Geometria e Topologia

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Geometria dos espaços de gwistor

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Resumo: Breve introdução ao espaço de gwistor, a estrutura G_2 natural existente no fibrado de esferas tangente $\pi : SM \to M$ de qualquer variedade riemanniana M, orientável e de dimensão 4.

Abstract Brief introduction to gwistor space or the natural G_2-structure associated to any oriented riemannian 4-manifold.

palavras-chave: fibrado esferas; estrutura G_2; métrica de Einstein.

keywords: tangent sphere bundle; G_2 structure; Einstein manifold.

1 Estruturas G_2

Seja T um espaço euclidiano orientado de dimensão 4 e fixemos um vector $u \in S^3 \subset T$, onde S^3 denota a esfera de raio 1. Qualquer outro vector de T escreve-se de forma única como $\lambda u + X$, com $\lambda \in \mathbb{R},\ X \in u^\perp$.

Reparamos então que T suporta uma estrutura quaterniónica natural, ou seja, uma estrutura de álgebra de divisão isomorfa a $\mathbb{H}$, com a métrica euclidiana inicial e tal que u é o elemento unidade. Com efeito, a operação produto é dada por

$$(\lambda_1 u + X)(\lambda_2 u + Y) = (\lambda_1\lambda_2 - \langle X, Y\rangle)u + \lambda_2 X + \lambda_1 Y + X \times Y\ ,$$

sendo $\times$ um produto-cruzado, definido em $u^\perp$ pela identidade $\langle X \times Y, Z\rangle = \mathrm{vol}(u, X, Y, Z)$ para qualquer triplo $X, Y, Z \in u^\perp$. A operação de conjugação em T compatível é descrita obviamente por $\overline{\lambda u + X} = \lambda u - X$.

Lembremos agora que a *maior* álgebra de divisão normada com elemento unidade que existe, a álgebra dos octoniões $\mathbb{O} = \mathbb{H} \oplus e\mathbb{H}$, pode ser encontrada através do processo de Cayley-Dickson:

$$(z_1, z_2) \cdot (z_3, z_4) = (z_1 z_3 - \overline{z_4} z_2, z_4 z_1 + z_2 \overline{z_3}), \qquad \forall z_i \in \mathbb{H}.$$

Note-se que e é um novo elemento de quadrado -1, como o é i em $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ ou j em $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$; porém, não só por si definem tais vectores a estrutura desejada (há muitos quádruplos ortonormados i, j, k, e na mesma relação). A operação de produto é suficiente. Por outras palavras, e voltando ao espaço T acima, interessam-nos as estruturas de módulo de T sobre $\mathbb{H}$ e a de $T \oplus T \simeq \mathbb{R}.1 \oplus \mathbb{R}^7$ sobre $\mathbb{O}$. Finalmente temos

$$\mathrm{G}_2 = \mathrm{Aut}\,\mathbb{O}$$

que é grupo de Lie simples, compacto, simplesmente conexo, de dimensão 14. Elementos de G_2 preservam a unidade 1 e a métrica, bem como a orientação, logo $\mathrm{G}_2 \subset \mathrm{SO}(7)$.

Tal grupo de Lie encontra-se entre as poucas classes de grupos de Lie simples dando origem a uma geometria riemanniana, dita excepcional, actualmente muito em voga em geometria e física. Surge como um dos possíveis grupos de holonomia irredutível de variedades riemannianas não simétricas (classificação descrita num bem conhecido resultado de M. Berger de 1955). Mas $\mathrm{G}_2 \subset \mathrm{SO}(7)$ só aparece de modo irredutível em variedades de dimensão 7. Sabe-se que existem mesmo variedades com aquele grupo de holonomia (R. Bryant, 1987).

Uma **estrutura** G_2 numa variedade $\mathcal{S}$, de dimensão 7, é definida por uma 3-forma ϕ *estável*, i.e.

- $\exists$ um produto vectorial tal que $\phi(X, Y, Z) = \langle X \cdot Y, Z \rangle$, o qual corresponde ao produto octoniónico dos imaginários puros $\mathbb{R}^7 \subset \mathbb{O}$, ou
- $\phi = e^{456} + e^{014} + e^{025} + e^{036} - e^{126} - e^{234} - e^{315}$ nalgum referencial (por definição, $e^{ijk} = e^i \wedge e^j \wedge e^k$), ou
- ϕ pertence a certa $\mathrm{GL}(7)$-órbita aberta de $\Lambda^3 T^*\mathcal{S}$ (dim $35 = 49 - 14$).

A existência de tal 3-forma implica a redução do grupo de estrutura $\mathrm{GL}(7)$ da variedade $\mathcal{S}$ para $\mathrm{SO}(7) \supset \mathrm{G}_2$, de forma única. Com efeito:

ϕ determina a orientação e uma única métrica sobre $\mathcal{S}$.

Como acontece noutras geometrias, a holonomia da variedade riemanniana e orientada $(\mathcal{S}, \phi)$ reduz-se a $\mathrm{G}_2 = \{g \in \mathrm{SO}(7) : \ g^*\phi = \phi\}$ se $\nabla^g \phi = 0$, onde ∇^g denota a conexão de Levi-Civita. Neste caso, $\mathcal{S}$ diz-se uma **variedade** G_2. A topologia de $\mathcal{S}$ é condicionada, cf. [6].

Um teorema de A. Gray garante: $\nabla^g \phi = 0$ se e só se $\mathrm{d}\phi = 0$ e $\mathrm{d} * \phi = 0$. Tais equações são raramente satisfeitas nos espaços que se conhecem actualmente; interessam-nos, por outro lado, parte das equações. ϕ diz-se *calibrada* se $\mathrm{d}\phi \in \Lambda^4 T^*\mathcal{S}$ se anula. E ϕ diz-se *cocalibrada* se $\mathrm{d} * \phi$ se anula.

2 O espaço de gwistor

Seja (M, g) uma variedade riemanniana, orientada, de dimensão 4 e seja

$$SM = \{u \in TM : \ \|u\| = 1\}.$$

Seja $\pi : TM \longrightarrow M$ o fibrado vectorial tangente. Tem-se $\mathrm{d}\pi : TTM \longrightarrow \pi^*TM$, morfismo de fibrados sobre TM. Como bem se sabe da geometria clássica de TM, identifica-se $\ker \mathrm{d}\pi \simeq \pi^*TM$ de forma natural. A conexão ∇^g sobre M permite escrever $TTM = H^{\nabla^g} \oplus \ker \mathrm{d}\pi \ \simeq \ \pi^*TM \oplus \pi^*TM$. O fibrado, dito vertical, $\ker \mathrm{d}\pi$ contém uma secção canónica, U, que em cada ponto $u \in SM$ toma o valor $u \in \pi^*TM$. Não é dificil provar que $TSM = U^\perp \subset TTM$. Reproduzindo a construção algébrica da secção 1, com estrutura métrica óbvia em $(\pi^*TM)_u = T$, fica demonstrado o

Teorema 2.1 *SM admite uma estrutura G_2 natural.*

A tal estrutura damos o nome **espaço G_2-twistor** ou **gwistor** de M.

Sobre o espaço de gwistor podemos sempre construir, localmente, um referencial móvel ortonormado directo, $e_0 = u$, $e_1, e_2, e_3 \in H^{\nabla^g}$, sistema depois reproduzido como $U_u, e_4, e_5, e_6 \in \ker \mathrm{d}\pi$ no subespaço vertical, de modo a descrever as seguintes formas globais de SM:

$$\alpha = \text{3-forma volume nas fibras de } SM = e^{456} \ , \qquad \theta = e^0 \ ,$$
$$\alpha_1 = e^{156} + e^{264} + e^{345} \ , \qquad \alpha_2 = e^{126} + e^{234} + e^{315} \ ,$$
$$\alpha_3 = e^{123} \ , \qquad \mathrm{vol} = \pi^*\mathrm{vol}_M = e^{0123} = \theta \wedge \alpha_3 \ .$$

Sendo U e e_0 campos definidos globalmente, em boa verdade a estrutura de SM reduziu-se a SO(3). Note-se que já se conheciam a métrica (de Sasaki) em SM, o campo vectorial $e_0 \in \mathfrak{X}_{SM}$ e logo a 1-forma θ, a qual verifica $(\mathrm{d}\theta)^3 \wedge \theta \neq 0$, implicando o resultado que diz que $(SM, \frac{1}{4}g, \frac{1}{2}\theta, 2e_0)$ constitui uma estrutura métrica de contacto (Y. Tashiro).

