Generating functions of lattice paths

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Abstract We recall the main types of *lattice paths*, which are sequences in the lattice of integer coordinates points in the plane. We start with the fundamental *central lattice paths* and *Dyck paths* and proceed in elementary terms through recently introduced lattice paths.

For every type, we consider the respective *generating function*. In fact, through our approach (via *Riordan arrays*), various entries of the On-Line Encyclopedia of Integer Sequences are unified, clarified, and simplified.

1 Introduction

Lattice paths form a classical[2] well-studied and important subject in Enumerative Combinatorics [3]. Yet, the fundamental paths considered here are not difficult to count. In fact, we may count all of them, including recently introduced variants, by using the same basic argument, as we show in this article. This was one of the main reasons for our contribution. The other one is to present the generating functions (also very important in Enumerative Combinatorics) associated to these paths, also by using a unifying argument, which clarifies, and in some cases simplifies, previously known formulas.

A *lattice path* is a sequence of points of integer coordinates in the plane such that the difference in coordinates of two consecutive points belongs to a given (small) set of vectors. In our case, all paths start at $(0,0)$ and end at a point of the x-axis. A *central lattice path* of length $2n$ for some $n \in \mathbb{N}$ is a path that starts at $(0,0)$ and ends at $(2n,0)$, such that two subsequent points in the sequence either differ by $D = (1,-1)$ (a down step) or by $U = (1,1)$ (an up step). The path may be seen as a sequence of letters D and U in equal number, n, and the position of the n letters D (or of the n

letters U) within the $2n$ letters determines bijectively the path. Hence, the number of central lattice paths of length $2n$ is $\binom{2n}{n}$. This forms a sequence that we can find in the On-Line Encyclopedia of Integer Sequences (OEIS), with reference

$$\text{A000984:}\left(\binom{2n}{n}\right)_{n\geqslant 0} = (1, 2, \mathbf{6}^1, 20, 70, 252, 924, 3\,432, 12\,870, \dots)\,.$$

Let us now consider the correspondent *generating function*, by definition

$$f(x) = \sum_{n\geqslant 0}\binom{2n}{n}x^n\,.$$

Generating functions were very successively used by Euler [2] with few explanations, in a way that we might think of as "infinite polynomials", with operations based on those of polynomials —the generating functions associated with quasi-zero sequences— with similar operational algebraic properties. For example, we define $\left(\sum_{n\geqslant 0} a_n\, x^n\right)(0) := a_0$,

$$\Big(\sum_{n\geqslant 0} a_n\, x^n\Big) + \Big(\sum_{n\geqslant 0} b_n\, x^n\Big) := \sum_{n\geqslant 0} (a_n + b_n)x^n$$

and

$$\Big(\sum_{n\geqslant 0} a_n\, x^n\Big)\Big(\sum_{n\geqslant 0} b_n\, x^n\Big) := \sum_{n\geqslant 0} \big(\textstyle\sum_{i=0}^{n} a_i\, b_{n-i}\big)x^n\,.$$

Thus, e.g., $\dfrac{1}{1-x} = \sum_{n\geqslant 0} x^n$ since $1 = \big(\sum_{n\geqslant 0} x^n\big)(1-x)$ in terms of generating functions, or

$$1 + \sum_{n\geqslant 1} 0\cdot x^n = \Big(\sum_{n\geqslant 0} 1\cdot x^n\Big)\Big(1 - x + \sum_{n\geqslant 2} 0\cdot x^n\Big)\,.$$

We also write $g(x) = \sqrt{f(x)}$ for generating functions $f(x)$ and $g(x)$ such that $f(0), g(0) \geqslant 0$ with the obvious meaning that $g(x)\,g(x) = f(x)$. Note that although some generating functions correspond to convergent series, at least in some neighborhoods of 0, others do not, like for example, $f(x) = \sum_{n\geqslant 0} n!x^n$. Yet, in some cases, we may use this correspondence.

[1] See Figure 2.

[2] To count *partitions*, that is, to count the number of ways of writing a positive number as the sum of smaller positive numbers.

