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Application of the BenTaher-Rachidi method in numerical sequences

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Resumo: Este artigo apresenta a aplicação do estudo sobre o método de BenTaher-Rachidi para a resolução de sequências numéricas lineares e recorrentes de ordem superior. Assim, obtém-se a fórmula de Binet, pelo método de BenTaher-Rachidi, nas sequências de Lucas, Pell, Leonardo, Mersenne, Oresme, Jacobsthal, Padovan, Perrin e Narayana.

Abstract This article presents the application of the study on the BenTaher-Rachidi method for the solving of linear and recurrent numerical sequences of higher order. Thus, Binet's formula is obtained, using the BenTaher-Rachidi method, in the sequences of Lucas, Pell, Leonardo, Mersenne, Oresme, Jacobsthal, Padovan, Perrin and Narayana.

palavras-chave: fórmula de Binet; método BenTaher-Rachidi; método tradicional; sequências numéricas.

keywords: Binet's formula; BenTaher-Rachidi method; traditional method; numeric sequences.

1 Introduction

Sequences have been extensively studied in mathematical literature over the years due to their wide applicability. A linear recursive sequence is defined as one that has an infinite number of terms, generated by a linear recurrence, called the recurrence formula, which allows the calculation of its immediate predecessor terms. However, this recurrence is not the only way to define linear recursive sequences, and it is still necessary to know its initial elements.

You can extract several properties and theorems from the recurrence of a sequence. There are several methods of solving a recurrence taking into account its characteristics: linear, non-linear, homogeneous and non-homogeneous order [8]. It is possible to obtain the terms of a sequence without the need to apply the recurrence formula, that is, through the generating matrix or other mechanisms, such as Binet's formula.

Usually, when using Binet's formula, the solving of a linear system is necessary, however [15] present a technique that consists of finding the necessary terms without using a linear system for cases in which there is a matrix of Vandermonde. With that, [8] presented a comparison between the traditional method of solving a recurrence with the BenTaher-Rachidi method.

Continuing the work of [8], we will introduce the traditional and BenTaher-Rachidi's methods of solving a recurrence, and we will also present linear sequences that fulfilled a recurrence, whose characteristic polynomial has simple roots, namely: Lucas sequence, Pell, Leonardo, Mersenne, Jacobsthal, Padovan, Perrin and Narayana. The BenTaher-Rachidi method will also be applied to these sequences in order to obtain their respective Binet's formulas using this alternative method presented.

2 Methods of resolving a recurrence

Primarily, Binet's formula was described in terms of another formula, introducing the notion of Binet's factorial formula. However, its resolution is still linked to the resolution of linear systems, which can be solved using the Vandermonde system. Thus, we have that each numerical sequence of the linear and recurrent type presents its respective characteristic polynomial. Therefore, it is necessary to know their respective roots to perform the calculation of the Binet's formula. Generally, for this calculation using the traditional method, one has to solve the Vandermonde system or invert the associated matrices, making the calculation difficult. Therefore, another method, called the BenTaher-Rachidi method, is then studied, facilitating this mathematical calculation [15].

2.1 Traditional method

In [8], we can find two methods of solving a recurrence. The first, called **traditional method**, consists of using the formula,

$$V_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i - 1} \beta_{i,j} n^j \right) \lambda_i^n, \tag{1}$$

for $n \geq 0$, which $\beta_{i,j}$ are coefficients determined through a linear system of r equations that are used as boundary condition the coefficients $(V_j)_{0 \leq j \leq r-1}$, and s is the number of distinct roots of the polynomial characteristic of the given recurrence.

As a way of exemplifying, the resolution for the Fibonacci sequence, presenting its recurrence formula $F_n = F_{n-1} + F_{n-2}, n \geq 2$, with $F_0 = 0, F_1 = 1$ characteristic polynomial $\lambda^2 - \lambda - 1 = 0$, whose roots are $\lambda_1 = \frac{1-\sqrt{5}}{2}, \lambda_2 = \frac{1+\sqrt{5}}{2}$. Using the formula presented in Equation 1, we have that: $F_n = \beta_{1,0}\lambda_1^n + \beta_{2,0}\lambda_2^n$. For this, it is necessary to calculate the values of the coefficients $\beta_{1,0}$ and $\beta_{2,0}$. Therefore, it is possible to integrate the data of the polynomial with the recurrence formula, to then assemble a system of equations, such as

$$\begin{cases} \beta_{1,0} + \beta_{2,0} = F_0 \\ \lambda_1\beta_{1,0} + \lambda_2\beta_{2,0} = F_1 \end{cases}$$

For that, it is necessary that the system of equations be solved, which presents difficulties for the calculation of sequences of higher order than the second order.

Thus, solving the determined system, we have that

$$\beta_{1,0} = -\frac{1}{\lambda_2 - \lambda_1} = -\frac{1}{\sqrt{5}} \qquad \text{and} \qquad \beta_{2,0} = \frac{1}{\lambda_2 - \lambda_1} = \frac{1}{\sqrt{5}}$$

Finally, the Binet's formula of the Fibonacci sequence is given by

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

In [15], it is possible to establish some new expressions of the Binet's formula for the sequences without using the resolution-based approach of the linear system, therefore approaching a method based only on combinatorial expression of sequences.

In this work, it is possible to find variations in the resolutions of the Binet's formula, be they: for the simple roots of the equation, for roots with multiplicities and for sequences of orders greater than or equal to two. Nevertheless, applications of these new expressions of Binet's formula are presented to solve the usual linear systems of Vandermonde equations. Furthermore, explicit formulas are obtained for the inverse entries of their associated matrices. Illustrative examples and a comparison is made with two current methods and some numerical aspects of the results that have been presented.

2.2 BenTaher-Rachidi method

The second method discussed is the **BenTaher-Rachidi method**, which is a technique presented in [15]. This method consists of finding the coefficients $\beta_{i,0}$ of the linear system (2), without the need to use a linear system for cases in which A is a Vandermonde matrix.

$$\begin{cases} \beta_{1,0} + \beta_{2,0} + \cdots + \beta_{r,0} = V_0 \\ \lambda_1\beta_{1,0} + \lambda_2\beta_{2,0} + \cdots + \lambda_r\beta_{r,0} = V_1 \\ \qquad\qquad \vdots \\ \lambda_1^{r-1}\beta_{1,0} + \lambda_2^{r-1}\beta_{2,0} + \cdots + \lambda_r^{r-1}\beta_{r,0} = V_{r-1} \end{cases} \tag{2}$$

This linear system can be written as $Ax = b$, where A is a Vandermonde matrix, x is the unknown vector $\beta_{i,0}$ and b is the vector of conditions recurrence contour.

Furthermore, it is worth noting that a Vandermonde matrix is defined by a square matrix in which each column (or row) is a geometric progression where the first term is 1.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ & & \vdots & \\ \lambda_1^m & \lambda_2^m & \cdots & \lambda_m^m \end{bmatrix}.$$

Thus, the solution of a homogeneous linear recurrence whose characteristic polynomial has only simple roots is given by the equation,

$$V_n = \sum_{i=1}^{r} \frac{1}{p'(\lambda_i)} \left(\sum_{p=0}^{r-1} \frac{A_p}{\lambda_i^{p+1}} \right) \lambda_i^n, \tag{3}$$

for $n \geq r$, where $A_p = a_{r-1}V_p + \cdots + a_pV_{r-1}$.

It is noteworthy that this BenTaher-Rachidi method does not require the resolution of a linear system. Thus, in the article [8], this method is applied to solve the polynomial characteristic of the Fibonacci sequence, thus finding the Binet's formula of this sequence. Furthermore, a comparison was made with the traditional method to show that it is possible to arrive at the same solution regardless of the method used.

With this, next, we will solve the polynomial characteristic of other linear sequences to present a new obtainment of the Binet's formula of these sequences.

3 Application of the BenTaher-Rachidi method

In this section, the BenTaher-Rachidi method will be applied to the numerical sequences of linear and recurring character, establishing a new alternative for the calculation of the Binet formula. Emphasizing that the respective application for the Fibonacci sequence has already been carried out in the work of [8], then the Lucas, Pell, Leonardo, Mersenne, Oresme, Jacobsthal, Padovan, Perrin and Narayana sequences are addressed.

3.1 Lucas Sequence

Lucas's sequence was developed by French mathematician Édouard Anatole Lucas (1842-1891), in which he made some mathematical contributions such as the well-known Tower of Hanoi [2]. And yet, the mathematician performed tests for prime numbers based on linear and recurring sequences, thus establishing a relationship of the twelfth prime number of Mersenne, a 39-digit number that remained the largest prime number for many years, and being the highest prime number found without the aid of computational and technological resources [14].

Lucas studied the Fibonacci sequence and in one of his generalizations, created the Lucas sequence, where he changed only the two initial values to 2 and 1, remaining with the same recurrence. The Lucas numbers form a second order sequence, linear and recurring, having its recurrence formula $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$, with $L_0 = 2$ and $L_1 = 1$. Its characteristic polynomial is identical to that of Fibonacci, $x^2 - x - 1$, having the same roots and presenting the same relationship with the gold number. Thus, Binet's formula for the Lucas sequence is given by, [13],

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{4}$$

Using the characteristic polynomial $p(\lambda) = \lambda^2 - \lambda - 1$, deriving p we get the polynomial $p'(\lambda) = 2\lambda - 1$ and we can calculate $p'(\lambda_1) = -\sqrt{5}$ and $p'(\lambda_2) = \sqrt{5}$.

Using the boundary conditions, we can calculate $A_0 = a_1 L_0 + a_0 L_1 = 3$ and $A_1 = a_1 L_1 = 1$, where $a_0 = 1$ and $a_1 = 1$, given by the coefficients of the recurrence relation. With that, using the formula (3), recurrence is given by

$$L_n = \frac{1}{p'(\lambda_1)}.\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}.\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}}\right)\lambda_2^n$$

By replacing the previously obtained values, it is possible to obtain

$$\begin{aligned} L_n &= \frac{1}{-\sqrt{5}}\left(\frac{3}{\left[\frac{1-\sqrt{5}}{2}\right]^1} + \frac{1}{\left[\frac{1-\sqrt{5}}{2}\right]^2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &\qquad + \frac{1}{\sqrt{5}}\left(\frac{3}{\left[\frac{1+\sqrt{5}}{2}\right]^1} + \frac{1}{\left[\frac{1+\sqrt{5}}{2}\right]^2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\ &= \frac{1}{-\sqrt{5}}\left(\frac{3.2(1-\sqrt{5})+4}{(1-\sqrt{5})^2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{3.2(1+\sqrt{5})+4}{(1+\sqrt{5})^2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\ &= \left(\frac{6\sqrt{5}-10}{6\sqrt{5}-10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{10+6\sqrt{5}}{10+6\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\ &= \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n \end{aligned}$$

resulting in the same answer obtained in Equation (4).

3.2 Pell Sequence

Pell's sequence carries this name in honor of the English mathematician John Pell (1611-1685), known for being extremely reserved, which made him recognized as one ot the most enigmatic mathematicians of the 17th century [9]. In [2], Pell acquired credit for the development of the study of Pell's equations, or Diophantine equation, described by $x^2 - Ay^2 = 1$, with x and y numbers integers and A not squared whole.

Pell's sequence was already known in Greek antiquity around 100 years after Christ, as part of an ancient algorithm to create successive approximations to $\sqrt{2}$, known as Theon's ladder. This sequence has the recurrence formula defined by $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$ and its initial values are $P_0 = 0$ and $P_1 = 1$.

This sequence has a characteristic polynomial defined by $x^2 - 2x - 1 = 0$ where one root is positive, known as the silver number (2.41), and its other root is a negative number [2]. This silver number represents the convergence relationship between the neighboring terms of the sequence. From the polynomial characteristic of the Pell sequence, it is possible to obtain its Binet's formula, by the traditional method, as was done in [11]. With that, we have that its Binet's formula is presented as

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \tag{5}$$

Now, using the BenTaher-Rachidi method and using the characteristic polynomial $p(\lambda) = \lambda^2 - 2\lambda - 1$, deriving p we get the polynomial $p'(\lambda) = 2\lambda - 2$ and we can calculate $p'(\lambda_1) = -2\sqrt{2}$ and $p'(\lambda_2) = 2\sqrt{2}$.

Using the boundary conditions, we can calculate $A_0 = a_1P_0 + a_0P_1 = 2$ and $A_1 = a_1P_1 = 1$, where $a_0 = 2$ and $a_1 = 1$, given by the coefficients of the recurrence relation. Thus, using the formula (3), the recurrence is given by

$$P_n = \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}}\right)\lambda_2^n$$

By replacing the previously obtained values, it is possible to obtain:

$$\begin{aligned} P_n &= \frac{1}{-2\sqrt{2}}\left(\frac{2}{(1-\sqrt{2})^1} + \frac{1}{(1-\sqrt{2})^2}\right)(1-\sqrt{2})^n \\ &\qquad + \frac{1}{2\sqrt{2}}\left(\frac{2}{(1+\sqrt{2})^1} + \frac{1}{(1+\sqrt{2})^2}\right)(1+\sqrt{2})^n \\ &= \frac{1}{-2\sqrt{2}}\left(\frac{2-2\sqrt{2}+1}{(1-\sqrt{2})^2}\right)(1-\sqrt{2})^n + \frac{1}{2\sqrt{2}}\left(\frac{2+2\sqrt{2}+1}{(1+\sqrt{2})^2}\right)(1+\sqrt{2})^n \\ &= \frac{1}{-2\sqrt{2}}\left(\frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right)(1-\sqrt{2})^n + \frac{1}{2\sqrt{2}}\left(\frac{3+2\sqrt{2}}{3+2\sqrt{2}}\right)(1+\sqrt{2})^n \\ &= \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n - (1-\sqrt{2})^n] \end{aligned}$$

resulting in the same answer obtained in Equation (5).

3.3 Leonardo Sequence

Historically, little is known about Leonardo's sequence. The authors of [2] believe that these numbers were studied by Leonardo de Pisa, known as Leonardo Fibonacci, and, therefore, has not been proven in any work in the literature, due to the lack of research related to that sequence. This sequence is very similar to the Fibonacci sequence, including a relationship between Leonardo's numbers and Fibonacci numbers. This relationship is defined by [7] as $Le_n = 2F_{n+1} - 1$.

Leonardo's sequence was initially presented by [7], in which there are two recurrences for this sequence, namely: $Le_n = Le_{n-1} + Le_{n-2} + 1$ and $Le_n = 2Le_{n-1} - Le_{n-3}$, for $n \geq 2$, being $Le_0 = Le_1 = 1$. Its characteristic polynomial is given by $x^3 - 2x^2 + 1 = 0$, in which there are three real roots, one equal to 1 and the other two equal to the roots of the characteristic Fibonacci equation, $x_2 = \dfrac{1+\sqrt{5}}{2}$ and $x_3 = \dfrac{1-\sqrt{5}}{2}$ [2, 17]. It is worth mentioning that these Leonardo numbers have their convergence relation between the neighboring terms of the sequence as being the gold number (1.61), as well as the result of one of its real roots. As for their Binet's formula, [7] define it using the relationship $Le_n = 2F_{n+1} - 1$ and the Binet's formula of the Fibonacci sequence. With that, we have that the Binet formula for Leonardo's sequence is given by

$$Le_n = 2\left(\frac{x_2^{n+1} - x_3^{n+1}}{x_2 - x_3}\right) - 1, \tag{6}$$

on what $x_2 = \dfrac{1+\sqrt{5}}{2}$ and $x_3 = \dfrac{1-\sqrt{5}}{2}$ are the roots of the polynomial characteristic of the sequence.