Finalmente,

$$\phi = \alpha_2 - \alpha + \theta \wedge \mathrm{d}\theta \ .$$

Observamos ainda que se pode generalizar a construção do espaço de gwistor com H^∇ provindo de qualquer conexão métrica ∇ em M.

Agora, sobre SM definem-se $\underline{r} = \underline{r}_u = r^{\nabla^g}(u, u)$, $\rho = r^{\nabla^g}(\ ,U) = (\mathrm{ric}\, U)^\flat$. Tem-se que $\mathrm{d} * \phi = \rho \wedge \mathrm{vol}$ e que $\mathrm{d}\phi = 2\theta \wedge \alpha_1 - \underline{r}\,\mathrm{vol} - \mathcal{R}^U\alpha + (\mathrm{d}\theta)^2$ onde

$$\mathcal{R}^U\alpha := \mathrm{d}\alpha = \sum_{0 \le i < j \le 3} R_{ij01}e^{ij56} + R_{ij02}e^{ij64} + R_{ij03}e^{ij45} \ .$$

Teorema 2.2 *Tem-se sempre* $\mathrm{d}\phi \neq 0$*; Temos* $\mathrm{d} * \phi = 0$ *se e só se* (M, g) *é variedade de Einstein.*

Exemplos. 1. se $M = \mathcal{H}^4$ é o espaço hiperbólico real com curvatura seccional -2, então $SM = S\mathcal{H}^4 = \frac{\mathrm{SO}_0(4,1)}{\mathrm{SO}(3)}$ é de tipo puro $W_3 = \{\tau \in \Lambda^3 : \ \tau \wedge \phi = \tau \wedge *\phi = 0\}$:

$$\mathrm{d}\phi = *\tau = *(2\theta \wedge \mathrm{d}\theta + 6\alpha) \ , \qquad \mathrm{d} * \phi = 0 \ .$$

2. espaços simétricos de rank 1 geram espaços de gwistor homogéneos,

$$S\mathbb{S}^4 = V_{5,2} \ , \qquad S\mathbb{CP}^2 = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)} = N_{1,1} \ , \qquad S\mathcal{H}^2_{\mathbb{C}} = \frac{SU(2,1)}{U(1)} \ .$$

Outros resultados sobre G_2 e novos desenvolvimentos na teoria dos espaços de gwistor encontram-se na bibliografia.

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Counting maps from curves to projective space via graph theory

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1 Brill–Noether theory on reducible curves

In Brill–Noether theory, one studies linear series on curves, in order to understand when a curve C of genus g comes equipped with a nondegenerate morphism of degree d to $\mathbb{P}^r$. For a *general* curve $[C] \in \mathcal{M}_g$, a basic answer is provided by the *Brill–Noether theorem* of Griffiths and Harris, which establishes that C admits such a morphism if and only if the invariant

$$\rho(d,g,r) = g - (r+1)(g-d+r)$$

is nonnegative, in which case ρ also computes the dimension of the space of linear series g^r_d of degree d and rank r on C. The Brill–Noether question also admits natural extensions, obtained by imposing incidence conditions on the images of the linear series in question. Namely, given integers $m \geq d$ and $s \geq d-r$, let $\mu(d,r,s) := d - r(s+1-d+r)$ denote the virtual dimension of space of inclusions

$$g^{s-d+r}_{m-d} + p_1 + \cdots + p_d \hookrightarrow g^s_m \tag{1}$$

on a fixed curve. When the curve C in question is smooth, and the g^s_m is a subspace $V \subset H^0(C,L)$ of global sections of a line bundle L, such inclusions correspond to d-tuples of points $p_1, \ldots, p_d \in C$ for which the natural evaluation map

$$\mathrm{ev} : V \to H^0(C, L/L(-p_1 - \cdots - p_d)) \tag{2}$$

satisfies $\mathrm{rank}(\mathrm{ev}) = d - r$. Geometrically, such d-tuples determine d-secant $(d-r-1)$-planes to the image of the g^s_m. In [3], we showed that when $\rho = 0$ and $\mu < 0$, there are *no* inclusions (1) on a general curve:

Theorem 1.1. *If $\rho = 0$ and $\mu < 0$, then a general curve C admits no linear series g^s_m with d-secant $(d-r-1)$-planes.*

Our proof of Theorem 1.1 is a natural generalization of the Brill–Noether proof given in [5, Ch. 5] and is based on an analysis of (limit) linear series on certain reducible curves of compact type.

2 Counting secant planes via graph theory

An immediate corollary of Theorem 1.1 is that *when $\rho=0$ and $\mu=-1$, curves with linear series g^s_m with d-secant $(d-r-1)$-planes determine a divisor in $\mathcal{M}_g$.* The case $r=1$ is particularly natural: in that case, exceptional secant planes correspond to d-tuples of points for which the evaluation maps (2) fail to be *surjective*. We show [4, Thm 2]:

Theorem 2.1. *The coefficients of the homology classes of secant-plane divisors in $\overline{\mathcal{M}}_g$, realized as linear combinations of standard generators over $\mathbb{Q}$, are explicit linear combinations of hypergeometric series of type ${}_3F_2$.*

The key ingredient for proving Theorem 2, which is of interest in its own right, is the following auxiliary result [3, Thm 4]:

Theorem 2.2. *The generating series for the virtual number N_d of d-secant $(d-2)$-planes to a degree-m curve C of genus g in $\mathbb{P}^{2d-2}$ is*

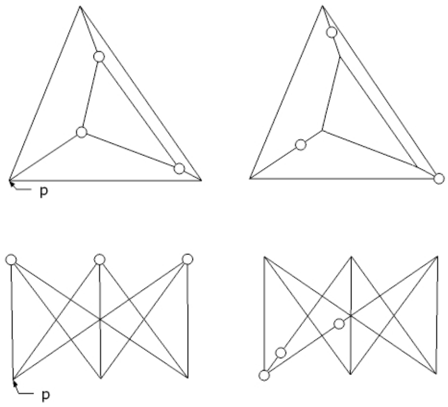
$$\sum_{d\geq 0} N_d(g,m) = \left(\frac{2}{(1+4z)^{1/2}+1}\right)^{2g-2-m} \cdot (1+4z)^{\frac{g-1}{2}}. \tag{3}$$

Two ingredients enter into our proof of Theorem 2.2. The first is *Porteous' formula*, which computes the homology class of the locus of d-secant $(d-2)$-planes as a determinant in the Chern classes of the so-called dth *tautological* bundle $L^{[d]}$ over the dth Cartesian product C^d, whose fiber over $(p_1,\dots,p_d)\in C^d$ is $H^0(L/L(-p_1-\cdots-p_d))$. The second is a combinatorial analysis of the resulting intersection-theoretic formula, which amounts to a weighted count of subgraphs of the complete graph K_d on d vertices.