For example, note that

$$\begin{aligned}\binom{-1/2}{n} &= \frac{(-1/2)(-3/2)\cdots(-1/2-n+1)}{n!}\\ &= \Big(-\frac{1}{2}\Big)^n \frac{(2n)!}{2^n n!\, n!}\,,\\ &= \Big(-\frac{1}{4}\Big)^n \binom{2n}{n}\end{aligned}$$

and hence, by the generalized Newton binomial theorem, for $|x| < \frac{1}{4}$,

$$(1-4x)^{-1/2} = \lim_{p\to+\infty} \sum_{n=0}^{p} \binom{2n}{n} x^n$$

and so, also *as generating functions*, $\left(\sum_{n\geqslant 0} \binom{2n}{n} x^n\right)^2 = \frac{1}{1-4x}$ and

$$f(x) = \frac{1}{\sqrt{1-4x}}\,.^{3} \tag{1}$$

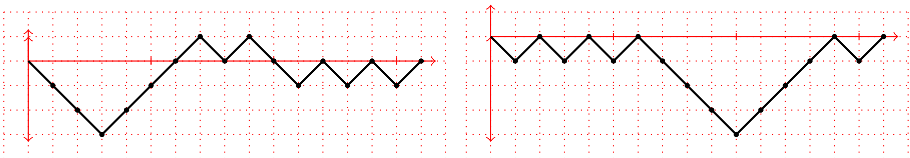

Figura 1: A *central lattice path* and a *Dyck path*

A central lattice path without any point above the x-axis is called a *Dyck path*. We may count Dyck paths by counting *non-Dick* central lattice paths $\mathcal{D}$ that *do* have a first point P above the x-axis: build a path $\mathcal{D}'$ with the same steps as $\mathcal{D}$ up to P and with the opposite steps afterward. Then $\mathcal{D}'$ is a lattice path from $(0,0)$ to $(2n,2)$, and every lattice path from $(0,0)$ to $(2n,2)$ can be thus obtained. Since there are $\binom{2n}{n-1}$ lattice paths (with $n-1$ down steps and $n+1$ up steps) from $(0,0)$ to $(2n,2)$, the number of Dyck paths is the *Catalan number of order n*,

$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}\,.$$

[3]In particular, since $\Big(\sum_{n\geqslant 0}\binom{2n}{n}x^n\Big)^2 = \sum_{n\geqslant 0} 4^n x^n$, $\sum_{\substack{0\leqslant i,j\leqslant n\\ i+j=n}} \binom{2i}{i}\binom{2j}{j} = 4^n$.

This forms the sequence in OEIS

$$\text{A000108:}\left(C_n\right)_{n\geqslant 0} = (1, 1, \mathbf{2}, 5, 14, 42, 132, 429, 1\,430, \dots)\,.$$

Since

$$C_n = 2\binom{2n}{n} - \frac{1}{2}\binom{2(n+1)}{n+1}\,,$$

we have that

$$\begin{aligned} 2x\sum_{n\geqslant 0}\frac{1}{n+1}\binom{2n}{n}x^n &= 4x\sum_{n\geqslant 0}\binom{2n}{n}x^n - \sum_{n\geqslant 1}\binom{2n}{n}x^n \\ &= 1 + (4x-1)\sum_{n\geqslant 0}\binom{2n}{n}x^n \\ &= 1 - \sqrt{1-4x}\,. \end{aligned}$$

Hence,

$$\sum_{n\geqslant 0} C_n x^n = \frac{2}{1+\sqrt{1-4x}}\,. \tag{2}$$

2 Other lattice paths

2.1 Central Delannoy paths and Schröder paths

In a *central Delannoy paths* [1] from $(0,0)$ to $(2n,0)$, for $n \in \mathbb{N}$, two subsequent points in the sequence either differ by $D = (1,-1)$ or by $U = (1,1)$, as before, or by a *horizontal step* $H = (2,0)$. The central Delannoy paths that remain below the diagonal are called *Schröder paths*.