Now, using the BenTaher-Rachidi method and using the characteristic polynomial

$$p(\lambda) = \lambda^3 - 2\lambda^2 + 1,$$

deriving p we get the polynomial $p'(\lambda) = 3\lambda^2 - 4\lambda$ and we can calculate

$$p'(\lambda_1) = -1, \quad p'(\lambda_2) = \frac{5-\sqrt{5}}{2} \qquad \text{and} \qquad p'(\lambda_3) = \frac{5+\sqrt{5}}{2}.$$

Using the boundary conditions, we can calculate

$$A_0 = a_2 Le_0 + a_1 Le_1 + a_0 Le_2 = 5 \qquad \text{and} \qquad A_1 = a_2 Le_1 + a_1 Le_2 = -1,$$

where $a_0 = 2$, $a_1 = 0$ and $a_2 = -1$, as given by the coefficients of the recurrence relation. Thus, using the formula (3), the recurrence is given by,

$$Le_n = \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}}+\frac{A_1}{\lambda_1^{1+1}}+\frac{A_2}{\lambda_1^{2+1}}\right)\lambda_1^n$$
$$+\frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}}+\frac{A_1}{\lambda_2^{1+1}}+\frac{A_2}{\lambda_2^{2+1}}\right)\lambda_2^n+\frac{1}{p'(\lambda_3)}\left(\frac{A_0}{\lambda_3^{0+1}}+\frac{A_1}{\lambda_3^{1+1}}+\frac{A_2}{\lambda_3^{2+1}}\right)\lambda_3^n$$

By replacing the previously obtained values, it is possible to obtain

$$\begin{aligned}
Le_n &= \frac{1}{-1}\left(\frac{5}{1^1}+\frac{(-1)}{1^2}+\frac{(-3)}{1^3}\right)1^n \\
&\quad +\frac{1}{\frac{5-\sqrt{5}}{2}}\cdot\left(\frac{5}{(\frac{1+\sqrt{5}}{2})^1}+\frac{(-1)}{(\frac{1+\sqrt{5}}{2})^2}+\frac{(-3)}{(\frac{1+\sqrt{5}}{2})^3}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n+ \\
&\quad \frac{1}{\frac{5+\sqrt{5}}{2}}\cdot\left(\frac{5}{(\frac{1-\sqrt{5}}{2})^1}+\frac{(-1)}{(\frac{1-\sqrt{5}}{2})^2}+\frac{(-3)}{(\frac{1-\sqrt{5}}{2})^3}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\
&= -1+\left(\frac{1}{\left(\frac{5-\sqrt{5}}{2}\right)}\right)\left(\frac{5\left(\frac{1+\sqrt{5}}{2}\right)^2-\left(\frac{1+\sqrt{5}}{2}\right)-3}{\left(\frac{1-\sqrt{5}}{2}\right)^3}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\
&+ \left(\frac{1}{\left(\frac{5+\sqrt{5}}{2}\right)}\right)\left(\frac{5\left(\frac{1-\sqrt{5}}{2}\right)^2-\left(\frac{1-\sqrt{5}}{2}\right)-3}{\left(\frac{1-\sqrt{5}}{2}\right)^3}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\
&= -1+\left(\frac{2}{(5-\sqrt{5})}\right)\left(\frac{5\left(\frac{6+2\sqrt{5}}{4}\right)-\frac{1}{2}-\frac{\sqrt{5}}{2}-3}{\left(\frac{6-2\sqrt{5}}{4}\right)\left(\frac{1-\sqrt{5}}{2}\right)}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\
&+ \left(\frac{2}{(5+\sqrt{5})}\right)\left(\frac{5\left(\frac{6-2\sqrt{5}}{4}\right)-\frac{1}{2}+\frac{\sqrt{5}}{2}-3}{\left(\frac{6-2\sqrt{5}}{4}\right)\left(\frac{1-\sqrt{5}}{2}\right)}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n
\end{aligned}$$

$$
\begin{aligned}
&= -1+\left(\frac{2}{(5-\sqrt{5})}\right)\left(\frac{\frac{15}{2}+\frac{5\sqrt{5}}{2}-\frac{1}{2}-\frac{\sqrt{5}}{2}-\frac{6}{2}}{\left(\frac{16+8\sqrt{5}}{8}\right)}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n \\
&+ \left(\frac{2}{(5+\sqrt{5})}\right)\left(\frac{\frac{15}{2}-\frac{5\sqrt{5}}{2}-\frac{1}{2}+\frac{\sqrt{5}}{2}-\frac{6}{2}}{\left(\frac{16+8\sqrt{5}}{8}\right)}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n
\end{aligned}
$$

By doing algebraic manipulations, it is possible to write the equation as

$$
\begin{aligned}
Le_n &= -1+\frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\
&= \frac{2}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]-1
\end{aligned}
$$

resulting in the same answer obtained in Equation (6).

3.4 Mersenne Sequence

The Mersenne numbers make up the Mersenne sequence, such numbers honoring the Frenchman Marin Mersenne (1588-1648). Marin Mersenne was a Franciscan who offered his home for meetings with contemporary scientists, such as Descartes, Galileo, Fermat, Pascal and Torricelli with an interest in discussing and studying mathematics and scientific subjects [2]. Mersenne contributed to number theory, specifically Mersenne's prime numbers, which are all natural numbers in the form $M_n = 2^n - 1$ where n is a natural number.

The Mersenne sequence has as its recurrence formula $M_n = 3M_{n-1} - 2M_{n-2}$, for $n \geq 2$, being $M_0 = 0$ and $M_1 = 1$ their initial values. And yet, this sequence has a second degree polynomial, $x^2 - 3x + 2 = 0$, where they have two real roots, one equal to 2 and the other equal to 1. We have that the Binet's formula of the Mersenne sequence is presented by [4], where it is defined as

$$M_n = 2^n - 1, n \geq 0. \tag{7}$$

Using the characteristic polynomial $p(\lambda) = \lambda^2 - 3\lambda + 2$, deriving p we get the polynomial $p'(\lambda) = 2\lambda - 3$ and we can calculate $p'(\lambda_1) = -1$ and $p'(\lambda_2) = 1$.

Using the boundary conditions, we can calculate $A_0 = a_1V_0 + a_0V_1 = a_1M_0 + a_0M_1 = 3$ and $A_1 = a_1V_1 = a_1M_1 = -2$, where $a_0 = 3$ and $a_1 = -2$, given by the coefficients of the recurrence relation. Thus, using the formula

(3), the recurrence is given by,

$$M_n = \frac{1}{p'(\lambda_1)}.\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}.\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}}\right)\lambda_2^n$$

By replacing the previously obtained values, it is possible to obtain:

$$\begin{aligned} M_n &= \frac{1}{-1}.\left(\frac{3}{1^1} + \frac{-2}{1^2}\right)1^n + \frac{1}{1}.\left(\frac{3}{2^1} + \frac{(-2)}{2^2}\right)2^n \\ &= -1(3-2) + 2^n\left(\frac{3}{2} - \frac{1}{2}\right) \\ &= 2^n - 1 \end{aligned}$$

resulting in the same answer obtained in Equation (7).

3.5 Oresme Sequence

The Oresme sequence was created by the German philosopher Nicole Oresme (1320 - 1382), being a linear and recurrent second order sequence [3]. This sequence exposes a graphical representation of qualities and speeds, it is also believed that Oresme used primitive ideas, known today as the improper integral, to perform the sum of the infinite series, obtaining a value of 2.

The Oresme sequence is defined for every $n \geqslant 2$ by

$$O_{n+2} = O_{n+1} + \left(-\frac{1}{4}\right)O_n,$$

and the initial conditions $O_0 = 0, O_1 = O_2 = \frac{1}{2}$. Its characteristic polynomial is given by $x^2 - x + \frac{1}{4}$, where $\lambda = \frac{1}{2}$ being its real root of multiplicity 2. Using the traditional method via the Binet formula, we have

$$O_n = \alpha_0\left(\frac{1}{2}\right)^n + \alpha_1 n\left(\frac{1}{2}\right)^n,$$

for $n \geqslant 0$. Given the initial conditions, we obtain $\alpha_0 = 0$ and $\alpha_1 = 1$. Thus, we get $O_n = n\left(\frac{1}{2}\right)^n$.

Now we will apply the Bentaher-Rachidi method in the general setting to the Oresme sequence. That is, as matter of fact, we need to appeal the general sitting of this method, when the associated characteristic polynomial of the linear recursive sequence owns distincts roots of multiplicity $\geqslant 1$, introducing the Stirling numbers of the first kind. Indeed, applying the Theorem 2.9 [15] in this case, we have

$$O_n = A_0 V_{n-2} + A_1 V_{n-3},$$

for every $n \geqslant 2$, where $V_n = (c_0 + c_1 n)\left(\frac{1}{2}\right)^n$, with $c_0 = 1$ and $c_1 = S_{1,1} = 1$ ($S_{1,1}$ are the first kind Stirling numbers), $A_0 = -\frac{1}{4}V_0 + V_1$ and $A_1 = -\frac{1}{4}V_1$. Thereby, a direct calculation yields

$$O_n = \frac{1}{2}\left(\frac{1}{2}\right)^n (1 + n - 2) - \frac{1}{8}\left(\frac{1}{2}\right)^{n-3}(1 + n - 3) = \frac{n}{2^n}$$

3.6 Jacobsthal Sequence

The Jacobsthal sequence was defined by the German mathematician Ernest Erich Jacobsthal (1882-1965), this sequence has a great similarity with the Fibonacci sequence and presents several applications of which we can exemplify the use of these numbers in the area of computing in directives to change the program execution flow [2].

Jacobsthal sequence is defined by recurrence $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ and being $J_0 = 0$ and $J_1 = 1$ their initial conditions. This sequence carries many mathematical properties, highlighting its characteristic polynomial $x^2 - x - 2 = 0$, having two real roots, $x_1 = -1$ and $x_2 = 2$ [2], where the root equal to 2 also represents the convergence relationship between neighboring terms of the sequence. Due to the characteristic polynomial, we have the Binet's formula for the Jacobsthal sequence is given by [6], being defined as

$$J_n = \frac{2^n - (-1)^n}{3} \tag{8}$$

From the characteristic polynomial $p(\lambda) = \lambda^2 - \lambda - 2$, deriving p we get the polynomial $p'(\lambda) = 2\lambda - 1$ and we can calculate $p'(\lambda_1) = -3$ and $p'(\lambda_2) = 3$. Using the boundary conditions, we can calculate $A_0 = a_1 J_0 + a_0 J_1 = 1$ and $A_1 = a_1 J_1 = 2$, where $a_0 = 1$ and $a_1 = 2$, given by the coefficients of the recurrence relation. Thus, using the formula (3), the recurrence is given by,

$$J_n = \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}} + \frac{A_1}{\lambda_1^{1+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}} + \frac{A_1}{\lambda_2^{1+1}}\right)\lambda_2^n$$

By replacing the previously obtained values, it is possible to obtain

$$\begin{aligned}
J_n &= \frac{1}{-3}\left(\frac{1}{-1^1} + \frac{2}{(-1)^2}\right)(-1)^n + \frac{1}{3}\left(\frac{1}{2^1} + \frac{(2)}{2^2}\right)2^n \\
&= -\frac{1}{3}(-1+2)(-1)^n + \frac{1}{3}\left(\frac{1}{2} + \frac{1}{2}\right)2^n \\
&= -\frac{1}{3}(-1)^n + \frac{1}{3}2^n \\
&= \frac{1}{3}[2^n - (-1)^n]
\end{aligned}$$

resulting in the same answer presented in Equation (8).

3.7 Padovan Sequence

This sequence was created by the Italian architect Richard Padovan (1935-), it is considered as a cousin of the Fibonacci sequence [1], the first being a linear, recurring, third order and integer sequence. And yet, in the work of [16, 18] there is an emphasis on the mathematical historical process of this sequence, the Dutchman Hans Van Der Laan (1904 - 1991), who stands out after the Second World War, used the early Christian abbey basilica as an example to train architects in rebuilding churches [19]. The process of rebuilding the churches had been carried out by Lan and his brother, eventually discovering a new standard of measurement given by an irrational number, a number known as a plastic number or radiant number, and was first studied by Gérard Cordonnier.

Padovan sequence is defined by recurrence $Pa_n = Pa_{n-2} + Pa_{n-3}$, for $n \geq 3$ and being $Pa_0 = Pa_1 = Pa_2 = 1$ it is initial terms, still presenting its respective characteristic polynomial $x^3 - x - 1 = 0$, having three roots

$$x_1 = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{27}}} \approx 1,32$$

$$x_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} \approx -0,66 + 0,56i$$

$$x_3 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} \approx -0,66 - 0,56i.$$

With that, we will use the notations x_1, x_2, x_3 to facilitate the calculations, since this sequence has complex roots and with higher algebraic values. The relationship between the value of 1.32, which is presented as the real solution of the characteristic polynomial, and the convergence relationship between the neighboring terms of the sequence is also emphasized, thus creating a similarity.

Thus, we have Padovan's Binet's formula, as being:

$$Pa_n = \frac{(x_2-1)(x_3-1)}{(x_1-x_2)(x_1-x_3)}x_1^n + \frac{(x_1-1)(x_3-1)}{(x_2-x_1)(x_2-x_3)}x_2^n + \frac{(x_1-1)(x_2-1)}{(x_3-x_1)(x_3-x_2)}x_3^n \quad (9)$$

Using the BenTaher-Rachidi method, we have that from the polynomial

$$p(\lambda) = \lambda^3 - \lambda - 1,$$

the derivative is then calculated, resulting in $p'(\lambda) = 2\lambda^2 - 1$. Therefore, according to the formula of the method and the sequence coefficients, we have that

$$A_0 = a_2Pa_0 + a_1Pa_1 + a_0Pa_2 = 2$$
$$A_1 = a_2Pa_1 + a_1Pa_2 = 2$$
$$A_2 = a_2Pa_2 = 1.$$

Therefore, by considering Formula 3, we get:

$$\begin{aligned} Pa_n &= \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}}+\frac{A_1}{\lambda_1^{1+1}}+\frac{A_2}{\lambda_1^{2+1}}\right)\lambda_1^n \\ &\quad +\frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}}+\frac{A_1}{\lambda_2^{1+1}}+\frac{A_2}{\lambda_2^{2+1}}\right)\lambda_2^n+\frac{1}{p'(\lambda_3)}\left(\frac{A_0}{\lambda_3^{0+1}}+\frac{A_1}{\lambda_3^{1+1}}+\frac{A_2}{\lambda_3^{2+1}}\right)\lambda_3^n \end{aligned}$$

Performing the replacement of previously calculated values and using Girard relations $x_1x_2x_3 = 1$,$x_1 + x_2 + x_3 = 0$ and $x_1x_2 + x_1x_3 + x_2x_3 = -1$, we have that

$$\begin{aligned} Pa_n &= \frac{1}{3x_1^2-1}\left(\frac{2}{x_1}+\frac{2}{x_1^2}+\frac{1}{x_1^3}\right)(x_1)^n+\frac{1}{3x_2^2-1}\left(\frac{2}{x_2}+\frac{2}{x_2^2}+\frac{1}{x_2^3}\right)(x_2)^n \\ &\quad +\frac{1}{3x_3^2-1}\left(\frac{2}{x_3}+\frac{2}{x_3^2}+\frac{1}{x_3^3}\right)(x_3)^n \\ &= \left(\frac{2x_1^2+2x_1+1}{3x_1^5-x_1^3}\right)(x_1)^n+\left(\frac{2x_2^2+2x_2+1}{3x_2^5-x_2^3}\right)(x_2)^n \\ &\quad +\left(\frac{2x_3^2+2x_3+1}{3x_3^5-x_3^3}\right)(x_3)^n \\ &= \left[\frac{2x_1^2+2x_1+x_1x_2x_3}{x_1(3x_1^4-x_1^2)}\right](x_1)^n+\left[\frac{2x_2^2+2x_2+x_1x_2x_3}{x_2(3x_2^4-x_2^2)}\right](x_2)^n \\ &\quad +\left[\frac{2x_3^2+2x_3+x_1x_2x_3}{x_3(3x_3^4-x_3^2)}\right](x_3)^n \\ &= \left[\frac{2x_1+x_1x_2x_3-x_1x_2-x_1x_3}{x_1(3x_1^3-x_1)}\right](x_1)^n+\left[\frac{2x_2+x_1x_2x_3-x_1x_2-x_2x_3}{x_2(3x_2^3-x_2)}\right](x_2)^n \\ &\quad +\left[\frac{2x_3+x_1x_2x_3-x_1x_3-x_2x_3}{x_3(3x_3^3-x_3)}\right](x_3)^n \\ &= \left[\frac{2+x_2x_3+x_1}{x_1(3x_1^2-1)}\right](x_1)^n+\left[\frac{2+x_2x_3+x_2}{x_2(3x_2^2-1)}\right](x_2)^n+\left[\frac{2+x_1x_2+x_3}{x_3(3x_3^2-1)}\right](x_3)^n \\ &= \left(\frac{x_2x_3-x_2-x_3+1}{3x_1^2-1}\right)(x_1)^n+\left(\frac{x_1x_3-x_1-x_3+1}{3x_2^2-1}\right)(x_2)^n \\ &\quad +\left(\frac{x_1x_2-x_1-x_2+1}{3x_3^2-1}\right)(x_3)^n \end{aligned}$$

that is

$$Pa_n = \left[\frac{(x_2-1)(x_3-1)}{2x_1^2+x_1^2-1}\right](x_1)^n + \left[\frac{(x_1-1)(x_3-1)}{2x_2^2+x_2^2-1}\right](x_2)^n + \left[\frac{(x_1-1)(x_2-1)}{2x_3^2+x_3^2-1}\right](x_3)^n$$

$$= \left[\frac{(x_2-1)(x_3-1)}{x_1^2-x_1x_2-x_1x_3+x_2x_3}\right](x_1)^n + \left[\frac{(x_1-1)(x_3-1)}{x_2^2-x_2x_3-x_1x_2+x_1x_3}\right](x_2)^n + \left[\frac{(x_1-1)(x_2-1)}{x_3^2-x_1x_3-x_3x_3+x_1x_2}\right](x_3)^n$$

$$= \frac{(x_2-1)(x_3-1)}{(x_1-x_2)(x_1-x_3)}(x_1)^n + \frac{(x_1-1)(x_3-1)}{(x_2-x_1)(x_2-x_3)}(x_2)^n + \frac{(x_1-1)(x_2-1)}{(x_3-x_1)(x_3-x_2)}(x_3)^n$$

obtaining the formula presented in the Equation (9).