3 Linear series on metric graphs

In the preceding section, graphs naturally arose in connection with counting (secant planes to) morphisms via the formalism of intersection theory. But graph theory also intervenes in a natural way as a result of degeneration, via the passage from a nodal curve to the *dual graph* recording the incidences of its components. There is a theory of complete linear series on metric graphs with $\mathbb{R}$-valued edge lengths due to Baker–Norine [1] and Mikhalkin–Zharkov [6]. Concretely, a (complete) linear series $|D|$ on a metric graph Γ is a configuration D of points in Γ, modulo an equivalence relation defined by piecewise-linear functions. Moreover, there is an explicit combinatorial *burning algorithm* due to Dhar for computing the rank of a configuration of points $D\in \mathrm{Div}(\Gamma)$; see [2].

Contrasting examples in genus four. Figure (a) shows two metric graphs of genus 4 (here, as in the remainder of the article, we assume that all weights on vertices are 0). The top graph Γ_1 pictured is planar, and the 3 circles determine a degree-3 configuration D_1 of trivial rank. Indeed, a fire that burns from p will be repelled by the 3 points in the support of D_1, which then evolve at equal velocity against the incoming fire. Assuming the planar graph has generic edge lengths, a single point p_1 of D_1 will arrive at a vertex v_1 of Γ_1, at which point a fire burning from p will approach v_1 (and p_1) from 2 distinct directions and all of Γ_1 will burn. By contrast, the configuration D_2 of 3 points on the complete bipartite graph $\Gamma_2 = K_{3,3}$ evolves in such a way that at at no time will any fire based at any point p approach any point in the support of D_2 along two distinct directions. It follows that $r(D_1) \geq 1$, and in fact the rank of D_1 is precisely 1.



4 The gonality of tree-decomposed graphs

The contrast between the behavior of degree-3 configurations on the planar genus-4 graph Γ_1 and on $\Gamma_2 = K_{3,3}$ is instructive. In fact, it is not hard to check that Γ_1 and Γ_2 each admit two degree-3 configurations of rank 1, as predicted by Brill–Noether theory for curves of genus 4. However, on Γ_1, these configurations depend strongly on the metric structure: each is obtained by placing 2 points on 2 out of 3 inner (resp., outer) “rim" vertices, and a third point along a “spoke" at distance from an outer (resp., inner) vertex at distance equal to the length of the shortest spoke. On Γ_2, on the other hand, each rank-1 configuration is associated to a choice of one of the two sets of 3 vertices along which Γ_2 decomposes as a union of three 4-edged trees.

Definition/construction. Let $V = \{v_1, \dots, v_n\}$, $n \geq 3$ denote a fixed set of vertices, and let T_1, T_2, and T_3 denote three trees each containing V as vertices but which are otherwise pairwise disjoint. The three trees $T_i, 1 \leq i \leq 3$ glue naturally to a graph Γ; we say that Γ admits a *tree decomposition* (T_1, T_2, T_3) *rooted along* V.

Some of the most famous graphs of genus at most 10 admit such tree decompositions: besides $K_{3,3}$, the examples of the so-called Petersen, Heawood, and Pappus graphs in genera 6, 8, and 10 (respectively) are tree-decomposable.

Theorem 4.1 (Existence of rank-one series on tree-decomposed graphs)**.** *Suppose that the metric graph Γ admits a tree decomposition rooted on $n \geq 3$ vertices V. Then V determines a rank-one, degree-n divisor D on Γ.*

Proof. The result follows from the burning algorithm. Namely, fix any choice of base point p from which to burn, say $p \in T_1$ without loss of generality. Any fire burning from p along T_1 is repelled by the points $p_1, \dots, p_n$ of D supported along V, which then evolve at equal velocity along T_1 away from V. The burning process iterates until ultimately the fire is extinguished by at least one of the points p_i, which proves that $r(D) \geq 1$. Similarly, to prove that $r(D) < 2$, it suffices to allow *two* successive fires to burn from p_1 : the first fire simply has the effect of canceling out p_1, while the second burns through all of Γ. $\square$

Definition. A graph (or a curve) Γ of genus g is *n-gonal* whenever $n = \min\{j \in \mathbb{Z}_{>0} : \exists \text{ a } g_j^1 \text{ on } \Gamma\}$.

Theorem 4.2. *$K_{3,3}$, Petersen, Heawood, and Pappus are 3-gonal, 4-gonal, 5-gonal, and 6-gonal graphs, respectively.*

Proof sketch. It is easy to exhibit tree decompositions of these graphs rooted on $n = 3, 4$, 5, and 6 vertices, respectively. Whence, by Theorem 4.1, it suffices to prove that each of these n-rooted tree-decomposed graphs admits no degree-$(n-1)$ configurations D of positive rank. Replacing D by a linearly equivalent configuration if necessary, we may assume that each point in $\mathrm{Supp}(D)$ appears with multiplicity at most 2. It remains to carry out a case-by-case inspection using the burning algorithm. $\square$

It is not hard to produce graphs that decompose as unions of trees rooted on $n \geq 3$ vertices but are α-gonal with $\alpha < n$. So additional conditions are needed to ensure that n-gonality is achieved. Theorem 4.2 and experimentation give some evidence that it suffices to maximize the minimal cycle length, or *girth*, of Γ.

Conjecture 4.1. *A metric graph Γ that admits a tree-decomposition (T_1, T_2, T_3) rooted on n vertices is n-gonal provided girth(Γ) is maximal for the combinatorial type of (T_1, T_2, T_3).*

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Spectral theory for toric orbifolds

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Abstract This informal note is a written version of the talk I gave at the Geometry/Topology session of the Portuguese Mathematical Society meeting. The talk addressed the toric version of the classical Riemannian geometry question "How much about a toric manifold can you recover from the spectrum of its Laplacian for a toric metric?". This problem was first proposed by Miguel Abreu. This is a very informal report on some progress towards answering that question. This is joint work with E. Dryden and V. Guillemin

1 Introduction

We start by giving a somewhat precise statement of the result we shall discuss.

Theorem 1.1. *The equivariant spectrum of a toric Kahler metric on a generic toric orbifold determines the toric orbifold up to symplectomorphisms and two possibilities.*

One of the goals of this note is to explain the words in the above theorem and the context in which it arises. But mainly what we want to underline here is that the above translates into a very elementary count. One that could essentially be carried out by a motivated high-school student.

2 Combinatorics

We start by describing the elementary count.

Question 2.1. *How many convex polytopes in* $\mathbb{R}^2$ *are there with given*
(1) Number of sides d,
(2) Length of sides,
(3) Direction of sides up to sign?

Note that 3 and 2 determine the edge vectors up to sign. A priori it seems that there are $2^d d!$ choices as one needs to choose signs and an ordering for the edge vectors.

Claim 2.2. *Generically there are at most* 2 *polytopes with data 3, 2 and 1.*

To back this claim we will describe two small propositions. The next one is fun!

Proposition 2.3 (The most obtuse angle lemma)**.** *Let P be a convex polytope in*$\mathbb{R}^2$*. Let* $e_1, \cdots, e_d$ *denote the ordered edge vectors of P where we take the negative orientation in* *the plane to orient ∂P. The angle between e_1 and e_2 is the most obtuse angle among the angles*

$$\langle e_1, e_i \quad i = 2, \cdots d$$

Proof. Draw a picture! Assume e_1 is $(1,0)$. Draw all the edges at the end of e_1 as a spray. If you choose the second edge to be one whose angle is not the most obtuse angle with e_1 you will eventually have to put the edge corresponding to that angle in (after the one you chose for second). You won't be able to do that without violating convexity. □

Using the above notation, this implies that there is a unique ordering of the edge vectors $e_1, \cdots, e_d$ that closes up into a convex polytope. Next we deal with the choice of sign. Using the notation in the statement of the proposition we have

$$\sum_{i=1}^{d} e_i = 0,$$

in $\mathbb{R}^2$. Suppose there was another choice for the signs of $e_1, \cdots, e_d$ that led to a closed convex polytope say $s_1 e_1, \cdots, s_d e_d$ where $s_i = \pm 1$, $i = 1, \cdots d$. Then we would have

$$\sum_{i=1}^{d} s_i e_i = 0.$$

Unless all the s_i's are 1 or they are all -1, by adding these two relations we would get

$$\sum_{i \in I} e_i = 0,$$

where I is a proper subset of $\{1, \cdots, d\}$. This means that there is a proper subset of the set of edges op P that closes up to give a convex polytope. If this is the case we say that P has subpolytopes. The point is

Proposition 2.4. *Generic convex polytopes in $\mathbb{R}^2$ have no subpolytopes*

Note that there are polytopes with subpolytopes. A particularly simple (and important) instance of this is the case when P has parallel edges. Carefully putting these two propositions together one can see that generically the answer to question 2.1 is 2 (or 0). There are two choices for the signs of the polytopes' edges and a single choice for the ordering of the sides. The two choices are P and P flipped. We could make the question harder by allowing for parallel edges and prescribing the sum of lengths of sides with a given direction. We can only handle the case when there are two or three pairs of parallel sides: the more pairs of sides there are the harder the question gets. These questions are related to the Minkowsi problem. See [K] for more details.

