(\|D^s u\|_{L^2(\mathbb{R}^N)} \|D^s v\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)} \|v\|_{L^2(\mathbb{R}^N)} \right) \\ &\leq C \left(\|u\|_{H^s(\mathbb{R}^N)} \|v\|_{H^s(\mathbb{R}^N)} \right), \end{aligned}$$

where C is a positive constant that depends on s, and on the L^∞ norms of A and b. For the coercivity we just need to observe that, for some $\beta > 0$ depending on λ_1 and λ_2,

$$\begin{aligned} B[u, u] &\geq \min\{\lambda_1, \lambda_2\} \int_{\mathbb{R}^N} |D^s u|^2 + |u|^2 \, dx = \min\{\lambda_1, \lambda_2\} \|u\|_{X^{s,2}(\mathbb{R}^N)} \\ &\geq \beta \|u\|_{H^s(\mathbb{R}^N)}, \end{aligned}$$

by the equivalence of norms.

Then by the Lax-Milgram theorem we conclude that there exists a unique $u \in H^s(\mathbb{R}^N)$ such that $B[u, v] = \langle f, v\rangle$ for all $v \in H^s(\mathbb{R}^N)$. ☐

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