3.8 Perrin Sequence

The Perrin sequence was developed by French engineer Olivier Raoul Perrin (1841-1910), who, in his spare time liked to produce scientific works, specifically for the area of mathematics. It is believed that in 1876 this sequence was mentioned implicitly by Édouard Lucas, known for creating the Lucas sequence and Lucas numbers. One can find applicability of this sequence in graph theory, and it has recently been used to discover the coordinates of taxis in urban networks in a confidential way [10].

This sequence has a great similarity with the Padovan sequence, presenting the same recurrence relation, differing only the initial terms, and even a characteristic polynomial. Thus, its recurrence is defined as $Pe_n = Pe_{n-2} + Pe_{n-3}$, for $n \geq 3$, being $Pe_0 = 3$, $Pe_1 = 0$ and $Pe_2 = 2$ its initial terms, as this sequence has the same recurrence as the Padovan sequence, the same characteristic polynomial can be presented $x^3 - x - 1 = 0$, having the same roots seen previously. Therefore, Perrin's Binet's formula is given by

$$Pe_n = \frac{(3x_2x_3+2)}{(x_1-x_2)(x_1-x_3)}x_1^n + \frac{(3x_1x_3+2)}{(x_2-x_1)(x_2-x_3)}x_2^n + \frac{(3x_1x_2+2)}{(x_3-x_1)(x_3-x_2)}x_3^n$$

Thus, this sequence changes only its initial values to $Pe_0 = 3, Pe_1 = 0$ and $Pe_2 = 2$, resulting in $A_0 = a_2Pe_0 + a_1Pe_1 + a_0Pe_2 = 3$, $A_1 = a_2Pe_1 + a_1Pe_2 = 2$ and $A_2 = a_2Pe_2 = 2$.

Using the BenTaher-Rachidi method, we have that from the polynomial $p(\lambda) = \lambda^3 - \lambda - 1$, the derivative is then calculated, resulting in $p'(\lambda) =$

$2\lambda^2-1$. Therefore, according to the formula of the method and the sequence coefficients, we have that

$$A_0 = a_2Pa_0 + a_1Pa_1 + a_0Pa_2 = 2$$
$$A_1 = a_2Pa_1 + a_1Pa_2 = 2 \text{ and}$$
$$A_2 = a_2Pa_2 = 1.$$

That done, we have that, from the Formula (3), we get

$$Pe_n = \frac{1}{p'(\lambda_1)}\left(\frac{A_0}{\lambda_1^{0+1}}+\frac{A_1}{\lambda_1^{1+1}}+\frac{A_2}{\lambda_1^{2+1}}\right)\lambda_1^n + \frac{1}{p'(\lambda_2)}\left(\frac{A_0}{\lambda_2^{0+1}}+\frac{A_1}{\lambda_2^{1+1}}+\frac{A_2}{\lambda_2^{2+1}}\right)\lambda_2^n + \frac{1}{p'(\lambda_3)}\left(\frac{A_0}{\lambda_3^{0+1}}+\frac{A_1}{\lambda_3^{1+1}}+\frac{A_2}{\lambda_3^{2+1}}\right)\lambda_3^n$$

Using Girard relations,

$$x_1x_2x_3 = 1, \quad x_1+x_2+x_3 = 0, \quad \text{and} \quad x_1x_2+x_1x_3+x_2x_3 = -1$$

and the operations previously presented, we have that

$$Pe_n = \frac{1}{3x_1^2-1}\left(\frac{3}{x_1}+\frac{2}{x_1^2}+\frac{2}{x_1^3}\right)(x_1)^n + \frac{1}{3x_2^2-1}\left(\frac{3}{x_2}+\frac{2}{x_2^2}+\frac{2}{x_2^3}\right)(x_2)^n + \frac{1}{3x_3^2-1}\left(\frac{3}{x_3}+\frac{2}{x_3^2}+\frac{2}{x_3^3}\right)(x_3)^n$$

$$= \left(\frac{3x_1^2+2x_1+2}{3x_1^5-x_1^3}\right)(x_1)^n + \left(\frac{3x_2^2+2x_2+2}{3x_2^5-x_2^3}\right)(x_2)^n + \left(\frac{3x_3^2+2x_3+2}{3x_3^5-x_3^3}\right)(x_3)^n$$

$$= \left[\frac{3x_1^2+2x_1+2}{x_1(3x_1^4-x_1^2)}\right](x_1)^n + \left[\frac{3x_2^2+2x_2+2}{x_2(3x_2^4-x_2^2)}\right](x_2)^n + \left[\frac{3x_3^2+2x_3+2}{x_3(3x_3^4-x_3^2)}\right](x_3)^n$$

$$= \left[\frac{3x_1-2x_1x_3-2x_1x_2}{x_1(3x_1^3-x_1)}\right](x_1)^n + \left[\frac{3x_2-2x_1x_2-2x_2x_3}{x_2(3x_2^3-x_2)}\right](x_2)^n + \left[\frac{3x_3-2x_1x_3-2x_2x_3}{x_3(3x_3^3-x_3)}\right](x_3)^n$$

$$= \left[\frac{3-2x_3-2x_2}{x_1(3x_1^2-1)}\right](x_1)^n + \left[\frac{3+x_1-2x_3}{x_2(3x_2^2-1)}\right](x_2)^n + \left[\frac{3-2x_1-2x_2}{x_3(3x_3^2-1)}\right](x_3)^n$$

$$= \left(\frac{3x_2x_3+2}{3x_1^2-1}\right)(x_1)^n + \left(\frac{3x_1x_2+2}{3x_2^2-1}\right)(x_2)^n + \left(\frac{3x_1x_2+2}{3x_3^2-1}\right)(x_3)^n$$

$$= \left(\frac{3x_2x_3+2}{2x_1x_1+x_1^2-1}\right)(x_1)^n + \left(\frac{3x_1x_2+2}{2x_2x_2+x_2^2-1}\right)(x_2)^n + \left(\frac{3x_1x_2+2}{2x_3x_3^3+x_3^2-1}\right)(x_3)^n$$

that is

$$Pe_n = \left(\frac{3x_2x_3+2}{x_1^2-x_1x_2-x_1x_3+x_2x_3}\right)(x_1)^n + \left(\frac{3x_1x_2+2}{x_2^2-x_1x_2-x_2x_3+x_1x_3}\right)(x_2)^n + \left(\frac{3x_1x_2+2}{x_3^2-x_1x_3-x_2x_3+x_1x_2}\right)(x_3)^n$$

$$= \left(\frac{3x_2x_3+2}{(x_1-x_2)(x_1-x_3)}\right)(x_1)^n + \left(\frac{3x_1x_2+2}{(x_2-x_1)(x_2-x_3)}\right)(x_2)^n + \left(\frac{3x_1x_2+2}{(x_3-x_1)(x_3-x_2)}\right)(x_3)^n$$

However, there is the formula presented in Equation (10).

3.9 Narayana Sequence

The Narayana sequence was introduced by the Indian mathematician Narayana Pandita (1340 - 1400) and, similarly to the Fibonacci sequence, it is derived from a problem that presents the numbers of Narayana is that of the herd of cows and calves that was proposed by Narayana in the 14th century, in which: *"A cow gives birth to a calf every year. In turn, the calf gives birth to another calf when it is three years old. What is the number of progenies produced for twenty years by a cow?"* [12]. When answering this problem, one finds the terms that make up the Narayana sequence, which are $1, 1, 1, 2, 3, 4, 6, 9, 13 \cdots$.

The Narayana sequence is a third order numerical sequence, presenting its recurrence formula $N_n = N_{n-1} + N_{n-3}$, for $n \geq 3$ and with the initial values $N_0 = N_1 = N_2 = 1$. Its respective characteristic polynomial is given by the equation $x^3 - x^2 - 1 = 0$, with three roots $\alpha \approx 1,465$, $\beta = 0,108(0,866i - 0,5) + 1,02(-0,866i - 0,5) + 0,3$ and $\gamma = 1,023(0,866i - 0,5) + 0,108(-0,866i - 0,5) + 0,3$. Next, Narayana's Binet's formula given by

$$N_n = \left(\frac{\alpha}{3\alpha-2}\right)(\alpha)^n + \left(\frac{\beta}{3\beta-2}\right)(\beta)^n + \left(\frac{\gamma}{3\gamma-2}\right)(\gamma)^n \qquad (10)$$

Applying the BenTaher-Rachidi method, we have that from the polynomial $p(\lambda) = \lambda^3 - \lambda^2 - 1$, the derivative is calculated, resulting in $p'(\lambda) = 3\lambda^2 - 2\lambda$. Thus, from the formula of the method and the sequence coefficients, we have that: $A_0 = a_2N_0 + a_1N_1 + a_0N_2 = 2$, $A_1 = a_2N_1 + a_1N_2 = 1$ and $A_2 = a_2N_2 = 1$. From the Formula 3, Girard's relations $\alpha\beta\gamma = 1$,$\alpha + \beta + \gamma = 1$

and $\alpha\beta + \alpha\gamma + \beta\gamma = 0$ and the operations previously presented, we have that

$$\begin{aligned}
N_n &= \frac{1}{3\alpha^2 - 2\alpha}\left(\frac{2}{\alpha} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3}\right)(\alpha)^n + \frac{1}{3\beta^2 - 2\beta}\left(\frac{2}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3}\right)(\beta)^n \\
&\qquad + \frac{1}{3\gamma^2 - 2\gamma}\left(\frac{2}{\gamma} + \frac{1}{\gamma^2} + \frac{1}{\gamma^3}\right)(\gamma)^n \\
&= \left(\frac{2\alpha^2 + \alpha + 1}{3\alpha^5 - 2\alpha^4}\right)(\alpha)^n + \left(\frac{2\beta^2 + \beta + 1}{3\beta^5 - 2\beta^4}\right)(\beta)^n + \left(\frac{2\gamma^2 + \gamma + 1}{3\gamma^5 - 2\gamma^4}\right)(\gamma)^n \\
&= \left[\frac{2\alpha + \alpha\beta\gamma - \alpha\beta - \alpha\gamma}{\alpha(3\alpha^3 - 2\alpha^2)}\right](\alpha)^n + \left[\frac{2\beta + \alpha\beta\gamma - \alpha\beta - \beta\gamma}{\beta(3\beta^3 - \beta^2)}\right](\beta)^n \\
&\qquad + \left[\frac{2\gamma + \alpha\beta\gamma - \alpha\gamma - \beta\gamma}{\gamma(3\gamma^3 - \gamma^2)}\right](\gamma)^n \\
&= \left[\frac{2 + \beta\gamma - \beta - \gamma}{\alpha(3\alpha^2 - 2\alpha)}\right](\alpha)^n + \left[\frac{2 + \alpha\gamma - \alpha - \gamma}{\beta(3\beta^2 - 2\beta)}\right](\beta)^n + \left[\frac{2 + \alpha\beta - \alpha - \beta}{\gamma(3\gamma^2 - 2\gamma)}\right](\gamma)^n \\
&= \left[\frac{-\alpha\beta - \alpha\gamma + \alpha}{\alpha(3\alpha - 2)}\right](\alpha)^n + \left[\frac{-\alpha\beta - \beta\gamma + \beta}{\beta(3\beta - 2)}\right](\beta)^n + \left[\frac{-\alpha\gamma - \beta\gamma + \gamma}{\gamma(3\gamma - 2)}\right](\gamma)^n \\
&= \left(\frac{\alpha}{3\alpha - 2}\right)(\alpha)^n + \left(\frac{\beta}{3\beta - 2}\right)(\beta)^n + \left(\frac{\gamma}{3\gamma - 2}\right)(\gamma)^n
\end{aligned}$$

Finally, there is the formula presented in Equation (10).

4 Conclusion

Arising from the junction of the resolution of the Binet's formula with the Vandermonde system, this worked presented the application of the resolution through the BenTaher-Rachidi method. Thus, based on the work of [8, 15], it was possible to discuss this new way of obtaining the Binet's formula of the Lucas, Pell, Leonardo, Mersenne, Oreseme, Jacobsthal, Padovan, Perrin and Narayana sequences.

Thus, this method makes the resolution simpler, despite presenting the calculation of the derivative of a function, presenting itself as an alternative way of solving the Binet's formula. For future work, research is projected from different perspectives, such as its application in the area of computing, applied science and others.

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Bessel Potentials and Lions-Calderón Spaces

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Resumo: O principal objetivo deste trabalho é apresentar um gradiente fracionário, utilizado por Tien-Tsan Shieh e Daniel E. Spector em 2015 para estudar uma nova classe de equações com derivadas parciais. Esse gradiente fracionário permite dar uma caracterização para os espaços de Lions-Calderón (que são também conhecidos na literatura como espaços de potenciais de Bessel) idêntica à caracterização usual dos espaços de Sobolev. Para além desta interessante caracterização, apresentamos ainda dois resultados, um sobre existência e unicidade de soluções fracas para uma equação com derivadas parciais fracionárias e uma caracterização do dual dos espaços de Lions-Calderón em termos de derivadas parciais fracionárias, que correspondem a uma generalização dos correspondentes resultados no caso clássico (i.e. $s = 1$) para o caso fracionário (i.e. $s \in (0, 1)$).

Abstract The main goal of this short survey is to present a fractional gradient, used by Tien-Tsan Shieh and Daniel E. Spector in 2015, to study a new class of partial differential equations. This fractional gradient allows us to provide a characterization of the Lions-Calderón spaces (also known in the literature as Bessel potential spaces) similar to the usual characterization of the Sobolev spaces. In addition to this characterization, we also present two results, one about existence and uniqueness of weak solutions to a fractional partial differential equation and a characterization of the dual space of the Lions-Calderón spaces with the help of fractional partial derivatives, which correspond to a generalization of the corresponding results in the classical case (i.e., $s = 1$) to the fractional case (i.e. $s \in (0, 1)$).

palavras-chave: Espaços de Lions-Calderón; Potenciais de Bessel; Gradiente fracionário de Riesz distribucional.

keywords: Lions-Calderón spaces; Bessel potentials; Distributional Riesz fractional gradient.