3 Toric Geometry

This section is essentially to set up notation. See [A] and [G] for more details.

Definition 3.1. *Let (X^{2n}, ω, J) be a Kahler manifold/orbifold where ω is a symplectic form and J is a compatible almost complex structure on X. Then (X, ω, J) is said to be toric if it admits a Hamiltonian holomorphic $\mathbb{T}^n$- action.*

Such an action admits a moment map $\phi : X \to \mathrm{Lie}(\mathbb{T}^n)^* = \mathbb{R}^n$, $\phi(X)$ is a convex polytope in $\mathbb{R}^n$ and is called the moment polytope of X. In the manifold case, it is actually a special type of polytope and it satisfies an integrality condition (which we will not describe here) called the Delzant condition. This polytope determines the symplectic structure of the underlying manifold. In the orbifold case, you need extra information in the form of integral weights attached to each facet that encodes the orbifold structure (see [LT]). Again, the weighted polytope of a toric orbifold determines it up to symplectomorphism. Note that (ω, J) determine a torus-invariant Kähler metric on X.

4 Spectral geometry

A Riemannian manifold (X, g) admits a Laplace-Beltrami operator $\triangle_g : \mathcal{L}^2 \to \mathcal{L}^2$. It generalizes the usual Laplace operator on $\mathbb{R}^n$.

Definition 4.1. *The spectrum of (X, g) is the set of eigenvalues of the operator $\triangle_g : \mathcal{L}^2 \to \mathcal{L}^2$*

It is well know that when X is compact the spectrum of (X, g) is a discrete subset of $\mathbb{R}^+$ and each eigenvalue admits a finite dimensional eigenspace. An isometry between two Riemannian manifolds (X, g), (Y, h) is a map $F : X \to Y$ such that $F^*h = g$. We have

$$\triangle(f \circ F) = (\triangle f) \circ F,\ f \in \mathcal{L}^2(Y).$$

Therefore if two manifolds are isometric they have the same spectrum. Is the reverse true? More precisely

Question 4.2. *Does the spectrum of (X, g) determine (X, g) up to isometry?*

This is a very classical question in Riemannian geometry. The answer is know to be no in this much generality. In fact the spectrum of $(X; g)$ does even determine the underlying X up to diffeomorphism. On the other hand it is known that the spectrum of (X, g) determines the dimension and the volume of X among other things. The question we are addressing here is

Question 4.3. *Does the spectrum of a toric Kahler metric on a compact toric manifold/orbifold determine the toric manifold/orbifold itself?*

This question was first posed by Abreu. Together with Emily Dryden and Victor Guillemin we used equivariant spectrum to make some progress on this. We now give a precise definition of equivariant spectrum. Let (X, ω, J) be a toric orbifold. For each $\theta \in \mathbb{T}^n$ we have an isometry of X, F_θ. This commutes with the Laplacian and induces a a $\mathbb{T}^n$ representation for each eigenvalue. Such a representation is determined by an integral weight $\alpha \in \mathbb{Z}^n$, which one can obtain by diagonalizing the representation.

Definition 4.4. *The* equivariant spectrum *of a toric orbifold is the list of all the eigenvalues of the Laplacian on the orbifold together with the weights of the action induced by $\mathbb{T}^n$ on the corresponding eigenspaces.*

In [DGS2] we prove

Theorem 4.5. *Let X be a generic toric orbifold with a fixed torus action and a toric Kähler metric. Then the equivariant spectrum of X determines the moment polytope P of $\mathcal{O}$, and hence the equivariant symplectomorphism type of X, up to two choices and up to translation.*

Our main tool is an asymptotic expansion for the heat kernel on an orbifold in the presence of an isometry. This turns out to "concentrate" on the fixed point set of the isometry. A manifold version of this was first discovered by Donnely. By applying this to our $\mathbb{T}^n$-family of isometries we show that the equivariant spectrum actually determines the directions orthogonal to the facets of the moment polytope as well as the corresponding volumes. We then use results similar to the ones described in section 2 to get our results.

5 Concluding remarks

Together with Emily Dryden and Victor Guillemin we also show that the equivariant spectrum determines if the toric Kahler metric is extremal and obtain some results in the manifold setting. It would of course be very interesting to understand if equivariant spectrum contains strictly more information than spectrum or not.

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Monads on Segre varieties

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Resumo: Construímos uma família de mónadas sobre a variedade de Segre do tipo

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^m}(-1,-1)^a \longrightarrow \mathcal{O}^b_{\mathbb{P}^l \times \mathbb{P}^m} \longrightarrow \mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^m}(1,1)^c \longrightarrow 0$$

e damos uma caracterização cohomológica de feixes livres de torsão que são cohomologia destas mónadas.

Abstract We construct a family of monads on the Segre variety of type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^m}(-1,-1)^a \longrightarrow \mathcal{O}^b_{\mathbb{P}^l \times \mathbb{P}^m} \longrightarrow \mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^m}(1,1)^c \longrightarrow 0$$

and give a cohomological characterisation of torsion free sheaves that are the cohomology of these monads.

palavras-chave: mónadas; variedade de Segre; caracterização cohomológica.

keywords: monads; Segre variety; cohomological characterisation.

1 Introdution

Given a smooth projective variety X over an algebraically closed field $\mathbb{K}$ of characteristic 0, a *monad* on X is a complex

$$M_\bullet\colon\ 0 \longrightarrow A \xrightarrow[\alpha]{} B \xrightarrow[\beta]{} C \longrightarrow 0$$

of coherent sheaves on X, with α an injective map and β surjective. The coherent sheaf $E := \ker\beta/\operatorname{im}\alpha$ is called the *cohomology (sheaf)* of the monad $M_\bullet$.

Monads were introduced by Horrocks in the sixties, in [3], and since then they have proved very useful objects for constructing vector bundles and studying their properties. For example, in [4], Horrocks gave a characterization of vector bundles on $\mathbb{P}^n$ in terms of monads, proving that every vector bundle E on $\mathbb{P}^n$ is the cohomology sheaf of a monad, with A and C being a direct sum of line bundles and B satisfying $H^1\big(B(k)\big) = H^{n-1}\big(B(k)\big) = 0$, for all $k \in \mathbb{Z}$, and $H^i\big(B(k)\big) = H^i\big(E(k)\big)$, for all $1 < k < n-1$.
When studying monads a first essential step is to determine their existence. Fløystad classified for which a, b and c and n there exist monads of type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \longrightarrow \mathcal{O}^b_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0, \tag{1.1}$$

and Costa and Miró-Roig proved a similar classification for monads on the quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$ of the form

$$0 \longrightarrow \mathcal{O}_{Q_n}(-1)^a \longrightarrow \mathcal{O}^b_{Q_n} \longrightarrow \mathcal{O}_{Q_n}(1)^c \longrightarrow 0.$$

(see [2] and [1], respectively). The generalisation of these results, namely, to existence of monads over other varieties is proposed in [5].