Any such path $\mathfrak{d}$ with d down steps must present also d up steps and $n-d$ forward steps. Of course, $\mathfrak{d}$ is determined by the length $2d$ central lattice path $\mathfrak{c}$ formed by the Ds and the Us, and by the positions on $\mathfrak{c}$ where the $n-d$ letters H are placed. Hence, with the same central lattice path $\mathfrak{c}$, there are

$$\binom{2d+(n-d)}{n-d} = \binom{n+d}{n-d}$$

central Delannoy paths. Note that $\mathfrak{d}$ is a Schröder path if and only if $\mathfrak{c}$ is a Dyck path. The numbers of central Delannoy paths form the OEIS sequence

$$\text{A001850:}\left(\sum_{d=0}^{n}\binom{n+d}{n-d}\binom{2d}{d}\right)_{n\geqslant 0} = (1, 3, \mathbf{13}, 63, 321, 1\,683, 8\,989, 48\,639, \dots)$$

whereas the numbers of Schröder paths form the OEIS sequence

$$\text{A006318: } \left(\sum_{d=0}^{n} \tfrac{1}{d+1} \binom{n+d}{n-d} \binom{2d}{d} \right)_{n \geqslant 0} = (1, 2, \mathbf{6}, 22, 90, 394, 1\,806, 8\,558, \ldots)$$

Below, we represent the central Delannoy paths for $n = 2$, where the paths drawn in red are Schröder paths. The first six paths are the central lattice paths.

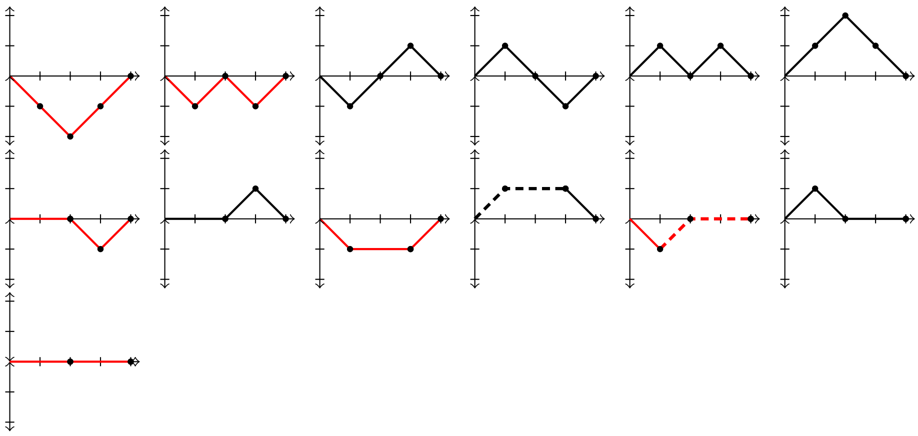

Figura 2: Delannoy and Schröder paths with $n = 2$

2.2 [Big] Motzkin paths

If, instead of allowing steps with $H = (2, 0)$, we allow steps with $F = (1, 0)$, the path $\mathfrak{m}$ is called a *big Motzkin path* [4]. A big Motzkin path is simply a *Motzkin path* if the path remains below the x-axis. Note that, if the path contains d down steps, then it contains also d up steps and $n - 2d$ forward steps. Hence, for a given central lattice path $\mathfrak{c}$, there are

$$\binom{2d + (n - 2d)}{n - 2d} = \binom{n}{n - 2d} = \binom{n}{2d}$$

Motzkin paths. Again, Motzkin paths occur when $\mathfrak{c}$ is a Dyck path. The numbers of big Motzkin paths form the *sequence of central trinomial coefficients*, the OEIS sequence

$$\text{A002426: } \left(\sum_{d=0}^{n} \binom{n}{2d} \binom{2d}{d} \right)_{n \geqslant 0} = (1, 1, 3, 7, 19, \mathbf{51}, 141, 393, 1\,107, 3\,139, \ldots)$$

whereas the numbers of Motzkin paths form the OEIS sequence

$$\text{A001006: } \left(\sum_{d=0}^{n} \frac{1}{d+1}\binom{n}{2d}\binom{2d}{d}\right)_{n\geqslant 0} = (1, 1, 2, 4, 9, \mathbf{21}, 51, 127, 323, 835, \dots)$$

We first represent the 21 Motzkin paths and then the remaining 30 big Motzkin paths of length 5.