1 Bessel Potentials

In Harmonic Analysis and as well as in potential theory two important potentials are studied, the Riesz potential that is related to the powers of the Laplacian, i.e., $(-\Delta)^{-s/2}$ for $0 < s < N$, and the Bessel potential, which was studied primarily by Aronszajn and Smith in [2] and [3], and is related to the powers of the Helmholtz operator, i.e., $(I - \Delta)^{-s/2}$. Without further ado we are going to introduce the two potentials here, although the Riesz potential will only be used explicitly in the second section.

Definition 1.1. *Let $0 < s < N$ and $x \in \mathbb{R}^N$. The Riesz potential of order s, $\mathcal{I}_s$, is defined by*

$$\mathcal{I}_s f(x) = (I_s * f)(x) = \frac{1}{\gamma(N,s)} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-s}} \, dy$$

where

$$I_s(x) = \frac{1}{\gamma(N,s)|x|^{N-s}}$$

is called the Riesz kernel, and

$$\gamma(N,s) = \frac{\pi^{N/2} 2^s \Gamma(s/2)}{\Gamma((N-s)/2)}.$$

We point out without giving a proof (but the interested reader can see it in [14]), that one has the identity $\mathscr{F} I_s(\xi) = (2\pi|\xi|)^{-s}$ and so, for $\varphi \in \mathscr{S}(\mathbb{R}^N)$ (or $\varphi \in \mathscr{S}'(\mathbb{R}^N)$), one has $\mathscr{F}(\mathcal{I}_s \varphi)(\xi) = (2\pi|\xi|)^{-s} \mathscr{F}\varphi(\xi)$.[1]

Definition 1.2. *Let $s \in \mathbb{R}$ and $x \in \mathbb{R}^N$. We define the Bessel kernel G_s of order s as being*

$$G_s(x) = \mathscr{F}^{-1}\left((1 + 4\pi^2|\xi|^2)^{-s/2}\right)(x),$$

and consequently, for $u \in \mathscr{S}'(\mathbb{R}^N)$ we define the Bessel potential $\mathscr{J}_s$ of order s of u as

$$\mathscr{J}_s u = G_s * u = \mathscr{F}^{-1}\left((1 + 4\pi^2|\xi|^2)^{-s/2} \mathscr{F} u(\xi)\right).$$

[1] $\mathscr{S}(\mathbb{R}^N)$ will always denote the Schwartz space whose elements are usually called rapidly decreasing functions, and consequently, its dual space, often called the space of tempered distributions, will be denoted by $\mathscr{S}'(\mathbb{R}^N)$. The interest of this spaces concerns mainly with the fact that they are useful when one wants to apply the Fourier transform $\mathscr{F}\varphi(\xi) = \int_{\mathbb{R}^N} \varphi(x) e^{-2\pi i x \cdot \xi} \, dx$. One should also note that some coefficients that appear when one takes the Fourier transform of a function might differ depending on the definition of this transform.

We note that the map $\mathscr{J}_s : \mathscr{S}(\mathbb{R}^N) \to \mathscr{S}(\mathbb{R}^N)$ is in fact onto because if $\psi \in \mathscr{S}(\mathbb{R}^N)$, then $\mathscr{F}\psi \in \mathscr{S}(\mathbb{R}^N)$ and consequentely $\xi \mapsto \mathscr{F}\varphi(\xi) = (1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}\psi(\xi) \in \mathscr{S}(\mathbb{R}^N)$, which implies both that $\varphi \in \mathscr{S}(\mathbb{R}^N)$ and that $\psi = \mathscr{J}_s\varphi$.

Now we are going to present some properties about the Bessel potential on Lebesgue spaces. For that to make sense we must notice that when we take the Fourier transform of a function in L^p, with $p \neq 1$ or 2, what is actually happening is that we are taking it on the tempered distribution that is associated to that function.

Theorem 1.1 (Integral characterization of the Bessel kernel; see [12]). *Assume that $s > 0$. Then,*

(1) $G_s(x) = \frac{1}{(4\pi)^{N/2}\Gamma(s/2)} \int_0^\infty e^{-\frac{t}{4\pi}} e^{-\frac{|x|^2\pi}{t}} t^{\frac{s-N}{2}} \frac{dt}{t}$*; and*

(2) $G_s(x) \in L^1(\mathbb{R}^N)$.

Proof.

(1) Using a well-known property of the Γ-functions, we have for $a, s > 0$ the following equality holds

$$a^{-s/2} = \frac{1}{\Gamma(s/2)} \int_0^{+\infty} e^{-ta} t^{s/2} \frac{dt}{t}. \tag{1}$$

Setting $a = \frac{(1+4\pi^2|\xi|^2)}{4\pi}$ we obtain

$$(4\pi)^{s/2}(1+4\pi^2|\xi|^2)^{-s/2} = \frac{1}{\Gamma(s/2)} \int_0^{+\infty} e^{-\frac{t}{4\pi}} e^{-\pi t|\xi|^2} t^{s/2} \frac{dt}{t}. \tag{2}$$

By taking the inverse Fourier transform of (2), and applying the Tonelli theorem and the inverse Fourier transform of the Gaussian function, we get

$$\begin{aligned} G_s(x) &= \frac{1}{(4\pi)^{s/2}\Gamma(s/2)} \mathscr{F}^{-1}\left(\int_0^{+\infty} e^{-\frac{t}{4\pi}} e^{-\pi t|\xi|^2} t^{s/2} \frac{dt}{t}\right)(x) \\ &= \frac{1}{(4\pi)^{s/2}\Gamma(s/2)} \int_0^{+\infty} e^{-\frac{t}{4\pi}} \mathscr{F}^{-1}\left(e^{-\pi t|\xi|^2}\right)(x) t^{s/2} \frac{dt}{t} \\ &= \frac{1}{(4\pi)^{N/2}\Gamma(s/2)} \int_0^{\infty} e^{-\frac{t}{4\pi}} e^{-\frac{|x|^2\pi}{t}} t^{\frac{s-N}{2}} \frac{dt}{t}. \end{aligned}$$

(2) Using the integral characterization obtained in the previous item, the fact that $\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}$ and the identity (1), we obtain the following chain of equalities

$$\begin{aligned}\int_{\mathbb{R}^N} G_s(x)\,dx &= \frac{1}{(4\pi)^{s/2}\Gamma(s/2)}\int_{\mathbb{R}^N}\int_0^{+\infty} e^{-\frac{t}{4\pi}} e^{-\frac{\pi|x|^2}{t}} t^{\frac{s-N}{2}}\frac{dt}{t}\,dx\\ &= \frac{1}{(4\pi)^{s/2}\Gamma(s/2)}\int_0^{+\infty} e^{-\frac{t}{4\pi}} t^{\frac{s-N}{2}}\left(\int_{\mathbb{R}^N} e^{-\frac{\pi|x|^2}{t}}\,dx\right)\frac{dt}{t}\\ &= \frac{1}{(4\pi)^{s}\Gamma(s/2)}\int_0^{+\infty} e^{-\frac{t}{4\pi}} t^{\frac{s-N}{2}} t^{N/2}\frac{dt}{t} = 1.\end{aligned}$$

□

As a consequence of this theorem when $s > 0$ we have that $\mathscr{J}_s : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a continuous map, since by Young's inequality for convolution we have

$$\|\mathscr{J}_s u\|_{L^p(\mathbb{R}^N)} = \|G_s * u\|_{L^p(\mathbb{R}^N)} \le \|G_s\|_{L^1(\mathbb{R}^N)}\|u\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)}.$$

Lemma 1.1 (see [14]). *For $s \in \mathbb{R}$, $\mathscr{J}_s : L^p(\mathbb{R}^N) \to \mathscr{S}'(\mathbb{R}^N)$ is injective.*

Proof. Let $g_1, g_2 \in L^p(\mathbb{R}^N)$ such that $\mathscr{J}_s(g_1) = \mathscr{J}_s(g_2)$ and consider $\varphi \in \mathscr{S}(\mathbb{R}^N)$ a rapidly decreasing function. Applying Fubini twice we get that

$$\begin{aligned}\int_{\mathbb{R}^N} \mathscr{J}_s(g_1)(x)\varphi(x)dx &= \int_{\mathbb{R}^N\times\mathbb{R}^N} G_s(x-y)g_1(y)\varphi(x)\,dxdy\\ &= \int_{\mathbb{R}^N} g_1(y)\mathscr{J}_s(\varphi)(y)dy.\end{aligned}$$

After doing the same to g_2 we obtain that $\int_{\mathbb{R}^N}(g_1-g_2)\mathscr{J}_s(\varphi)\,dx = 0$ for all $\varphi \in \mathscr{S}(\mathbb{R}^{\mathrm{N}})$. We have already pointed out that $\mathscr{J}_s : \mathscr{S}(\mathbb{R}^N) \to \mathscr{S}(\mathbb{R}^N)$ is surjective so $\int_{\mathbb{R}^N}(g_1-g_2)\psi\,dx = 0$ for all $\psi \in \mathscr{S}(\mathbb{R}^N)$, and by density this means that $g_1 = g_2$ a.e.. □

For the next property that relates the Riesz potential with the Bessel potential we need the following lemma.

Lemma 1.2 (Wiener's Theorem; see [12]). *If $\varphi_1 \in L^1(\mathbb{R}^N)$ and $\mathscr{F}\varphi_1 + 1 \neq 0$ everywhere, then there exists $\varphi_2 \in L^1(\mathbb{R}^N)$ such that $(\mathscr{F}\varphi_1(\xi)+1)^{-1} = \mathscr{F}\varphi_2(\xi)+1$ for all $\xi \in \mathbb{R}^N$.*

Proof. See [12] pages 249-251. □

Theorem 1.2 (Relation between the Riesz and Bessel potentials; see [12] and [14]). *Let $s > 0$.*

(1) There exists a finite measure μ_s on $\mathbb{R}^N$ so that its Fourier transform[2] *is given by*

$$\mathscr{F}\mu_s(\xi) = \frac{(2\pi|\xi|)^s}{(1+4\pi^2|\xi|^2)^{s/2}}.$$

(2) There exists a finite signed measure ν_s on $\mathbb{R}^N$ so that

$$(1+4\pi^2|x|^2)^{s/2} = \mathscr{F}\nu_s(x)\,(1+(2\pi|x|)^s)\,.$$

Proof.

(1) The idea is to use the Taylor expansion $(1-t)^{s/2} = 1 + \sum_{m=1}^{\infty} A_{m,s} t^m$ where $A_{m,s} = \frac{(-s/2)(1-(s/2))\cdots(m-1-(s/2))}{m!}$, valid for all $|t| < 1$. Note that for $m > (s/2)+1$, $A_{m,s}$ has always the same sign and $(1-t)^{s/2}$ remains bounded as $t \to 1$, which imply that $\sum_{m=1}^{\infty} |A_{m,s}| < +\infty$. So, if we set $t = (1+4\pi^2|\xi|^2)^{-1}$ we obtain

$$\left(\frac{4\pi^2|\xi|^2}{1+4\pi^2|\xi|^2}\right)^{s/2} = 1 + \sum_{m=1}^{\infty} A_{m,s}(1+4\pi^2|\xi|^2)^{-m}.$$

By defining the signed measure $\mu_s = \delta_0 + \left(\sum_{m=1}^{\infty} A_{m,s} G_{2m}\right) dx$ we obtain the desired result.

(2) This proof is just a consequence of Lemma 1.2 and so we just need to check its hypothesis. Consider $\varphi_1(x) = G_s(x) + \sum_{m=1}^{\infty} A_{m,s} G_{2m}(x)$. The first thing we need to check is the integrability of φ_1, in fact we note that using the monotone convergence theorem we obtain that $\varphi_1 \in L^1(\mathbb{R}^N)$ because $\|G_j\|_{L^1(\mathbb{R}^N)} = 1$ for all $j > 0$ and the series $\sum_{m=1}^{\infty} A_{m,s}$ converges absolutely; the second and last thing that we need to check is that $\mathscr{F}\varphi_1(\xi) + 1$ is nowhere vanishing. Indeed $\mathscr{F}\varphi_1(\xi) = \mathscr{F}\mu_s(\xi) - \mathscr{F}\delta_0(\xi) + \mathscr{F}G_s(\xi)$ which implies that

$$\mathscr{F}\varphi_1(\xi) + 1 = \frac{(2\pi|\xi|)^s + 1}{(1+4\pi^2|\xi|^2)^{s/2}} > 0 \quad \forall \xi \in \mathbb{R}^N.$$

Then, we can apply Lemma 1.2 in order to conclude that there exists a function $\varphi_2 \in L^1(\mathbb{R}^N)$ such that $(1+4\pi^2|\xi|^2)^{s/2} = (1+(2\pi|\xi|^s))(\mathscr{F}\varphi_2(\xi)+1)$ which gives the desired result with $\nu_s = \delta_0 + \varphi_2(x)dx$.

[2] the Fourier transform of a Radon measure μ is given by $\mathscr{F}\mu(y) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot y}\, d\mu(x)$

□

Remark. *Like we pointed out in the beginning of this section, the Riesz and Bessel potentials are related with some real powers of the Laplacian and the Helmholtz operators, respectively. Although the Riesz potential is only defined for* $0 < s < N$, *the Laplacian operator can be defined for all* s *as* $u \mapsto (-\Delta)^s u := \mathscr{F}^{-1}(|2\pi\xi|^{2s}\mathscr{F}u)$. *As a consequence, the previous theorem establishes, in the first point, a boundedness result for the formal quotient operator*

$$\frac{(-\Delta)^{s/2}}{(I-\Delta)^{s/2}}, \quad s > 0 \tag{3}$$

on every $L^p(\mathbb{R}^N)$ *with* $1 \leq p \leq +\infty$*; while the second point allows us to write the Fourier transform of the positive powers of the Helmholtz operator as the product of the Fourier transform of a measure with the sum of 1 (the identity of* $\mathbb{R}^N$*) and the Fourier transform of the Laplacian with the same power, which is interesting because the Helmholtz operator is the sum of the identity operator* I *with the Laplacian.*

2 Lions-Calderón Spaces

In this section we will deal with a space that we called Lions-Calderón space. This space was introduced by Aronzajn and Smith in [2] and in [3], in the Hilbertian case, $p = 2$, and later the ideas were generalized by A. P. Calderón in [4], J. L. Lions in [10] and J. L. Lions and E. Magenes in [11] to the non-Hilbertian case, $p \neq 2$. It is important to have in mind that A. P. Calderón's work has its foundations and is more directed towards Harmonic Analysis, while J. L. Lions' work has its foundations in the theory of complex interpolation in Functional Analysis (in fact, these spaces appear only as an example in [10]) and is later used in [11] to study the regularity of the non-homogeneous Dirichlet problem for an elliptic partial differential equation of order $2m$. Since from the beginning these spaces were more used in the theory of Harmonic Analysis rather than in the theory of partial differential equations (although, in recent years some interest has begun to emerge in the theory of partial differential equations), the notations and nomenclature used nowadays are generally the ones that came from Harmonic Analysis, which as we will explain later has some drawbacks. In order to make a historical appreciation and establish links between these two perspectives, we will introduce these spaces in the way that both authors did[3].