In the present paper we address the problem of the existence of monads on $X = \mathbb{P}^l \times \mathbb{P}^m$, with $l, m \geq 2$. Based on Fløystad's and Costa and Miró--Roig's ideas, we provide an example of monads of type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}(-1,-1)^a \longrightarrow \mathcal{O}^b_{\mathbb{P}^l\times\mathbb{P}^m} \longrightarrow \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}(1,1)^c \longrightarrow 0$$

which cannot be obtained as restrictions from monads on $\mathbb{P}^n$.

Furthermore, we give a cohomological characterization of torsion-free sheaves on X that are the cohomology of these monads.

2 Monads on Segre varieties

Let l and m be natural numbers, with $l, m \geq 2$. Consider the Segre embedding $\mathbb{P}^l \times \mathbb{P}^m \longrightarrow \mathbb{P}^{lm+l+m}$, with homogeneous coordinates $[z_{00} : \cdots : z_{lm}]$ in $\mathbb{P}^{lm+l+m}$. The Segre variety is defined by the polynomials

$$z_{ab}z_{cd} - z_{ad}z_{cb},$$

with $0 \leq a, c \leq l$ and $0 \leq b, d \leq m$. We wish to construct a monad using an even number of linear forms in $S := \mathbb{C}[z_{00} : \cdots : z_{lm}]$ and we will use these polynomials to do this. Since

$$z_{ab}z_{cd} = \tfrac{1}{4}\left(\left(z_{ab} + z_{cd}\right)^2 - \left(z_{ab} - z_{cd}\right)^2\right),$$

we can write $u_{abcd} = z_{ab} + z_{cd}$ and $v_{abcd} = z_{ab} - z_{cd}$ and get for any point in the Segre variety

$$u_{abcd}{}^2 - v_{abcd}{}^2 - u_{adcb}{}^2 + v_{adcb}{}^2 = 0.$$

Therefore, for any quadruple $q_{abcd} := (u_{abcd}, v_{abcd}, u_{adcb}, v_{adcb})$ if any three entries are zero, so is the fouth. Let $s = \left\lfloor \frac{l+1}{2} \right\rfloor$ and $t = \left\lfloor \frac{m+1}{2} \right\rfloor$ and let p be an integer such that $0 \le p \le st$ and $(l+1)(m+1) - p$ is even. Set $e := \frac{(l+1)(m+1)-p}{2}$ and consider the following list of quadruples as the one above:

$$\begin{aligned}
&q_{0011}, q_{0213}, \dots, q_{0,2t-2,1,2t-1},\\
&q_{2031}, q_{2233}, \dots, q_{2,2t-2,3,2t-1},\\
&\dots\\
&q_{2s-2,0,2s-1,1}, q_{2s-2,2,2s-1,3}, \dots, q_{2s-2,2t-2,2s-1,2t-1}.
\end{aligned}$$

Now choose the first p quadruples of this list and for $1 \le i \le p$, let w_{3i-2}, w_{3i-1}, and w_{3i} be the first three entries of each quadruple, and let $w_{3p+1}, \dots, w_{2e}$ be the variables z_{ab} not involved in the chosen quadruples.

Let $k \ge 1$ and let $A_1, A_2 \in M_{(k+e-1)\times k}(S)$ and $B_1, B_2 \in M_{k\times(k+e-1)}(S)$ be the matrices with entries in $S := K[z_{00}, \dots, z_{lm}]$, given by

$$B_1 = \begin{bmatrix} w_1 & \dots & w_e & & \\ & \ddots & & \ddots & \\ & & w_1 & \dots & w_e \end{bmatrix}, \quad B_2 = \begin{bmatrix} w_{e+1} & \dots & w_{2e} & & \\ & \ddots & & \ddots & \\ & & w_{e+1} & \dots & w_{2e} \end{bmatrix},$$

$A_1 = B_1{}^T$, and $A_2 = B_2{}^T$, and note that $B_1A_2 = B_2A_1$.

Let $A = \begin{bmatrix} A_2 & -A_1 \end{bmatrix}^T$ and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, and let

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}(-1,-1)^k \xrightarrow[\alpha]{} \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}^{2k+lm+l+m-1-p} \xrightarrow[\beta]{} \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}(1,1)^k \longrightarrow 0$$

be the sequence with maps α and β defined by matrices A and B, respectively. Now A and B fail to have maximal rank k if and only if $w_0, \dots, w_{2e}$ are all zero, which, as we have seen, cannot happen in the Segre variety. In particular, α is injective and β is surjective, and since $BA = 0$, this sequence yields a monad. Moreover, its cohomology E is a locally free sheaf of rank $lm + l + m - 1 - p$.

More generally, torsion-free sheaves on $\mathbb{P}^l \times \mathbb{P}^m$ with the same rank and Chern polynomial as E are uniquely determined by the following result (see [6]):

Theorem 2.1. *Let E be a torsion-free sheaf on $\mathbb{P}^l \times \mathbb{P}^m$ and let H be the divisor corresponding to $\mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}(1,1)$. Then E is the cohomology sheaf of a monad of type*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}(-1,-1)^a \longrightarrow \mathcal{O}^b_{\mathbb{P}^l\times\mathbb{P}^m} \longrightarrow \mathcal{O}_{\mathbb{P}^l\times\mathbb{P}^m}(1,1)^c \longrightarrow 0$$

if and only if $\mathrm{rk}(E) = b-a-c$, $c_t(E) = \frac{1}{(1+Ht)^a(1-Ht)^c}$, and, for $0 \le j \le l+m$, $-l+1 \le s \le 1$, and $-m+1 \le t \le 1$ satisfying $s+t = -j+2$, at most one of the groups $H^q\left((\Omega^{-s}_{\mathbb{P}^l}(-s) \boxtimes \Omega^{-t}_{\mathbb{P}^m}(-t)) \otimes E\right)$ is non-zero.

Remark 2.2. Fløystad proved that there is a monad of type (1.1) if and only if $b \ge 2c+n-1$ and $b \ge a+c$, or $b \ge a+c+n$. Any monad on $\mathbb{P}^n$, with $n = lm+l+m$, restricts to a monad on $\mathbb{P}^l \times \mathbb{P}^m$ embedded in $\mathbb{P}^n$ as a Segre variety. However, if we take $a = c = k$ and $b = 2k+lm+l+m-1-p$, the first condition above is satisfied if and only if $p = 0$ and the second never holds, so we see that for $0 < p \le st$ the monads described above cannot be obtained as restrictions from monads on $\mathbb{P}^n$.

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Towards classifying Hamiltonian torus actions with isolated fixed points

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Abstract Let (M, ω) be a compact symplectic manifold, and T a compact real torus. A natural question to ask is the following: what are the topological implications of the existence of an effective Hamiltonian T action on (M, ω)? In particular, is it possible to classify all the possible cohomology rings and Chern classes that can arise?

We address this question when the number of fixed points is finite, and analyze closely the case in which the number of fixed points is minimal. In this case, we answer the questions above when $\dim(M) \leqslant 6$ and $T = S^1$, recovering results obtained by Karshon and Tolman, and when $\dim(M) = 8$ and $T = (S^1)^2$.

keywords: Torus actions; fixed points; cohomology ring; Chern classes.

1 Introduction

Given a compact manifold M and a compact Lie group G, it is a natural question to ask whether M admits a (smooth, non-trivial) G action. Another way of looking at the same problem is to try to characterize the topological implications of such an action. For example, what can we say about the cohomology ring and the Pontrjagin classes of M?