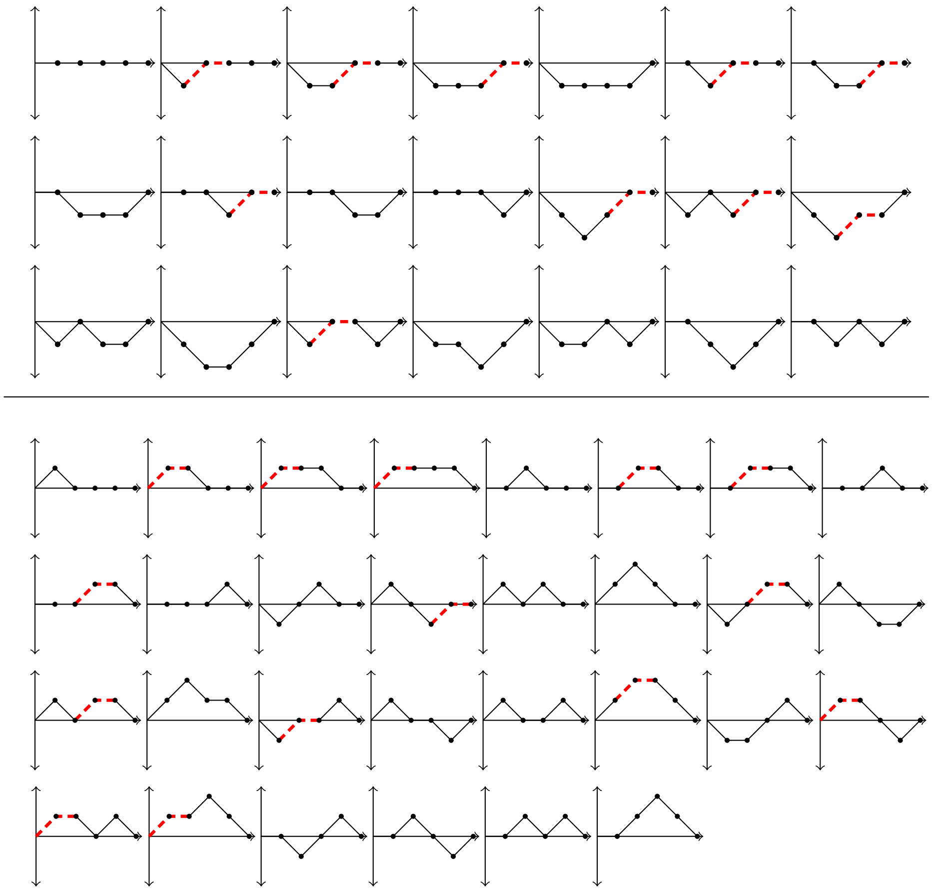

Figura 3: Motzkin and big Motzkin paths

2.3 Restricted central Delannoy and big Motzkin paths

We now count paths where given subsequences of paths, namely the subsequences *UH* and *UF*, are not allowed. The first ones are, respectively, the

UH-avoiding big Motzkin paths and the *UH-avoiding* Motzkin paths. In Figure 2 *UH* subsequences and in Figure 3 *UF* subsequences are dashed. Note that these paths are determined by sequences formed only by the d down steps and the $n-2d$ forward steps, and hence its numbers are, respectively, $\sum_{d=0}^{n}\binom{n-d}{n-2d}\binom{2d}{d}$ and $\sum_{d=0}^{n}\frac{1}{d+1}\binom{n-d}{n-2d}\binom{2d}{d}$. Thus, UH-avoiding big Motzkin paths and UH-avoiding Motzkin paths, respectively, form the OEIS sequences

$$\text{A026569:}\Big(\sum_{d=0}^{n}\binom{n-d}{d}\binom{2d}{d}\Big)_{n\geqslant 0}=(1,1,3,5,13,\mathbf{27},67,153,375,893,2\,189,\dots)$$
$$\text{A090344:}\Big(\sum_{d=0}^{n}\frac{1}{d+1}\binom{n-d}{d}\binom{2d}{d}\Big)_{n\geqslant 0}=(1,1,2,3,6,\mathbf{11},23,47,102,221,493,\dots)$$