Calderón presented this space in the following way:"Let s be a real number and $1 \leq p \leq +\infty$. We define $L^p_s(\mathbb{R}^N)$ to be the image of $L^p(\mathbb{R}^N)$ under $\mathscr{J}_s$. If $f \in L^p_s(\mathbb{R}^N)$ then $f = \mathscr{J}_s g$ for some $g \in L^p(\mathbb{R}^N)$. This g is unique; we define the norm $\|f\|_{p,s}$ of $f \in L^p_s(\mathbb{R}^N)$ by $\|f\|_{p,s} = \|g\|_p$." In the last years some authors (for example, [13] and [14]) adopted the notation $L^{p,s}(\mathbb{R}^N)$ and called this space, Bessel potential space.[4]

In contrast, Lions presented it in the following way: "We will indicate with $H^{s,p}(\mathbb{R}^N)$, $1 < p < +\infty$, s real, the (Banach) space of tempered distributions u such that $\mathscr{F}^{-1}((1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}u) \in L^p(\mathbb{R}^N)$ with the norm $\|u\|_{H^{s,p}(\mathbb{R}^N)} = \|\mathscr{F}^{-1}((1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}u)\|_{L^p(\mathbb{R}^N)}$."[5]

At first sight $L^p_s(\mathbb{R}^N)$ and $H^{s,p}(\mathbb{R}^N)$ may be different spaces, however we note that they are exactly the same, since

$$\begin{aligned} u \in H^{s,p}(\mathbb{R}^N) &\Leftrightarrow \exists f \in L^p(\mathbb{R}^N) : f = \mathscr{F}^{-1}\left((1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}u\right) \\ &\Leftrightarrow \exists f \in L^p(\mathbb{R}^N) : u = \mathscr{F}^{-1}\left((1+4\pi^2|\xi|^2)^{-s/2}\mathscr{F}f\right) \\ &\Leftrightarrow u \in L^p_s(\mathbb{R}^N). \end{aligned}$$

Having said that both spaces are equal, we point out that both notations have some problems. The first is related to the fact that there are more mathematical relevant spaces than letters in the latin alphabet, for example $L^{s,p}$ is already used to denote the Lorentz space (see [7] and [15]), and $H^{s,p}$ is used for example to denote Nikol'skii spaces (see [1]). In order to overcome this problem, for the rest of this work we will use the notation $\Lambda^{s,p}(\mathbb{R}^N)$ to denote these spaces[6] with the norm $\|u\|_{\Lambda^{s,p}(\mathbb{R}^N)} = \|\mathscr{F}^{-1}((1+4\pi^2|\xi|^2)^{s/2}\mathscr{F}u)\|_{L^p(\mathbb{R}^N)} = \|\mathscr{J}_{-s}u\|_{L^p(\mathbb{R}^N)}$. Another important aspect that we touch here is the nomenclature of these spaces. There has been a solid tradition in functional analysis and in the theory of partial differential equations to name the spaces in honor to the authors that introduced them (we can point out several examples such as the Lebesgue spaces, Sobolev spaces, Besov spaces and Triebel-Lizorkin spaces), and so,

[3]With minor alterations to make it compatible with our notation, for example in [4] the author uses E_n to denote the n-dimensional euclidean space $\mathbb{R}^n$ and u as the real number that we have been denoting by s.

[4]It is also interesting to note that some authors with formation in Harmonic Analysis also call this spaces generalized Sobolev spaces as in [8].

[5]Translated freely from the french.

[6]to our knowledge this notation does not coincide with the notation of any other space

in order to continue this long standing tradition we propose the name Lions-Calderón spaces for $\Lambda^{s,p}$.

After this introduction, we are going to present now some properties about Lions-Calderón spaces.

Lemma 2.1 (see [9]). *Let $s \in \mathbb{R}$ and $p \in [1, +\infty]$. Then $\Lambda^{s,p}(\mathbb{R}^N)$ is a Banach space.*

Proof. First we observe that $\Lambda^{s,p}(\mathbb{R}^N)$ with $s \in \mathbb{R}$ and $p \in [1, +\infty]$ is a vector space because $\mathscr{J}_s$ is linear; and is a normed vector space thanks to the properties of the norms in $L^p(\mathbb{R}^N)$ and injectivity of the Bessel potential. However we still need to prove that $\Lambda^{s,p}(\mathbb{R}^N)$ is complete. For that consider a Cauchy sequence $\{f_m\}_{m\in\mathbb{N}} \subset \Lambda^{s,p}(\mathbb{R}^N)$, by the definition of the norm $\|\cdot\|_{\Lambda^{s,p}(\mathbb{R}^N)}$, $\{\mathscr{J}_{-s} f_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^N)$, and so there is a $g \in L^p(\mathbb{R}^N)$ such that $\|\mathscr{J}_{-s} f_m - g\|_{L^p(\mathbb{R}^N)} \to 0$ as $m \to +\infty$. And again using the properties of the norm $\|\cdot\|_{\Lambda^{s,p}(\mathbb{R}^N)}$ we see that $\|f_m - \mathscr{J}_s g\|_{\Lambda^{s,p}(\mathbb{R}^N)} = \|\mathscr{J}_{-s} f_m - g\|_{L^p(\mathbb{R}^N)}$ and so $\{f_m\}_{m\in\mathbb{N}}$ is convergent in $\Lambda^{s,p}(\mathbb{R}^N)$. □

Lemma 2.2 (see [14]). *Let $s \in \mathbb{R}$ and $1 \le p \le +\infty$. Then $\Lambda^{s+\varepsilon,p}(\mathbb{R}^N) \subset \Lambda^{s,p}(\mathbb{R}^N)$ with $\|f\|_{\Lambda^{s,p}(\mathbb{R}^N)} \le \|f\|_{\Lambda^{s+\varepsilon,p}(\mathbb{R}^N)}$ for all $\varepsilon > 0$.*

Proof. Let $f \in \Lambda^{s+\varepsilon,p}(\mathbb{R}^N)$. Then

$$\begin{aligned}\|f\|_{\Lambda^{s,p}(\mathbb{R}^N)} &= \|\mathscr{J}_{-s} f\|_{L^p(\mathbb{R}^N)} = \|\mathscr{J}_{\varepsilon}(\mathscr{J}_{-s-\varepsilon} f)\|_{L^p(\mathbb{R}^N)} \\ &\le \|\mathscr{J}_{-s-\varepsilon} f\|_{L^p(\mathbb{R}^N)} = \|f\|_{\Lambda^{s+\varepsilon,p}(\mathbb{R}^N)}.\end{aligned}$$

□

Lemma 2.3 (see [9]). *Let $s \in \mathbb{R}$ and $p \in [1, +\infty)$, then $\mathscr{S}(\mathbb{R}^N)$ is dense in $\Lambda^{s,p}(\mathbb{R}^N)$.*

Proof. Let $f \in \Lambda^{s,p}(\mathbb{R}^N)$. Since $\mathscr{S}(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for $p \in [1, +\infty)$, this means that for every $\varepsilon > 0$ there exists a $g \in \mathscr{S}(\mathbb{R}^N)$ such that

$$\|f - \mathscr{J}_s g\|_{\Lambda^{s,p}(\mathbb{R}^N)} = \|\mathscr{J}_{-s} f - g\|_{L^p(\mathbb{R}^N)} < \varepsilon.$$

At the same, using the fact $\mathscr{J}_s : \mathscr{S}(\mathbb{R}^N) \to \mathscr{S}(\mathbb{R}^N)$, we obtain that $\mathscr{J}_s g \in \mathscr{S}(\mathbb{R}^N)$ and then $\mathscr{S}(\mathbb{R}^N)$ is dense in $\Lambda^{s,p}(\mathbb{R}^N)$. □

In fact we have a stronger result, but first we need to state the following one.

Theorem 2.1 (see [4],[9] and [14])**.** *Suppose k is a positive integer and $1 < p < +\infty$. Then $W^{k,p}(\mathbb{R}^N) = \Lambda^{k,p}(\mathbb{R}^N)$ and the two norms are equivalent.*

The idea of the proof is the following: using a similar argument as in the proof of Theorem 2.4 below with $s = 1$ we are able to prove that $u \in \Lambda^{1,p}(\mathbb{R}^N)$ if and only if u and $\frac{\partial u}{\partial x_j}$ (taken in the sense of distributions),where $j = 1, ..., N$, are elements of $\Lambda^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$, and then the case with $k \geq 1$ follows immediately. For the complete proof we suggest [14].

Theorem 2.2 (Density; cf. [11])**.** *Let $s \geq 0$ and $p \in (1, +\infty)$, then $C_c^\infty(\mathbb{R}^N)$ is dense in $\Lambda^{s,p}(\mathbb{R}^N)$.*

Proof. Consider a function $f \in \Lambda^{s,p}(\mathbb{R}^N)$ and let $\varepsilon > 0$ arbitrary. Consider, by Lemma 2.3, a function $g \in \mathscr{S}(\mathbb{R}^N)$ such that $\|f - g\|_{\Lambda^{s,p}(\mathbb{R}^N)} \leq \varepsilon$. Then let $k > s$ be a positive integer and let $h \in C_c^\infty(\mathbb{R}^N)$ such that $\|h - g\|_{W^{k,p}(\mathbb{R}^N)} \leq \varepsilon$ (this being possible because $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}(\mathbb{R}^N)$). Since the norms of $W^{k,p}(\mathbb{R}^N)$ and $\Lambda^{k,p}(\mathbb{R}^N)$ are equivalent by Theorem 2.1, and $\|g - h\|_{\Lambda^{s,p}(\mathbb{R}^N)} \leq \|g - h\|_{\Lambda^{k,p}(\mathbb{R}^N)}$ by Lemma 2.2, then $\|g - h\|_{\Lambda^{s,p}(\mathbb{R}^N)} < C\varepsilon$, where C is a positive constant that does not depend on g or h, allowing us to conclude that $\|f - h\|_{\Lambda^{s,p}(\mathbb{R}^N)} \leq (1 + C)\varepsilon$ and consequently that $C_c^\infty(\mathbb{R}^N)$ is dense in $\Lambda^{s,p}(\mathbb{R}^N)$. ☐

Lemma 2.4. *If $s \in \mathbb{R}$, $p \in (1, +\infty)$ and $p' = \frac{p}{p-1}$, then $(\Lambda^{s,p}(\mathbb{R}^N))' \cong \Lambda^{-s,p'}(\mathbb{R}^N)$.*

Proof. We just need to prove that if $\varphi \in \left(\Lambda^{s,p}(\mathbb{R}^N)\right)'$ then $\varphi \in \Lambda^{-s,p'}(\mathbb{R}^N)$, taken the appropriate isomorphisms. Suppose that $\varphi \in \left(\Lambda^{s,p}(\mathbb{R}^N)\right)' \subset \mathscr{S}'(\mathbb{R}^N)$ and so

$$\|\varphi\|_{(\Lambda^{s,p}(\mathbb{R}^N))'} = \sup_{\substack{f \in \Lambda^{s,p}(\mathbb{R}^N) \\ f \neq 0}} \frac{|\langle \varphi, f \rangle|}{\|f\|_{\Lambda^{s,p}(\mathbb{R}^N)}} = \sup_{\substack{g \in L^p(\mathbb{R}^N) \\ g \neq 0}} \frac{|\langle \varphi, \mathscr{J}_s g \rangle|}{\|g\|_{\Lambda^p(\mathbb{R}^N)}}$$

$$= \sup_{\substack{g \in L^p(\mathbb{R}^N) \\ g \neq 0}} \frac{|\langle \mathscr{J}_s \varphi, g \rangle|}{\|g\|_{\Lambda^p(\mathbb{R}^N)}} = \|\mathscr{J}_s \varphi\|_{L^{p'}(\mathbb{R}^N)} = \|\varphi\|_{\Lambda^{-s,p'}(\mathbb{R}^N)}.$$

☐

Remark. *From now on we will use an abuse of notation and write $\Lambda^{-s,p'}(\mathbb{R}^N)$ instead of $\left(\Lambda^{s,p}(\mathbb{R}^N)\right)'$.*

Let us now introduce the fractional Sobolev space, as it is done in [1] or in [6]. The reason why we introduce this space is because later on this section we will establish a relationship between the Lions-Calderón spaces and a space of functions in $L^p(\mathbb{R}^N)$ with a notion of fractional derivatives also in $L^p(\mathbb{R}^N)$.

Definition 2.1. *Let $s \in (0,1)$ and $p \in [1, +\infty)$. We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ as*

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{s+\frac{N}{p}}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

with the intrinsic natural norm[7]

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{1/p}.$$

It is important to notice that the space $W^{s,p}(\mathbb{R}^N)$ can be seen as an intermediate Banach space between $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$, because of the way they are defined in terms of trace spaces (see for example [1]) and because the Gagliardo seminorm

$$[u]_{W^{s,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{1/p}$$

can be seen as a generalization of the Hölder continuity to L^p (see for example [14]).

Now we state some other interesting properties about the Lions-Calderón spaces without proof.

Theorem 2.3 (see [4] and [11])**.** *Let $1 \le p < +\infty$.*

(1) If $0 < s-t < N/p$ and $1 < p \le q \le \frac{Np}{N-(s-t)p}$, then $\Lambda^{s,p}(\mathbb{R}^N) \subset \Lambda^{t,q}(\mathbb{R}^N)$ continuously;

(2) If $0 \le \mu \le s - \frac{N}{p} < 1$, then $\Lambda^{s,p}(\mathbb{R}^N) \subset C^{0,\mu}(\mathbb{R}^N)$ continuously;

[7] the fact that we say that this norm is intrinsic comes from the fact that the norm depends only on the immediate properties of the element involved, unlike an equivalent norm to this called the "trace norm" (see for example [1]).

(3) *For any real* $s \in (0,1)$*, we have* $\Lambda^{s,2}(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$*, where the norms in the two spaces are equivalent. In particular, for any* $u \in W^{s,2}(\mathbb{R}^N)$

$$[u]^2_{W^{s,2}(\mathbb{R}^N)} = 2C \int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} |\mathscr{F}u(\xi)|^2 \, d\xi,$$

where C *is a constant depending only on* N *and* s*;*

(4) *If* $1 < p < +\infty$ *and* $\varepsilon > 0$*, then for every* s *we have* $\Lambda^{s+\varepsilon,p}(\mathbb{R}^N) \subset W^{s,p}(\mathbb{R}^N) \subset \Lambda^{s-\varepsilon,p}(\mathbb{R}^N)$*, where both inclusions are continuous.*

Remark. *In light of the item (3) of the previous theorem, we will denote the space* $\Lambda^{s,2}(\mathbb{R}^N)$ *when* $s \in (0,1)$ *as* $H^s(\mathbb{R}^N)$*, since this terminology is well established in the community of partial differential equations and functional analysis for the space* $W^{s,2}(\mathbb{R}^N)$*. In fact we go further and denote* $\Lambda^{s,2}(\mathbb{R}^N)$ *by* $H^s(\mathbb{R}^N)$ *for all* $s \in \mathbb{R}$*.*

In an interesting article [13] the authors considered the notion of distributional Riesz fractional gradient (for short fractional gradient) that generalizes the idea of derivatives of integer order to derivatives of fractional order. This notion of derivative turns out to be quite adequate to the theory of Calculus of Variations (see for example [5]).

Definition 2.2. *Let* $s \in (0,1)$ *and consider* $u \in L^p(\mathbb{R}^N)$ *with* $p \in (1,+\infty)$ *such that* $\mathcal{I}_{1-s}u$ *is well-defined. The distributional Riesz fractional gradient is given by*

$$(D^s u)_j = \frac{\partial^s u}{\partial x_j^s}, \quad j = 1, ..., N,$$

where $\frac{\partial^s u}{\partial x_j^s}$ *is taken in the distributional sense with*

$$\left\langle \frac{\partial^s u}{\partial x_j^s}, v \right\rangle = -\left\langle \mathcal{I}_{1-s}u, \frac{\partial v}{\partial x_j} \right\rangle = -\int_{\mathbb{R}^N} (\mathcal{I}_{1-s}u) \frac{\partial v}{\partial x_j} \, dx, \quad \forall v \in C_c^\infty(\mathbb{R}^N).$$

With this notion we can define the space of fractionallly differentiable functions $X^{s,p}(\mathbb{R}^N) = \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_{X^{s,p}(\mathbb{R}^N)}}$ where $1 < p < +\infty$, $s \in (0,1)$ and

$$\|u\|_{X^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \|D^s u\|^p_{L^p(\mathbb{R}^N)} \right)^{1/p}, \quad u \in C_c^\infty(\mathbb{R}^N).$$

The next theorem asserts that the Lions-Calderón space is essentially the same as the space of fractionally differentiable functions, but before stating and proving this theorem we need to introduce the Riesz transform[8] and some of its properties without proof.

The vector-valued Riesz transform for $f \in L^p(\mathbb{R}^N)$ with $1 \le p < +\infty$ is given by

$$\mathcal{R}(f)(x) = (\mathcal{R}_j(f)(x))_j = \left(c_n \,\text{p.v.} \int_{\mathbb{R}^N} \frac{y_j}{|y|^{N+1}} f(x-y)\,dy\right)_j, \quad j = 1, ..., N,$$

with $c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$. Two important properties of these transforms are:

(1) $\mathscr{F}(R_j f)(\xi) = -i\frac{\xi_j}{|\xi|}\mathscr{F} f$;

(2) if $f \in L^p(\mathbb{R}^N)$ with $1 < p < +\infty$, then $\|\mathcal{R}_j f\|_{L^p(\mathbb{R}^N)} \le C_p \|f\|_{L^p(\mathbb{R}^N)}$.