This question is strictly related to the Petrie conjecture. Indeed in [6] the author conjectured that if M is homotopically equivalent to the complex projective space $\mathbb{C}P^n$ and it is acted on by a circle, then its Pontrjagin classes agree with the ones of $\mathbb{C}P^n$; the general proof of this conjecture is still missing. More recently Tolman [7] addressed a similar question in the symplectic category: given a compact symplectic manifold (M,ω) of dimension $2n$ with a Hamiltonian circle action and $n+1$ fixed points [1], is it possible to characterize all the possible cohomology rings and characteristic classes that can arise? She proved that in order to answer this question, *it is sufficient to characterize the* S^1 *representations at the fixed point set.* More precisely, let J be an S^1-invariant almost complex structure compatible with ω, and M^{S^1} the set of fixed points. Then for every $P \in M^{S^1}$, the isotropy action of S^1 on the tangent space at P can be written as

$$\lambda \cdot (z_1, \dots, z_n) = (\lambda^{w_{1P}} z_1, \dots, \lambda^{w_{nP}} z_n)$$

(here we identify $T_P M$ with $\mathbb{C}^n$). So the S^1 action on $TM|_{M^{S^1}}$ is completely characterized by the multiset of *weights* of the S^1 action, i.e. the multiset of integers

$$W = \biguplus_{P \in M^{S^1}} \{w_{1P}, \dots, w_{nP}\},$$

which, by the result of Tolman mentioned above, determines the (equivariant) cohomology ring and (equivariant) total Chern class.

The problem of finding effective formulas involving these integers has been widely studied in literature (see [2, 4, 7]). However the equations known, which mostly come from localization formulas in equivariant cohomology and K-theory, are high-degree polynomial equations in the weights; thus they can be used to check whether a multi-set of integers can be a set of weights of the S^1 representation on $TM|_{M^{S^1}}$, but cannot be in general solved to find them.

2 An algorithm that determines the isotropy action

In [1], Godinho and I gave new tools to approach this problem. Namely, we introduced an explicit algorithm that gives linear relations among the weights, and it relies on the following crucial result. If $\mathsf{c}_j \in H^{2j}(M,\mathbb{Z})$ denotes the j-th Chern class of the tangent bundle, $2n$ the dimension of M, and N_p the number of fixed points with exactly p negative weights, for $p = 0, \dots, n$, then *the Chern number* $\mathsf{c}_1\mathsf{c}_{n-1}[M]$ *only depends on* n *and* N_p, *for* $p = 0, \dots, n$. (see [1, Theorem 1.2 and Corollary 3.1]).

[1] In this case, $n+1$ is exactly the minimal number of fixed points of the action, and the Betti numbers of M agree with those of $\mathbb{C}P^n$.

Based on this result, we construct an algorithm that uses a combination of `Mathematica` and `C++` to determine a family of vector spaces which contain the lattices of the weights of all possible S^1 actions on M, such that the sum of the absolute value of the weights at each fixed point is bounded by a constant, given as a data of the problem. (The files of the algorithm can be found at `http://www.math.ist.utl.pt/~lgodin/MinimalActions.html`.)

In many cases the last condition can be replaced by a certain positivity condition, under which the algorithm is automatically *finite*. Using this, we were able to classify quickly all the possible Hamiltonian circle actions with minimal number of fixed points on a compact symplectic manifold of dimension less than 8, recovering the results obtained in [3] and [7]. When $\dim(M) = 8$ we proved that

Theorem 1. *If the action extends to a 2-torus action or if none of the weights is one then the isotropy representations must agree with the ones of the standard circle action on the complex projective space* $\mathbb{C}P^4$*, and the cohomology ring and Chern classes agree with the ones of* $\mathbb{C}P^4$*.*

In dimension 8, the existence of an action that does not satisfy the hypotheses in Theorem 1 is strictly related to the existence of a fake projective space with a specific list of Chern numbers (see [8]) and of a compact symplectic non-Kahlerizable manifold with a Hamiltonian circle action and five fixed points. Since it is not known whether these manifolds exist, we are currently working on trying to determine a family of weights not satisfying the hypotheses of Theorem 1, and then constructing, as in [5], a symplectic manifold with such an action.

Since the algorithm works in any dimension and any number of fixed points, we do believe that this new computational approach will be very useful for the explicit construction of compact symplectic manifolds with very interesting properties.

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On Jaeger's HOMFLY-PT expansions, branching rules and link homology: a progress report

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Resumo: Descrevemos a expansão HOMFLY-PT de Jaeger para o polinómio de Kauffman e como a generalizar a outros invariantes quânticos utilizando as "regras de ramificação" para representações de álgebras de Lie. Apresentamos um programa para a construção de expansões de Jaeger para homologias de enlaces.

Abstract We describe Jaeger's HOMFLY-PT expansion of the Kauffman polynomial and how to generalize it to other quantum invariants using the so-called "branching rules" for Lie algebra representations. We present a program which aims to construct Jaeger expansions for link homology theories.

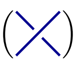
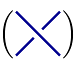
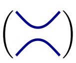
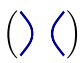
palavras-chave: Invariante quântico, regras de ramificação, homologia de enlaces.

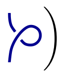
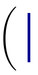
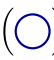
keywords: Quantum invariant, branching rules, link homology.

1 Link polynomials and Jaeger expansions

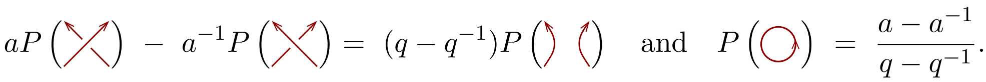
This story starts with two celebrated invariant polynomials of links.

Definition 1.1. The *Kauffman polynomial* $F = F(a, q)$ is the unique invariant of framed unoriented links satisfying

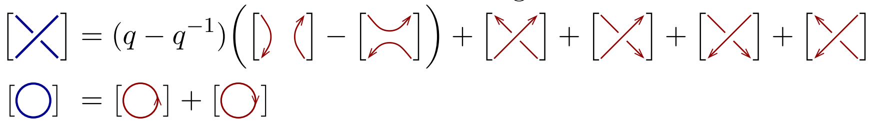
$F\Big($  $\Big) - F\Big($  $\Big) = (q - q^{-1})\Big(F\Big($  $\Big) - F\Big($  $\Big)\Big)$

$F\Big($  $\Big) = a^{-2}q\, F\Big($  $\Big)$ and $F\Big($  $\Big) = \dfrac{a^2q^{-1} - a^{-2}q}{q - q^{-1}} + 1.$

Definition 1.2. The *HOMFLY-PT polynomial* $P = P(a, q)$ is the unique invariant of oriented links satisfying



In 1989 François Jaeger showed that the Kauffman polynomial of a link L can be obtained as a weighted sum of HOMFLY-PT polynomials on certain links associated to L. Consider the following formalism



where the r.h.s. is evaluated to HOMFLY-PT polynomials completed with information about *rotation numbers*: $[\vec{D}] = (a^{-1}q)^{\operatorname{rot} D} P(\vec{D})$ (of course we only take the diagrams which are globally coherently oriented).

The proof of the following can be found in [1].

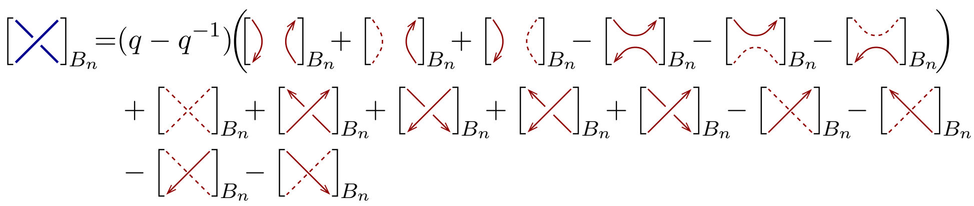
Theorem 1.3 (F. Jaeger, 1989)**.** *Let D be a diagram of a link L and $X(D)$ denote the set of crossings of D. The sum*

$$\sum_{\sigma \,\in\, X(D)} [\sigma] = F(D) \tag{1}$$

is a HOMFLY-PT expansion of the Kauffman polynomial of L.