Central Delannoy paths and Schröder paths with d down steps also contain d up steps, but now contain $n-d$ forward steps, and so the sequences are OEIS sequence

$$\text{A026375: }\Big(\sum_{d=0}^{n}\binom{n}{d}\binom{2d}{d}\Big)_{n\in\mathbb{N}_0}=(1,3,\mathbf{11},45,195,873,3\,989,18\,483,\dots)$$

and OEIS sequence

$$\text{A007317: }\Big(\sum_{d=0}^{n}\frac{1}{d+1}\binom{n}{d}\binom{2d}{d}\Big)_{n\in\mathbb{N}_0}=(1,2,\mathbf{5},15,51,188,731,2\,950,\dots)$$

We note that, for every lattice path sequence $V=\big(v_n\big)_{n\in\mathbb{N}_0}$ defined above, we have found a double infinite triangular array $T=\big(b_{n,d}\big)_{0\leqslant d\leqslant n}$ such that

$$v_n=\sum_{d=0}^{n}b_{n,d}\,u_d\quad\text{for every } n\in\mathbb{N}_0$$

or, in other words, such that $V^T=T\cdot U^T$, where $U=\big(u_n\big)_{n\in\mathbb{N}_0}$ is either the central lattice path sequence or the Dyck path sequence. In fact, by Lemma 1, below, in all cases, for two given generating functions $f(x)$ and $g(x)$,

$$b_{n,d}=[x^n]\big(f(x)(x\,g(x))^d\big)\quad\text{for } 0\leqslant d\leqslant n\,,$$

or, equivalently,

$$\sum_{n\geqslant d}b_{n,d}\,x^n=f(x)(x\,g(x))^d\,.$$

We shorten notations by writing the *Riordan array* $(f(x)\mid g(x))$ for $T=\big(b_{n,d}\big)_{0\leqslant d\leqslant n}$. We can now obtain the generating functions of these sequences by adequately transforming (1) and (2). In fact, if $B(x) = \sum_{n\geqslant 0} v_n x^n$ and $A(x) = \sum_{n\geqslant 0} u_n x^n$, then (see [6])

$$\begin{aligned} B(x) &= \sum_{n\geqslant 0} \Big(\sum_{d=0}^{n} b_{n,d}\, u_d \Big) x^n \\ &= \sum_{d\geqslant 0} u_d \Big(\sum_{n\geqslant d} b_{n,d}\, x^n \Big) \\ &= \sum_{d\geqslant 0} u_d\, f(x) \big(x\, g(x) \big)^d \\ &= f(x) \Big(\sum_{d\geqslant 0} u_d \big(x\, g(x) \big)^d \Big) , \end{aligned}$$

that is,

$$B(x) = f(x) A\big(x\, g(x) \big) . \tag{3}$$

We note that, in the case of the restricted central Delannoy and big Motzkin paths, this unifies and clarifies entries A026569, A090344, A026375, and A007317 of the OEIS (Cf. [7, 8]), and generally simplifies the corresponding generating functions.

Lemma 1.

$$\begin{aligned} \left(\frac{1}{1-x} \,\middle|\, \frac{1}{(1-x)^2} \right) &= \left(\binom{n+d}{n-d} \right)_{0\leqslant d\leqslant n} ; \\ \left(\frac{1}{1-x} \,\middle|\, \frac{x}{(1-x)^2} \right) &= \left(\binom{n}{2d} \right)_{0\leqslant d\leqslant n} ; \\ \left(\frac{1}{1-x} \,\middle|\, \frac{x}{1-x} \right) &= \left(\binom{n-d}{d} \right)_{0\leqslant d\leqslant n} ; \\ \left(\frac{1}{1-x} \,\middle|\, \frac{1}{1-x} \right) &= \left(\binom{n}{d} \right)_{0\leqslant d\leqslant n} . \end{aligned}$$