Theorem 2.4 (see [13])**.** *If $p \in (1, +\infty)$ and $s \in (0, 1)$, then $X^{s,p}(\mathbb{R}^N) = \Lambda^{s,p}(\mathbb{R}^N)$.*

Proof. We start by proving that $\Lambda^{s,p}(\mathbb{R}^N) \subset X^{s,p}(\mathbb{R}^N)$. For that, let us consider that $u \in \Lambda^{s,p}(\mathbb{R}^N)$, which means that there exists a function $f \in L^p(\mathbb{R}^N)$ such that $u = G_s * f$. Then, as it was pointed out before, since $s > 0$, $\|u\|_{L^p(\mathbb{R}^N)} \le \|f\|_{L^p(\mathbb{R}^N)}$, and so $u \in L^p(\mathbb{R}^N)$. The only thing left to be proved in this part is that $D^s u \in L^p(\mathbb{R}^N)$. For that, assume firstly that $f \in C_c^\infty(\mathbb{R}^N)$ and then we will argue by density to prove the desired result. Since $f \in C_c^\infty(\mathbb{R}^N)$ and $u \in L^p(\mathbb{R}^N)$, then for all $v \in C_c^\infty(\mathbb{R}^N)$

$$\begin{aligned}\left\langle \frac{\partial^s u}{\partial x_j^s}, v \right\rangle &= -\left\langle I_{1-s} * u, \frac{\partial v}{\partial x_j} \right\rangle = -\left\langle I_{1-s} * (G_s * f), \frac{\partial v}{\partial x_j} \right\rangle \\ &= -\left\langle G_s * (I_{1-s} * f), \frac{\partial v}{\partial x_j} \right\rangle = -\left\langle G_s * \frac{\partial^s f}{\partial x_j^s}, v \right\rangle.\end{aligned}$$

In this sense the Fourier transform of $\frac{\partial^s u}{\partial x_j^s}$ is given by

$$\begin{aligned}&\left\langle \mathscr{F}\frac{\partial^s u}{\partial x_j^s}, v \right\rangle = -\left\langle \mathscr{F}\left(G_s * \frac{\partial^s f}{\partial x_j^s}\right), v \right\rangle = -\left\langle G_s * \frac{\partial^s f}{\partial x_j^s}, \mathscr{F} v \right\rangle \\ &= -\left\langle G_s * (I_{1-s} * f), \frac{\partial \mathscr{F} v}{\partial x_j} \right\rangle = -\left\langle G_s * (I_{1-s} * f), \mathscr{F}(-2\pi i \xi_j v) \right\rangle \\ &= -\left\langle \mathscr{F} G_s \mathscr{F} I_{1-s} \mathscr{F} f, -2\pi i \xi_j v \right\rangle = \left\langle (1 + 4\pi^2|\xi|^2)^{-s/2}\left((2\pi)^s i\xi_j |\xi|^{-1+s}\right)\mathscr{F} f, v \right\rangle \\ &= \left\langle -i\frac{\xi_j}{|\xi|}\frac{(2\pi|\xi|)^s}{(1 + 4\pi^2|\xi|^2)^{s/2}}\mathscr{F} f, v \right\rangle.\end{aligned}$$

[8] The Riesz transform is one important example of the theory of singular integrals studied extensively in [15].

Note that everything is well-defined since $I_{1-s} \in \mathscr{S}(\mathbb{R}^N)'$ and $f \in \mathscr{S}(\mathbb{R}^N)$.

But by Theorem 1.2 and the Fourier transform of the Riesz transform we obtain that $\frac{\partial^s u}{\partial x_j^s} = \mu_s * \mathcal{R}_j f$. With this information and using Young's inequality for convolution (applied to Radon measures) and the boundedness properties of the Riesz transform we conclude that

$$\|D^s u\|_{L^p(\mathbb{R}^N)} \leq \|\mu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} \|\mathcal{R} f\|_{L^p(\mathbb{R}^N)} \leq C \|\mu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} \|f\|_{L^p(\mathbb{R}^N)},$$

where $\|\mu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} = \sup_{\varphi \in C_c(\mathbb{R}^N), \|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \varphi \, d\mu_s$. Now, using the fact that $C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, we are able to conclude the inclusion that we wanted to prove.

Now we prove the converse inclusion, $X^{s,p}(\mathbb{R}^N) \subset \Lambda^{s,p}(\mathbb{R}^N)$. We start by considering the function $f := \nu_s * \left(u + \sum_{j=1}^N \mathcal{R}_j \frac{\partial^s u}{\partial x_j^s} \right)$, where ν_s is the same measure that appears in the item (2) of the Theorem 1.2 and $u \in X^{s,p}(\mathbb{R}^N)$. Observe that $f \in L^p(\mathbb{R}^N)$ because

$$\|f\|_{L^p(\mathbb{R}^N)} \leq \|\nu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} \left\| u + \sum_{j=1}^N \mathcal{R}_j \frac{\partial^s u}{\partial x_j^s} \right\|_{L^p(\mathbb{R}^N)} \leq C_{N,p} \|\nu_s\|_{\mathcal{M}_b(\mathbb{R}^N)} \|u\|_{X^{s,p}(\mathbb{R}^N)}.$$

Assuming now that $u \in \mathscr{S}(\mathbb{R}^N) \cap X^{s,p}(\mathbb{R}^N)$ (which is dense in $X^{s,p}(\mathbb{R}^N)$ because $C_c^\infty(\mathbb{R}^N) \subset \mathscr{S}(\mathbb{R}^N)$). Then

$$\begin{aligned} \mathscr{F} G_s \mathscr{F} f &= \mathscr{F} G_s \mathscr{F} \nu_s \left(\mathscr{F} u + \sum_{j=1}^N \frac{-i\xi_j}{|\xi|} ((2\pi)^s i \xi_j |\xi|^{-1+s}) \mathscr{F} u \right) \\ &= \mathscr{F} G_s \mathscr{F} \nu_s \left(\mathscr{F} u + (2\pi|\xi|)^s \mathscr{F} u \right) = \mathscr{F} G_s \left(1 + 4\pi^2 |\xi|^2\right)^{s/2} \mathscr{F} u \\ &= \mathscr{F} (G_s * G_{-s}) \mathscr{F} u = \mathscr{F} u, \end{aligned}$$

and therefore $u = G_s * f$. Then by density, we get the desired result for $u \in X^{s,p}(\mathbb{R}^N)$. □

From the proof of this theorem we can notice two things:

1) in conjugation with the Theorem 2.1 we observe that for $s \in (0,1]$ and $t > 0$, $u \in \Lambda^{t+s,p}(\mathbb{R}^N)$ if and only if $u \in \Lambda^{t,p}(\mathbb{R}^N)$ and $\frac{\partial^s u}{\partial x_j^s} \in \Lambda^{t,p}(\mathbb{R}^N)$, for all $j = 1, ..., N$;

2) the norms $\|\cdot\|_{\Lambda^{s,p}(\mathbb{R}^N)}$ and $\|\cdot\|_{X^{s,p}(\mathbb{R}^N)}$ are equivalent, and so $X^{s,p}(\mathbb{R}^N)$ is a Banach space for every $s \in (0,1)$ and $1 < p < \infty$. Moreover $X^{s,2}(\mathbb{R}^N)$ is a Hilbert space for the inner product

$$(u, v)_{X^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} uv + D^s u \cdot D^s v \, dx.$$

3 Development and Applications

In this section we present a development of the theory of Lions-Calderón spaces that is related to the classic theory of Sobolev spaces, and an application to the theory of partial differential equations.

We start by introducing the notion of s-divergence as considered in [5, 16].

Definition 3.1. *Let $s \in (0,1)$. We define the s-divergence, which we will denote by* div^s*, of a smooth vector field $\varphi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ as*

$$\operatorname{div}^s \varphi(x) = \frac{1}{\gamma(N,s)} \int_{\mathbb{R}^N} \frac{\operatorname{div} \varphi(y)}{|x-y|^{N+s-1}}\, dy,$$

for all $x \in \mathbb{R}^N$.

It interesting to note that this notion of s-divergence is closely related to the notion of the Riesz fractional gradient, in the sense of the following lemma.

Lemma 3.1 (see [5]). *Let $s \in (0,1)$. Then for all $f \in C_c^\infty(\mathbb{R}^N)$ and $\varphi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} f \operatorname{div}^s \varphi\, dx = -\int_{\mathbb{R}^N} \varphi \cdot D^s f\, dx. \tag{4}$$

Proof. Using integration by parts, the Lebesgue's dominated convergence theorem and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} f \operatorname{div}^s \varphi\, dx &= \frac{1}{\gamma(N,s)} \int_{\mathbb{R}^N} f(x) \lim_{\varepsilon\to 0} \int_{\{|y|>\varepsilon\}} \frac{\operatorname{div}_y \varphi(x+y)}{|y|^{N+s-1}}\, dy\, dx \\
&= \frac{N+s-1}{\gamma(N,s)} \int_{\mathbb{R}^N} f(x) \lim_{\varepsilon\to 0} \int_{\{|y|>\varepsilon\}} \frac{y\cdot\varphi(x+y)}{|y|^{N+s+1}}\, dy\, dx \\
&= \frac{N+s-1}{\gamma(N,s)} \int_{\mathbb{R}^N} \lim_{\varepsilon\to 0} \int_{\{|x-y|>\varepsilon\}} f(x) \frac{(y-x)\cdot\varphi(y)}{|y-x|^{N+s+1}}\, dy\, dx \\
&= \frac{N+s-1}{\gamma(N,s)} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\{|x-y|>\varepsilon\}} f(x) \frac{(y-x)\cdot\varphi(y)}{|y-x|^{N+s+1}}\, dy\, dx \\
&= -\frac{N+s-1}{\gamma(N,s)} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\{|x-y|>\varepsilon\}} \varphi(y)\cdot \frac{(x-y)f(x)}{|x-y|^{N+s+1}}\, dx\, dy \\
&= -\frac{N+s-1}{\gamma(N,s)} \int_{\mathbb{R}^N} \varphi(y)\cdot \lim_{\varepsilon\to 0} \int_{\{|x|>\varepsilon\}} \frac{x f(x+y)}{|x|^{N+s+1}}\, dx\, dy \\
&= -\frac{N+s-1}{\gamma(N,s)} \int_{\mathbb{R}^N} \varphi(y)\cdot \lim_{\varepsilon\to 0} \int_{\{|x|>\varepsilon\}} \frac{D_x f(x+y)}{|x|^{N+s-1}}\, dx\, dy = -\int_{\mathbb{R}^N} \varphi\cdot D^s f\, dy.
\end{aligned}$$

□

This allow us to say that if $\varphi \in L^{p'}(\mathbb{R}^N;\mathbb{R}^N)$ for $p' \in (1,+\infty)$, the s-divergence of φ exists in the sense of distributions because on the one hand we have that for all $f \in C_c^\infty(\mathbb{R}^N) \subset \Lambda^{s,p}(\mathbb{R}^N)$, $D^s f \in L^p(\mathbb{R}^N)$ by Theorem 2.4 and so, by the Hölder inequality, the right hand side of (4) exists, while on the other hand by density of $C_c^\infty(\mathbb{R}^N)$ in $L^{p'}(\mathbb{R}^N)$ and by linearity of the right hand side of (4), we conclude that $\operatorname{div}^s \varphi \in \Lambda^{-s,p'}(\mathbb{R}^N)$.

Theorem 3.1. *Assume that $L \in \Lambda^{-s,p'}(\mathbb{R}^N)$, with $s \in (0,1)$, $p \in (1,+\infty)$ and $p' = p/(p-1)$. Then there are $v_0, v_1, ..., v_N \in L^{p'}(\mathbb{R}^N)$ such that*

$$\langle L, u\rangle = \int_{\mathbb{R}^N} uv_0 + \sum_{j=1}^{N} (D^s u)_j v_j \, dx, \quad \forall u \in \Lambda^{s,p}(\mathbb{R}^N).$$

Proof. Let $P : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N, \mathbb{R}^{N+1})$ defined as $P(u) = [u, D^s u]$. Note that

$$\|Pu\|_{L^p(\mathbb{R}^N,\mathbb{R}^{N+1})} = \|u\|_{X^{s,p}(\mathbb{R}^N)}$$

which means that P is an isometry of $X^{s,p}(\mathbb{R}^N)$ onto a subspace $W \subset L^p(\mathbb{R}^N, \mathbb{R}^{N+1})$. Then we start by defining the linear functional L^* on W as $\langle L^*, Pu\rangle = \langle L, u\rangle$, which by the isometric isomorphism between $X^{s,p}(\mathbb{R}^N)$ and W we obtain that $\|L^*\|_{W'} = \|L\|_{(X^{s,p}(\mathbb{R}^N))'}$. Now, using Hahn-Banach theorem, there exists an extension $\tilde{L}$ of L^* to all $L^p(\mathbb{R}^N, \mathbb{R}^{N+1})$ such that $\|L^*\|_{W'} = \|\tilde{L}\|_{(L^p(\mathbb{R}^N,\mathbb{R}^{N+1}))'}$. Knowing this, when we apply the Riesz representation theorem, we obtain that there exist $v_0, ..., v_N \in L^{p'}(\mathbb{R}^N)$ such that

$$\left\langle \tilde{L}, w\right\rangle = \int_{\mathbb{R}^N} \sum_{j=0}^{N} w_j v_j \, dx, \quad \forall w = (w_j)_{j=0}^N \in L^p(\mathbb{R}^N; \mathbb{R}^{N+1}),$$

and thus, for $u \in \Lambda^{s,p}(\mathbb{R}^N) = X^{s,p}(\mathbb{R}^N)$

$$\langle L, u\rangle = \langle L^*, Pu\rangle = \left\langle \tilde{L}, Pu\right\rangle = \int_{\mathbb{R}^N} uv_0 + \sum_{j=1}^{N} (D^s u)_j v_j \, dx.$$

□

Remark. *This result, together with the fact that $\operatorname{div}^s : L^{p'}(\mathbb{R}^N;\mathbb{R}^N) \to \Lambda^{-s,p'}(\mathbb{R}^N)$, allow us to state that the element $L \in \Lambda^{-s,p'}(\mathbb{R}^N)$ considered in the previous theorem can be identified with the distribution $v_0 - \operatorname{div}^s v$*

where $v = [v_1, ..., v_n] \in L^{p'}(\mathbb{R}^N, \mathbb{R}^N)$*. We also remark that this result is similar to the one that we find in the classical case* $s = 1$*, consisting, up to our knowledge, a new result even for the Hilbertian case* $p = 2$ *with* $0 < s < 1$*, where the Lions-Calderón spaces correspond to the classical fractional Sobolev spaces as pointed in the item (3) of Theorem 2.3, and also a new characterization for their dual spaces for negative* s*.*

Now we present an application of Lions-Calderón spaces to the theory of partial differential equations that is based on a result stated and proved in [13] and is just a simple consequence of the Lax-Milgram theorem.