It is not hard to see how to extend this expansion to tangles (see [5]). To explain this expansion we look into the representation theory of quantum enveloping algebras of the simple Lie algebras (QEAs). It is known that the HOMFLY-PT polynomial is related to the representation theory of the QEAs of type A_{n-1} (e.g. $\mathfrak{sl}_n$) and that in turn the Kauffman polynomial is related to the QEAs of types B_n, C_n and D_n ($\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$) respectively). Taking $a = q^n$ in $F(a,q)$ and $P(a,q)$ we obtain the $\mathfrak{so}_{2n}$ and the $\mathfrak{sl}_n$ polynomials respectively.

Following N. Reshetikhin and V. Turaev there is a functor from the category of tangles whose arcs are colored by irreducible finite dimensional (f.d.) representations of a QEA $\mathfrak{g}$ to the (tensor) category of f.d. representations of $\mathfrak{g}$ (see [3, 4]). In other words, for each of these tangles there is a $\mathfrak{g}$-invariant map which depends only on the (regular) isotopy class of the tangle which gives a full isotopy invariant in the cases we are interested in. So what we really have in Theorem 1.3 is an $\mathfrak{sl}_n$-expansion of the $\mathfrak{so}_{2n}$-polynomial (the case where all strands are colored by the fundamental representation)! There are other (2-variable) HOMFLY-PT expansions of $F(a,q)$ resulting in $\mathfrak{sl}_n$-expansions of the $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ polynomials after the specialization $a = q^n$. For example the assignment [6]



$$[\,|\,]_{B_n} = [\uparrow]_{B_n} + [\downarrow]_{B_n} + [\,\vdots\,]_{B_n}$$

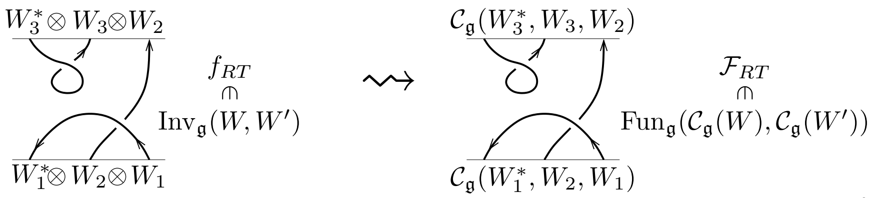
where $[\vec{D}]_{B_n} = a^{-\operatorname{rot} D} P(\vec{D})$ and a dashed line means the corresponding strand is to be erased, gives an $\mathfrak{sl}_n$-expansion of the $\mathfrak{so}_{2n+1}$-polynomial.

2 Branching rules, link homology and categorification

Let us give an explanation for this phenomenon. An inclusion $\mathfrak{l} \hookrightarrow \mathfrak{g}$ of Lie algebras (resp. QEAs) gives rise to functors (Ind and Res) between their categories of representations. In general Res does not send an irreducible over $\mathfrak{g}$ to an irreducible over $\mathfrak{l}$. The branching rules tell us how to express an irreducible over $\mathfrak{g}$ as a direct sum of irreducibles over $\mathfrak{l}$.

This is what we had before! For example, the expression $[\,|\,]_{D_n} = [\uparrow]_{D_n} + [\downarrow]_{D_n}$, which can be obtained from the extension of Theorem 1.3 to tangles, is a diagrammatic interpretation of the isomorphism $V_{fund}(\mathfrak{so}_{2n}) \cong V_{fund}(\mathfrak{sl}_n) \oplus V^*_{fund}(\mathfrak{sl}_n)$ for $\mathfrak{so}_{2n} \supset \mathfrak{sl}_n$, and $[\,|\,]_{B_n} = [\uparrow]_{B_n} + [\downarrow]_{B_n} + [\,\vdots\,]_{B_n}$ corresponds to $V_{fund}(\mathfrak{so}_{2n+1}) \cong V_{fund}(\mathfrak{sl}_n) \oplus V^*_{fund}(\mathfrak{sl}_n) \oplus V_{triv}(\mathfrak{sl}_n)$ for $\mathfrak{so}_{2n+1} \supset \mathfrak{sl}_n$.

The general picture of *categorification of quantum link invariants*, pioneered by M. Khovanov [2] upgrades the representations W appearing in the RT picture to categories $\mathcal{C}_{\mathfrak{g}}(W)$ (which are required to satisfy certain properties) and the RT map f_{RT} to a (derived) functor $\mathcal{F}_{RT}$ between (the derived categories of) these categories. Again, the isomorphism class of this functor depends only on the isotopy class of the tangle. The categorification of f_{RT} for general f.d. irreducible representations of QEAs was constructed by B. Webster in [7, 8] in its full generality.



We can try to use Webster's work to construct categorical $\mathfrak{l}$-expansions for the categorified $\mathfrak{g}$-RT invariants.

The categories $\mathcal{C}_{\mathfrak{g}}$ appearing in [7] extend to linear combinations of (arbitrary) f.d. irreducibles of $\mathfrak{g}$ which means that Webster's functors extend to (formal) linear combinations of tangles.

Definition 2.1. A *categorical Jaeger expansion* consists of (i) *categorified branching rules* i.e. a functor $\mathcal{C}_{\mathfrak{g}}(V^{\mathfrak{g}}) \to \mathcal{C}_{\mathfrak{l}}(\oplus_i V_i^{\mathfrak{l}})$ for $\mathfrak{l} \subset \mathfrak{g}$, which is full and bijective on objects, and (ii) its extension to corresponding decompositions of the "tangle functors". Here $V^{\mathfrak{g}}$ and each of the $V_i^{\mathfrak{l}}$ are irreducible f.d. representations of $\mathfrak{g}$ and $\mathfrak{l}$ respectively (resp. tensor products of such representations).

Although (ii) seems desirable from the topological point of view (work still in progress), the fulfillment of (i) is already very interesting, due to the potential applications to areas like representation theory and physics.

Theorem 2.2. *There are functors $\mathcal{C}_{\mathfrak{g}}(V^{\mathfrak{g}}) \to \mathcal{C}_{\mathfrak{l}}(\oplus_i V_i^{\mathfrak{l}})$ categorifying the branching rules for $\mathfrak{sl}_{n+1} \supset \mathfrak{sl}_n$ (for all representations and tensor products of minuscule representations), $\mathfrak{so}_{2n}$, $\mathfrak{so}_{2n+1} \supset \mathfrak{sl}_n$ (for fundamental representations and their tensor products).*

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Waldhausen decomposition and Systems of PDEs

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Resumo: Apresentamos uma generalização em dimensão superior da noção de sistema local rígido utilizando a decomposição de Waldhausen da 3-esfera $\mathbb{S}^3$ associada a uma curva plana.

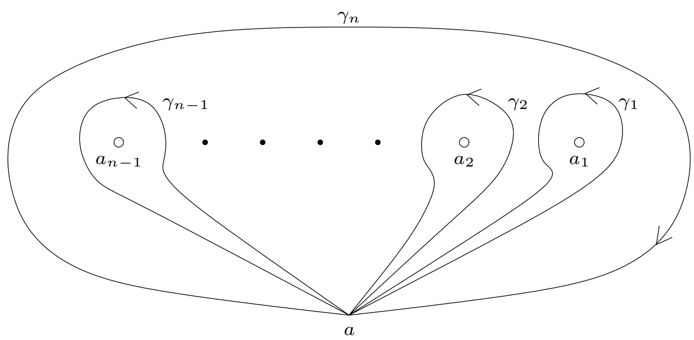
Abstract: We present a higher dimensional generalization of the notion of rigid local system using the Waldhausen decomposition of a 3-sphere $\mathbb{S}^3$ associated to a plane curve.

palavras-chave: Sistema Local Rígido; decomposição de Waldhausen.

keywords: Rigid Local System; Waldhausen decomposition.