Demonstração. Note that, by the generalized Newton binomial theorem again, if $\alpha = -n$, $n \in \mathbb{N}$, since $\binom{\alpha}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k \binom{n+k-1}{k}$,

$$(1-x)^{-n} = \sum_{k\geqslant 0} \binom{n+k-1}{k} x^k .$$

Let us prove the first identity. For $f(x) = \frac{1}{1-x}$ and $g(x) = \frac{1}{(1-x)^2}$,

$$\begin{aligned}[x^n]\big(f(x)(x\,g(x))^d\big) &= [x^n]\left(\frac{x^d}{(1-x)^{2d+1}}\right)\\ &= [x^{n-d}]\left(\frac{1}{(1-x)^{2d+1}}\right)\\ &= [x^{n-d}]\sum_{m\geqslant 0}\binom{2d+m}{m}x^m\\ &= \binom{2d+n-d}{n-d}\\ &= \binom{n+d}{n-d}.\end{aligned}$$

The other identities are proven similarly, being

$$\begin{aligned}[x^n]\left(\frac{1}{1-x}\left(\frac{x^2}{(1-x)^2}\right)^d\right) &= [x^{n-2d}]\left(\sum_{m\geqslant 0}\binom{2d+m}{m}x^m\right),\\ [x^n]\left(\frac{1}{1-x}\left(\frac{x^2}{1-x}\right)^d\right) &= [x^{n-2d}]\left(\sum_{m\geqslant 0}\binom{d+m}{m}x^m\right),\\ [x^n]\left(\frac{1}{1-x}\left(\frac{x}{1-x}\right)^d\right) &= [x^{n-d}]\left(\sum_{m\geqslant 0}\binom{d+m}{m}x^m\right). \qquad \square\end{aligned}$$

Then, the generating function of the sequence of the central Delannoy numbers is, by (3),

$$\frac{1}{1-x}\,\frac{1}{\sqrt{1-4\frac{x}{(1-x)^2}}} = \frac{1}{\sqrt{1-6x+x^2}}$$

whereas the generating function of the sequence of the central Schröder numbers is

$$\frac{1}{1-x}\,\frac{2}{1+\sqrt{1-4\frac{x}{(1-x)^2}}} = \frac{2}{1-x+\sqrt{1-6x+x^2}}\,.$$

The generating functions of the sequences of UF-avoiding central Delannoy numbers and of UF-avoiding Schröder numbers are, respectively

$$\frac{1}{1-x}\,\frac{1}{\sqrt{1-4\frac{x}{1-x}}} = \frac{1}{\sqrt{1-6x+5x^2}}$$

and

$$\frac{2}{(1-x)\left(1+\sqrt{1-4\frac{x}{1-x}}\right)} = \frac{2}{1-x+\sqrt{1-6x+5x^2}}\,.$$

Likewise, the generating function of the sequence of the big Motzkin numbers is

$$\frac{1}{1-x}\frac{1}{\sqrt{1-4\frac{x^2}{(1-x)^2}}}=\frac{1}{\sqrt{1-2x-3x^2}}$$

the generating function of the sequence of the Motzkin numbers is

$$\frac{2}{(1-x)\left(1+\sqrt{1-4\frac{x^2}{(1-x)^2}}\right)}=\frac{2}{1-x+\sqrt{1-2x-3x^2}}$$

and the generating function of the sequence of UH-avoiding big Motzkin and Motzkin numbers are, respectively

$$\begin{aligned}\frac{1}{1-x}\frac{1}{\sqrt{1-4\frac{x^2}{1-x}}}&=\frac{1}{\sqrt{1-x}\sqrt{1-x-4x^2}}\\&=\frac{1}{\sqrt{1-2x-3x^2+4x^3}}\end{aligned}$$

and

$$\frac{2}{(1-x)\left(1+\sqrt{1-4\frac{x^2}{1-x}}\right)}=\frac{2}{1-x+\sqrt{1-2x-3x^2+4x^3}}\,.$$

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