Theorem 3.2 (cf. [13])**.** *Let* $0 < s < 1$*. Suppose that* $f \in H^{-s}(\mathbb{R}^N)$ *and consider the functions* $A = [a^{ij}]_{N\times N}$ *where for each* $i, j = 1, ..., N$ *we have* $a^{ij} : \mathbb{R}^N \to \mathbb{R}$*, and* $b : \mathbb{R}^N \to \mathbb{R}$ *that are bounded and measurable such that*

$$A(x)y \cdot y \geq \lambda_1 |y|^2 \text{ and } b(x) \geq \lambda_2$$

for some $\lambda_1, \lambda_2 > 0$ *and for almost all* $x \in \mathbb{R}^N$ *and all* $y \in \mathbb{R}^N$*. Then there exists a unique* $u \in H^s(\mathbb{R}^N)$ *such that*

$$\int_{\mathbb{R}^N} A(x)D^s u \cdot D^s v + b(x)uv\, dx = \langle f, v\rangle, \quad \forall v \in H^s(\mathbb{R}^N).$$

Proof. Consider the bilinear form $B : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$B[u, v] = \int_{\mathbb{R}^N} A(x)D^s u \cdot D^s v + b(x)uv\, dx.$$

In order to apply the Lax-Milgram theorem we just need to check that B is continuous and coercive. For the continuity, when we apply the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} B[u, v] &\leq C \left(\|D^s u\|_{L^2(\mathbb{R}^N)} \|D^s v\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)} \|v\|_{L^2(\mathbb{R}^N)} \right) \\ &\leq C \left(\|u\|_{H^s(\mathbb{R}^N)} \|v\|_{H^s(\mathbb{R}^N)} \right), \end{aligned}$$

where C is a positive constant that depends on s, and on the L^∞ norms of A and b. For the coercivity we just need to observe that, for some $\beta > 0$ depending on λ_1 and λ_2,

$$\begin{aligned} B[u, u] &\geq \min\{\lambda_1, \lambda_2\} \int_{\mathbb{R}^N} |D^s u|^2 + |u|^2\, dx = \min\{\lambda_1, \lambda_2\} \|u\|_{X^{s,2}(\mathbb{R}^N)} \\ &\geq \beta \|u\|_{H^s(\mathbb{R}^N)}, \end{aligned}$$

by the equivalence of norms.

Then by the Lax-Milgram theorem we conclude that there exists a unique $u \in H^s(\mathbb{R}^N)$ such that $B[u, v] = \langle f, v\rangle$ for all $v \in H^s(\mathbb{R}^N)$. □

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33 anos do Projecto Matemática Ensino

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Resumo: Em 1989 três docentes da Universidade de Aveiro, decidem criar uma ferramenta que permitisse avaliar algumas centenas de estudantes de uma forma célere, recorrendo a computadores. Como primeira experiência criaram conteúdos simples, sobre equações lineares, e fizeram o desafio a uma escola da região para testar o programa desenvolvido. Foi um sucesso e assim nasceu a competição EQUAmat que ainda hoje desperta o interesse dos alunos do 3º ciclo do Ensino Básico. Ao longo destes anos o Projecto Matemática Ensino (PmatE) desenvolveu várias atividades, todas elas com o mesmo objetivo fundamental: promover a aprendizagem, usando o desafio dos jogos. É esta a história que se pretende contar nestas linhas.

Abstract: In 1989, three teachers from the University of Aveiro decided to create a tool that would allow a quick assessment of a few hundred students, using computers. As a first experience, they created simple content, about linear equations, and challenged a school in the region to test the developed program. It was a success and the EQUAmat competition born, which keeps attracting the interest of thousands of students in the 3rd cycle of Basic Education. Over these years, the Mathematics Teaching Project (PmatE) has developed several activities, all of them with the same fundamental objective: to promote learning, using the challenge of games. This is the story to be told in these lines.

palavras-chave: Matemática; Competições; Sucesso; Aprendizagem; Avaliação; Computadores.

keywords: Mathematics; Competitions; Success; Learning; Assessment; Computers.

1 Introdução

O Projecto Matemática Ensino (PmatE) é um projeto de investigação e desenvolvimento, fundado em 1989 na Universidade de Aveiro (UA) que pretende aliar as tecnologias digitais ao desenvolvimento de conteúdos e eventos para a promoção do sucesso escolar e da cultura científica. [1]

Inicialmente dedicado apenas à Matemática, daí o seu nome, ao longo dos anos tem vindo a ser alargado a várias áreas científicas como Português, Biologia, Geologia, Física, Química, Literacia Financeira e, mais recentemente, Inglês.

As Competições Nacionais de Ciência (CNC) são o evento pelo qual o PmatE é mais reconhecido. A primeira competição ocorreu em 1991, dedicada a alunos do 3º ciclo do Ensino Básico, e o entusiasmo dos participantes levou a que se repetisse o evento anualmente, até aos dias de hoje. Para a realização das CNC este projeto conta com uma Plataforma de Ensino Assistido (PEA), que se constitui como um espaço de intercâmbio e partilha de recursos. Esta ferramenta de apoio à avaliação, à aprendizagem e ao ensino disponibiliza um repositório de objetos de aprendizagem, destacando-se os Modelos Geradores de Questões (MGQ) [2] que são a base das CNC.

Para além das CNC o PmatE desenvolveu também um vasto número de atividades e projetos com o intuito de promover o interesse pela ciência e pelo conhecimento em geral. Destacam-se os projetos Exi@mat e Rede de Escolas no âmbito da intervenção escolar, que envolveram professores dos ensinos básico e secundário. Na comunicação e divulgação de ciência foram várias as iniciativas levadas a cabo, algumas das mais emblemáticas estão referidas na secção 5.

A cooperação com os Países Africanos de Língua Oficial Portuguesa (PALOP), em particular com Moçambique na formação de professores, foi também uma aposta do PmatE entre 2002 e 2013, com o projeto Pensas@moz. Um outro projeto, CPLP nas Escolas, apoiado pela Comunidade dos Países de Língua Portuguesa (CPLP) levou o PmatE a praticamente todos os países que integram a Comunidade.

Nas secções seguintes é feita uma incursão nas diversas atividades desenvolvidas pelo PmatE ao longo das suas 3 décadas de existência.

2 O PmatE e as Competições

Em 1989 a Secção Autónoma de Matemática (não reunia ainda condições para ser considerada Departamento) da Universidade de Aveiro lecionava disciplinas com um elevado número de estudantes (três das unidades curriculares tinham mais de 1000 estudantes inscritos) o que dificultava o processo

[1] O site do Projecto Matemática Ensino (`https://pmate.ua.pt`) disponibiliza informação sobre os projetos desenvolvidos ao longo dos anos.

de avaliação. O computador estava a ser introduzido no ensino em Portugal e o Professor João David Vieira desafia dois colegas mais novos, António Batel Anjo e Maria Paula Carvalho, para criarem um sistema informático de apoio à avaliação.

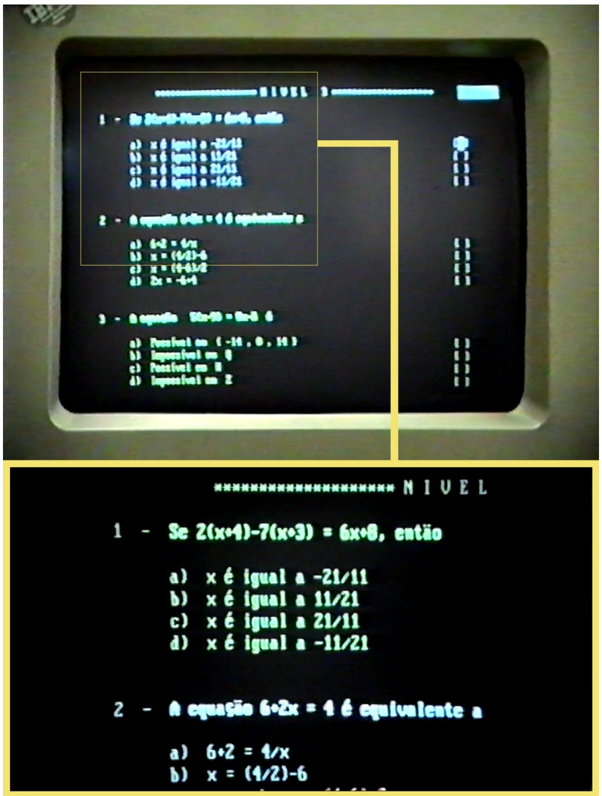

Figura 1: As primeiras competições

> "...Em fins de 1989 desafiei dois colegas mais novos, um, António Batel entusiasta da computação, e Maria Paula Carvalho, conhecedora da aplicação da computação ao ensino da matemática para comigo atacarmos o problema que acima referi [2]. O desafio foi aceite com entusiasmo." [1]

[2] Avaliar com fiabilidade os estudantes inscritos em unidades curriculares de Matemática da Universidade de Aveiro.

Contudo, as dificuldades eram muitas, quer com o equipamento informático disponível, quer com recursos humanos para elaborar conteúdos para avaliar estes estudantes.

> "...Decidimos, então, testar, se possível, as ideias já apresentadas, mas agora com matérias elementares. Propôs-se à escola (Secundária nº 1 de Aveiro) a participação na construção de uma base de dados para avaliação formativa dos alunos no 7.º ano de escolaridade. O tema escolhido, por sugestão dos professores estagiários, foi *Equações numéricas do 1.º grau em* $\mathbb{Q}$".[1]

Assim nasceu o que se tornou no ex-libris do Projecto – a competição matemática EQUAmat, cuja primeira edição ocorre em 1991 na Universidade de Aveiro, com conteúdos sobre equações de 1º grau (cf. Figura 1).

Figura 2: Diskettes usadas na competição de 1993.

As diskettes (cf. figura 2) com os conteúdos das provas eram enviadas para as escolas, permitindo que os alunos treinassem para as competições ao longo do ano letivo. O entusiasmo com que os estudantes participavam era contagiante e, frequentemente, os professores tinham que lecionar conteúdos programáticos mais avançados de modo a que os alunos progredissem no jogo.

Este entusiasmo alastrou a outras escolas e em 2000 (Ano Mundial da Matemática) a EQUAmat juntou 2000 participantes. Em 2002 as competições passam para o online e inicia-se um novo ciclo, com competições para outros níveis de escolaridade e outras áreas curriculares. Os níveis de participação nas competições (atualmente designadas por Competições Nacionais de Ciência) foram aumentando, rondando atualmente os 8000 participantes.

Figura 3: Alunos em competição.

Uma competição é organizada por níveis, sendo cada nível gerado por um Modelo Gerador de Questões (MGQ)[3] do tipo verdadeiro/falso generalizado. Uma das principais características de um MGQ é a sua aleatoriedade. Desta forma, existem várias concretizações possíveis para um mesmo MGQ de modo que dois computadores lado a lado dificilmente terão a mesma concretização para esse MGQ. Contudo, as afirmações incidirão sobre os mesmos objetivos e terão graus de dificuldade semelhantes [2].

O PmatE dispõe de uma Plataforma de Ensino Assistido, onde disponibiliza, durante todo o ano letivo, provas de treino para as competições. No final de um treino, o aluno pode consultar o seu desempenho e verificar as questões em que errou e um professor consultar o desempenho de todos os seus alunos. Esta interação fomenta nos alunos a vontade de aprender para ir mais longe e alcançar uma boa classificação na competição.

Nas edições de 2015 e 2017, foram inquiridos participantes das CNC (alunos e professores) sobre as razões da sua participação nestes eventos. Os professores afirmam que a participação nas competições fomenta a motivação dos alunos para aprender [8], e, apesar de na perspetiva dos alunos essa aprendizagem não ser muito significativa, o facto é que os resultados obtidos nas competições demonstram o seu empenho ao longo do ano letivo com a realização de um elevado número de treinos, por exemplo, a prova EQUAmat

[3]Um modelo gerador de questões é um gerador de questões sobre um determinado tema, obedecendo a uma classificação por objetivos científico-didáticos e por níveis de dificuldade. [2]

Figura 4: Nível 10 da EQUAmat.

para o 7º ano de escolaridade, registou em 2017 um total de 41147 treinos [8].

A geração aleatória de questões devido ao uso dos MGQ é também um fator a ter em conta na manutenção do interesse pela competição. Cada prova de treino apresenta uma nova concretização dos MGQ que a constituem, o mesmo acontecendo com a prova de competição, evitando assim a memorização de respostas.

A atualização dos conteúdos, acompanhando as diversas alterações curriculares, é uma preocupação do PmatE. Anualmente é feita uma revisão das grelhas de provas e são criados novos MGQ, tentando colmatar as falhas detetadas (atualmente o PmatE dispõe de aproximadamente uma dezena de milhar de geradores de questões, sendo que cada MGQ pode gerar inúmeras concretizações distintas). Contudo, a escassez de colaboradores para a elaboração de novos MGQ é um constrangimento sentido ao longo dos anos, sendo a produção de MGQ inferior ao desejável.

O testemunho de concorrentes em edições anteriores das CNC é a melhor forma de justificar a longevidade do projeto:

> "Permitiu-me aprender conceitos de matemática de forma informal, divertida e aplicada antes de eles serem lecionados nas aulas, o que me motivou para a disciplina e facilitou a posterior aprendizagem formal."
>
> "O Pmate é sem dúvida um concurso que não só desenvolve o nosso raciocínio matemático, mas que também desperta a nossa

pragmaticidade que podemos utilizar e que levamos certamente para a vida."

"Ver que alguém estava acima de mim no top100 obrigava-me a treinar e a aprender os conteúdos. De igual forma, levar o nome da minha escola/cidade a outra cidade/mesmo a nível nacional era sempre excelente e acabava por ser mais uma motivação."

3 O PmatE na intervenção escolar

O PmatE tornou-se conhecido pelas competições, contudo, o desenvolvimento de uma plataforma de ensino assistido (PEA) levou a que fossem consideradas iniciativas de intervenção escolar direta, potenciando a sua utilização para outros fins.

Duas dessas iniciativas envolveram professores e escolas numa partilha de experiências e trabalho colaborativo: o Projeto Gulbenkian Exi@mat e o Projeto Rede de Escolas.

A utilização da PEA para avaliação diagnóstica, de forma eficaz com feedback imediato, levou ainda a que fossem criados testes diagnóstico, disponibilizados anualmente para alunos em início de ciclo.

Projeto Gulbenkian Exi@mat

Em 2002 o projeto Exi@mat (exi de exigência e de êxito) foi apresentado à Fundação Calouste Gulbenkian: "Pretendia-se testar as potencialidades do software desenvolvido enquanto instrumento de apoio ao ensino e aprendizagem, via avaliação diagnóstica (auto e hetero) em situação de sala de aula e não apenas em situação de preparação para uma competição." [1]

A Fundação Calouste Gulbenkian aprovou o projeto, com uma duração de 4 anos e um financiamento de 93 000€.

O objetivo fundamental do projeto era combater o insucesso na Matemática, recorrendo ao auxílio do computador para a realização de provas de (auto-)avaliação e diagnóstico. Estiveram envolvidas 6 escolas do 3º ciclo do ensino básico, representadas por docentes de Matemática que lecionavam a disciplina aos 7º, 8º e 9º anos de escolaridade. Com a colaboração dos professores que participaram no projeto foram desenvolvidos vários MGQ sobre os conteúdos curriculares desses anos de escolaridade, que, posteriormente foram utilizados nas provas referidas.

A utilização da PEA permitia ao professor acompanhar o trabalho dos seus alunos em casa ou na escola, diagnosticar a aquisição de conhecimentos

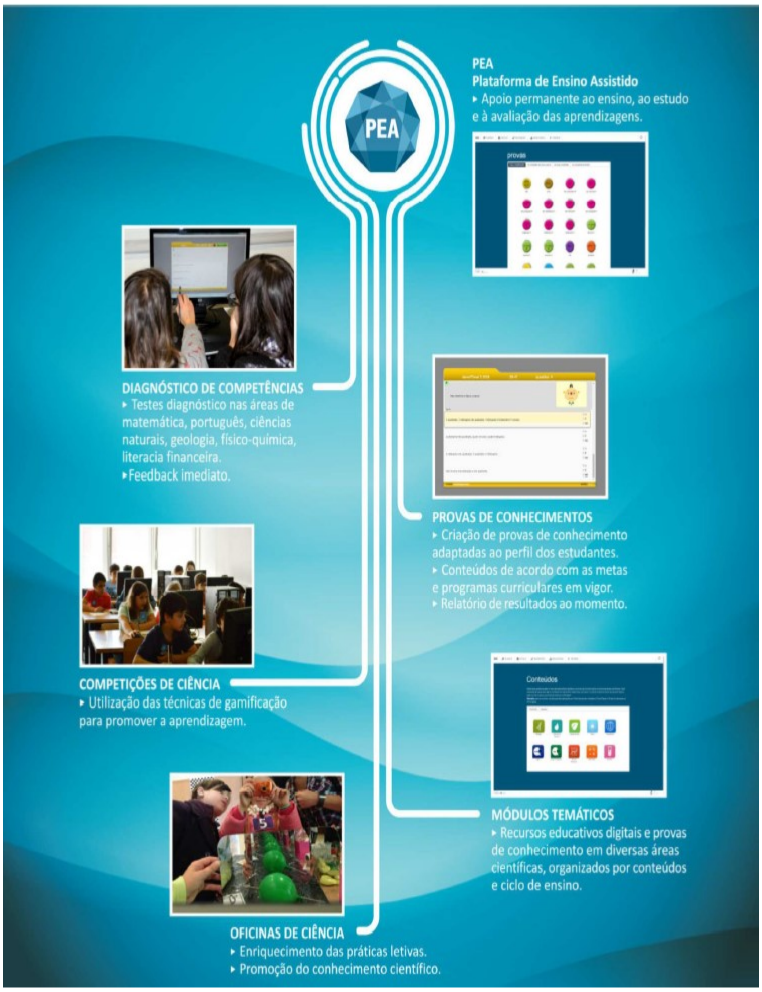

Figura 5: Ilustração das funcionalidades da PEA

e eventuais bloqueios, colmatando-os atempadamente. Os alunos, por sua vez, tinham a possibilidade de fazer o autocontrolo dos seus conhecimentos, realizando as provas que lhes eram disponibilizadas pelos seus professores e consultando as suas respostas.