1 Ordinary differential equations

Consider $a_1, \ldots, a_n \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Assume $a_n = \infty$. Set $U = \mathbb{CP}^1 \setminus \{a_1, \ldots, a_n\}$. Choose $a \in U$. There are generators $\gamma_1, \ldots, \gamma_n \in \pi_1(U, a)$ such that γ_i encircles a_i, $1 \leq i \leq n$, and $\gamma_1 \cdots \gamma_n = \mathrm{id}_{\pi_1}$.



Let
$$Pu = 0 \tag{1}$$
be a linear differential equation with holomorphic coefficients in $\mathbb{CP}^1$ and regular singularities at $a_1, \dots, a_n$. Let $f_1, \dots, f_k$ be a basis of the vector space L_a of the germs of solutions of (1) at a. Given $\gamma \in \pi_1(U, a)$ and $f \in L_a$, let $\gamma \cdot f$ be the element of L_a obtained by analytic continuation of f along γ. There is a matrix $\Phi_\gamma = (a_{ij}(\gamma))$ such that $\gamma \cdot f_i = \sum_j a_{ij}(\gamma) f_j$. The map $\gamma \mapsto \Phi_\gamma$ is a linear representation of $\pi_1(U, a)$, called the *monodromy* of (1). The Riemann-Hilbert correspondence states that the monodromy map defines a dictionary between linear differential equations on $\mathbb{P}^1$, with regular singularities at $a_1, \dots, a_n$, and linear representations of $\pi_1(U, a)$.

Set $A_i = \Phi_{\gamma_i}$, $i = 1, \dots, n$. The set $\{(X^{-1}A_1X, \dots, X^{-1}A_nX) : X \in GL_k(\mathbb{C})\}$ is called the *simultaneous conjugacy class* of $(A_1, \dots, A_n)$. We say that (1), Φ and $(A_1, ...A_n)$ are *rigid* if the simultaneous conjugacy class of $(A_1, \dots, A_n)$ is determined by the conjugacy classes of A_i, $i = 1, \dots, n$, i.e., by the *local monodromies* around a_i, $i = 1, \dots, n$. Although it is quite easy to calculate the local monodromies of (1), it is very hard to compute the global monodromy (unless (1) is rigid!). This fact was first noticed by Riemann in his study of the hypergeometric differential equation (see [1]). Equations with this type of property were called *free from accessory parameters.* The notion of rigidity, recently introduced by Katz (see [4]), is a very ambitious reformulation of the notion of equation free from accessory parameters and renovated the interest in this classical field.

2 Systems of partial differential equations

The Riemann-Hilbert correspondence has a higher dimensional generalization namely, a dictionary between regular holonomic $\mathcal{D}$-modules (which, locally, are systems of PDEs) and perverse sheaves. If we assume that the $\mathcal{D}$-module verifies some generic conditions, the associated perverse sheaf is determined by a linear representation of the fundamental group of the complement of the singular locus of the $\mathcal{D}$-module (see [2], [5]).

Appel and others considered classes of systems of PDEs in $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$, that can be reconstructed from their local monodromies, i.e., from their *Riemann schemes.* These systems of PDEs would be natural candidates of two dimensional rigid $\mathcal{D}$-modules, if we knew how to introduce such concept. As Haraoka noticed in [3], in order to generalize the concept of rigidity, we first need to have a good definition of *local monodromy* near a singular point of the singular locus of the $\mathcal{D}$-module. We propose here a solution for this problem (see [7]).

Let $Y \subset \mathbb{C}^2$ be the plane curve parametrized by $\varphi(t) = (t^4, t^6 + t^7)$. We call $y = x^{3/2} + x^{7/4}$ the *Puiseux expansion* of Y and $3/2, 7/4$ the *Puiseux exponents* of Y. They are topological invariants that determine the topology of Y. Set $\Delta_\varepsilon = \{(x, y) \in \mathbb{C}^2 : |x|, |y| < \varepsilon\}$. Consider the plane curves $Y_0 = \{y = 0\}$, $Y_1 = \{y = x^{3/2}\}$ and $Y_2 = Y$. The intersections of Y_i, $i = 0, 1, 2$, with the 3-dimensional sphere $\partial\Delta_\varepsilon$ define iterated torus knots denoted Γ_i, $i = 0, 1, 2$. We can choose tubular neighboorhoods T_i of Γ_i, $i = 0, 1, 2$, such that $T_{i+1} \subset int(T_i)$, $i = 0, 1$. The connected components of $\partial\Delta_\varepsilon \setminus \cup_{i=0}^2 \partial T_i$ have a very simple structure: they are Seifert manifolds. Roughly speaking, ∂T_i, $i = 0, 1, 2$, define a Waldhausen decomposition of $\partial\Delta_\varepsilon$ adapted to the knots Γ_i. Moreover, the tori surfaces ∂T_i, $i = 0, 1, 2$, are uniquely determined modulo isotopy (see [10]).

We say that a linear representation Φ of $\pi_1(\mathbb{C}^2 \setminus Y)$ is *rigid* if its isomorphism class is determined by the restriction of Φ to $\pi_1(\partial T_i)$, $i = 0, 1, 2$. Since $\partial T_i \simeq \mathbb{S}^1 \times \mathbb{S}^1$, this restriction is determined by conjugacy classes of pairs of commuting matrices, which we call *local monodromy* of Φ along ∂T_i.

When Y is weighted homogeneous and contains the x and y-axis, every irreducible representation Φ of $\pi_1(\mathbb{C}^2 \setminus Y)$ is isomorphic to the pullback of a representation Ψ of $\pi_1(\mathbb{CP}^1 \setminus \text{finite set})$ twisted by a representation of rank 1, via the morphism induced by $(x, y) \mapsto (y^k : x^n)$ for some pair of coprime integers k, n. Moreover, Φ is rigid with respect to the Waldhausen decompositions of $\partial\Delta_\varepsilon$ adapted to the link $\partial\Delta_\varepsilon \cap Y$ if and only if Ψ is rigid in the punctured Riemann sphere (see [7]), which shows the close relationship between the two definitions of rigidity.

Example 2.1 Consider the system of partial differential equations,

$$\vartheta u = \lambda u, \qquad Qu = 0, \tag{2}$$

where $\vartheta = 2x\partial_x + 3y\partial_y$, $Q = \partial_x^2 - (3/2)^2 x \partial_y^2$. Set $v = y^{-\lambda/3}u$. Since v is constant along the integral curves of ϑ, there is a multivalued holomorphic function f on $\mathbb{CP}^1$ such that

$$u = y^{\lambda/3} f(y^2/x^3). \tag{3}$$

Applying Q to the right hand side of (3) we conclude that f is the solution of an hypergeometric differential equation, and thus that (2) is rigid. Moreover, the solutions of (2) ramify along the weighted homogeneous curve $\{y^2 = x^3\}$.

The authors of [9] classified the $\mathcal{D}_{\mathbb{C}^2}$-modules "with simple characteristics" ramified along a weighted homogeneous curve Y. They found a definition of system of PDEs "free from acessory parameters" that is quite natural for this class of systems of PDEs. Moreover, the trick performed to system (2) of Example 2.1 can be extended to this class of systems. The system (2) is free from acessory parameters if and only if the associated differential equation on $\mathbb{CP}^1$ is free from accessory parameters. The authors of [9] remarked that they did not found as many systems free from acessory parameters as it would be expected. We show in [7] that replacing "simple characteristics" by "multiplicity one", a class only 'slightly bigger' (see [6]), we can find rigid $\mathcal{D}$-modules of multiplicity 1 along Y with prescribed local monodromies around the irreducible components of Y. We extend in [8] the previous result to arbitrary plane curves with irreducible tangent cone.

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