Projeto Rede de Escolas

Este projeto surgiu da vontade de alargar o Exi@mat a outros ciclos de escolaridade (nomeadamente 1º e 2º ciclos do Ensino Básico e ao Ensino Secundário) e a mais escolas. Teve início em outubro de 2006, envolvendo cerca de 30 escolas dos vários ciclos de ensino. Não tendo sido um projeto financiado, teve, contudo, o apoio do Ministério da Educação, através das suas direções regionais, concedendo aos professores envolvidos uma redução de horário para a participação no projeto. Seguindo as linhas orientadoras do Exi@mat, foram desenvolvidos conteúdos para a realização de provas de (auto-)avaliação e diagnóstico na PEA. Tornou-se também um espaço de partilha para os professores participantes, com a realização de seminários mensais de discussão das estratégias levadas a cabo por cada escola e a elaboração de outros materiais didáticos digitais, para além de MGQ, que foram disponibilizados numa plataforma de gestão de conteúdos (o MOODLE). [3]

Apesar de o projeto ter terminado em 2009 (por falta de recursos humanos no PmatE para continuar com a dinâmica exigida num envolvimento desta envergadura), nos anos subsequentes muitos dos professores que participaram no projeto continuaram a usar a plataforma de ensino assistido do PmatE para a realização de provas de avaliação e diagnóstico com os seus alunos.

Testes Diagnóstico

Uma outra vertente a destacar no domínio da intervenção escolar são os Testes Diagnóstico. No início de cada ano letivo o PmatE disponibiliza testes diagnóstico para os alunos dos anos de transição de ciclo (5º, 7º e 10º) nas áreas curriculares fundamentais. Estes testes são realizados online e os resultados são disponibilizados às escolas através de uma tabela dinâmica, constituída por grupos de objetivos didáticos, em que o professor pode criar grupos de alunos, nomeadamente por turma ou por escola.

Foram também aplicados Testes Diagnóstico avaliando as competências básicas em Matemática, aos estudantes que iniciavam os seus estudos superiores em Ciências e Engenharia na Universidade de Aveiro (UA) [5]. As

CONTEÚDOS | CONTEÚDOS V2 | PROVAS | RESULTADOS

por objetivo

Matemática_módulo III

VOLTAR

Copiar | CSV | Excel | PDF | Imprimir

pesquisar

Descrição	Total	Acertou	Errou	NR	%
1 - Matemática	172	136	36	0	58,14
1.1 - Organização e Tratamento de Dados. Estatística. Probabilidades	172	136	36	0	58,14
1.1.1 - Conceitos básicos de Estatística	124	90	34	0	45,16
1.1.2 - Organização e representação de dados	48	46	2	0	91,67
1.1.2.1 - Representação gráfica de dados	48	46	2	0	91,67
1.1.2.1.1 - Pictograma e gráfico de barras	48	46	2	0	91,67

Figura 6: Excerto de um relatório de um teste diagnóstico.

lacunas detetadas através destes testes foram objeto de intervenção nas primeiras aulas de Cálculo I para os estudantes da UA.

Ao longo dos 33 anos de existência do PmatE foram realizadas muitas outras iniciativas com o objetivo de promover o sucesso na disciplina de Matemática, como por exemplo a elaboração de manuais escolares para os 6º, 7º e 10º anos de escolaridade, com ligação à PEA para diagnóstico de conhecimentos, jogos didáticos online (por exemplo a Batalha Naval para consolidação do conceito de coordenadas no plano, o Jogo do Ratão que testava conhecimentos sobre operações com frações, etc.) e várias outras de menor impacto.

Figura 7: Capa de entrada no jogo Ratão

4 O PmatE e a cooperação com países de língua oficial portuguesa

A cooperação com países africanos de língua oficial portuguesa (PALOP) foi uma vertente explorada pelo PmatE entre 2002 e 2013, destacando-se o projeto Pensas@moz, focado na melhoria da qualidade do ensino em Moçambique, nomeadamente na aprendizagem dos alunos e na qualificação dos professores, através de atividades dedicadas aos estudantes, como as Competições Nacionais de Ciência e de cursos de formação para professores nas áreas de Matemática e Língua Portuguesa.

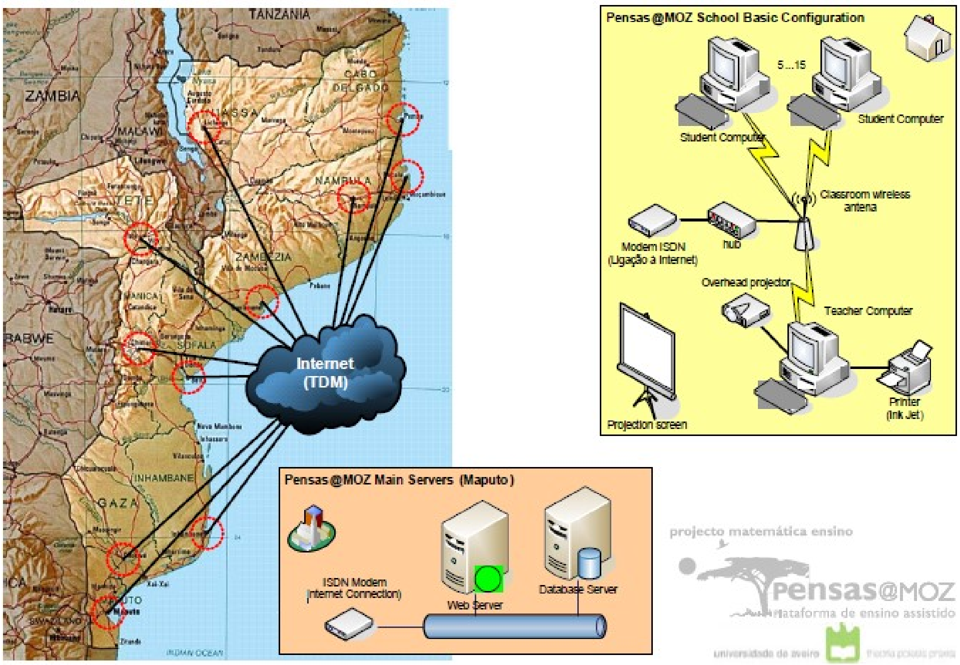

Figura 8: Arquitetura do projeto Pensas@moz

Neste âmbito foi criada uma rede de escolas distribuídas pelas diversas províncias de Moçambique, os centros Pensas, equipadas com computadores e ligação à internet (ver Figura 8), onde eram ministradas formações a professores de Matemática e Língua Portuguesa. Em alguns destes centros foram dinamizadas competições de ciência, similares às que ocorrem em Portugal, mas com os conteúdos adaptados ao curriculum moçambicano. [4]

A plataforma do projeto, uma réplica da PEA do PmatE, foi o instrumento usado nas sessões de formação de professores. Para além de fornecer aos professores conhecimentos curriculares das disciplinas de Matemática e Língua Portuguesa, serviu também para os dotar de competências informáticas. Para colmatar a dificuldade no acesso a manuais escolares, foram criadas brochuras sobre alguns dos tópicos abordados nas sessões de formação, disponibilizadas na plataforma.

Na cooperação refere-se também o Programa CPLP nas Escolas, resultante de uma parceria entre a Comunidade de Países de Língua Portuguesa (CPLP) e o PmatE/Universidade de Aveiro, que decorreu entre 2012 e 2014.

Este programa pretendia contribuir para realização do objetivo do milénio 8 (ODM8) com a criação de uma "...parceria global, com os países em desenvolvimento, a fim de formular e aplicar estratégias que proporcionem aos jovens um trabalho digno e produtivo e . . . tornar acessíveis os benefícios das tecnologias, em particular os das tecnologias da informação e comunicação" (https://www.instituto-camoes.pt/images/cooperacao/objectivos_desenvolv_milenio.pdf).

Figura 9: Excerto da plataforma do projeto CPLP nas Escolas

Sob os pilares educação para o desenvolvimento e educação para a cidadania o programa pretendia tornar possível, através da implementação de uma plataforma interativa on-line, a interação entre as comunidades educativas dos países envolvidos promovendo a partilha de conteúdos curriculares, linguísticos e culturais entre estudantes de diferentes países da comunidade de países de língua portuguesa. O programa contava ainda com uma vincada componente de sensibilização das crianças para a importância do desenvolvimento sustentável e do conhecimento intercultural.

5 O PmatE na comunicação e divulgação de ciência

A Educação Financeira foi uma aposta do PmatE entre 2009 e 2014. Durante este período realizaram-se 5 conferências internacionais de Educação Financeira que juntaram professores de vários graus de ensino e profissionais da área financeira, numa partilha de experiências que promovessem a capacitação dos cidadãos para enfrentar os problemas e desafios financeiros do dia a dia, com responsabilidade e autonomia.

Para os mais jovens, a exposição itinerante EDUCAÇÃO+Financeira [6] (2009/10 a 2013/14), com o apoio da Caixa Geral de Depósitos, circulou pelo país dispondo de três módulos distintos, destinados a diferentes faixas etárias:

Módulo 1 Dinheiro para quê? dedicado a alunos dos 1º e 2º ciclos do Ensino Básico;

Módulo 2 Como gastar o dinheiro? destinado ao 3º ciclo do Ensino Básico;

Módulo 3 Compro ou não compro? para alunos do Ensino Secundário e público em geral.

O projeto CaixaMat [7] (também em parceria com a Caixa Geral de Depósitos) foi pioneiro no âmbito da divulgação e comunicação de ciência, na modalidade de roadshow, recorrendo a uma infra-estrutura constituída por um camião especialmente preparado para acolher experiências no campo da Física, Biologia e Matemática, estando devidamente equipado com material informático. Este camião percorreu o país nos anos letivos 2005/06 a 2008/09.

Na divulgação de ciência, apoiados pela Agência Nacional Ciência Viva, os projetos Pais com Ciência e Escolher Ciência foram dois marcos na história do PmatE.

"Pais com Ciência" teve como objetivo unir as associações de pais, os agrupamentos de escolas e as instituições de ensino superior em prol da divulgação do conhecimento científico, promovendo a literacia científica junto das comunidades educativas. O PmatE dinamizou os projetos "Barreiro com Ciência", "Beja com Ciência", "Ciência na Planície" (Amareleja), "Marco na Ciência" e "Mira com Ciência", envolvendo um total de 1329 alunos.

No projeto "Em volta da Energia" (Escolher Ciência), partindo do facto de Portugal possuir vários recursos energéticos renováveis, em especial os que se relacionam com o sol, o vento e o calor geotérmico, a Universidade

Figura 10: Camião do projeto CaixaMat

de Aveiro (UA) pretendeu dar a conhecer a alunos do ensino secundário vertentes da Ciência e da Tecnologia que mostrassem a aplicabilidade da energia.

6 Conclusões

Este trabalho pretendeu apresentar a história resumida de um projeto pioneiro em Portugal na área da educação. Muitas foram as iniciativas do PmatE que ficaram fora deste trabalho, contudo, as suas vertentes fundamentais ao serviço da educação em Portugal (e nos PALOP) foram aqui referidas.

No ano em que surgiu, poucas ou nenhumas iniciativas deste tipo existiam em Portugal e, mesmo internacionalmente, são raros os registos de eventos como as Competições Nacionais de Ciência.

Os fundadores do projeto, João David Vieira, António Batel Anjo e Paula Carvalho, já não integram a equipa, mas foram eles os responsáveis pela sua divulgação junto da comunidade educativa, essencialmente em Portugal.

Atualmente as únicas vertentes que continuam ativas são as Competições Nacionais de Ciência, que continuam a motivar milhares de alunos do Ensino Básico e do Ensino Secundário, e os Testes Diagnóstico, embora com uma expressão muito reduzida. A proliferação de ferramentas digitais online e gratuitas ao serviço da educação e o parco investimento disponível para modernizar o PmatE, retiraram-no da ribalta relativamente a outras iniciativas distintas das competições.

O fenómeno EQUAmat (e as restantes competições) deve muito do seu êxito aos professores. Nos inquéritos realizados em 2015 (e 2017) ficou patente o interesse que este evento desperta nos professores que acompanham os alunos. O gráfico da figura 11 ilustra o número de participações de professores em edições das competições anteriores a 2015. Um projeto que já envolveu duas gerações de alunos!

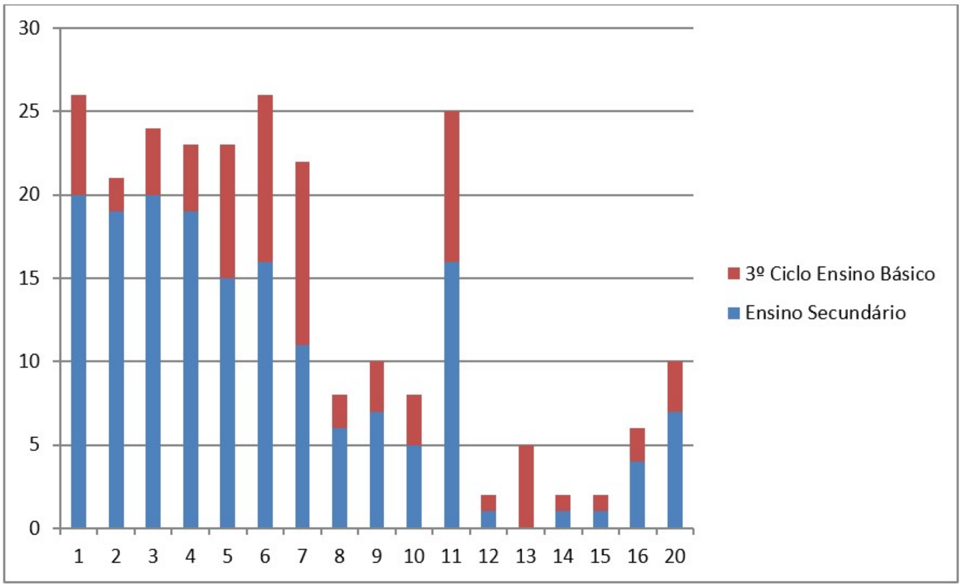

Figura 11: Número de participações de professores de Matemática do 3º ciclo do Ensino Básico e do Ensino Secundário em edições anteriores das CNC (dados de 2015)

Não existem dados concretos sobre o impacto do PmatE na aprendizagem dos alunos ao longo destas três décadas, contudo, o seu papel na promoção do interesse pelas disciplinas curriculares é inegável. São frequentes os comentários de indivíduos que participaram em algumas das iniciativas, recordando-as com satisfação e reconhecimento pelo interesse que essa participação neles despertou.

As referências que se encontram nos meios de comunicação, essencialmente locais, a atividades levadas a cabo ao longo do tempo pelo PmatE, são inúmeras e, quiçá, poderão ser um mote para um trabalho de História da (Educação) Matemática daqui a alguns anos.

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