From the Teukolsky equation to a system of wave equations on Schwarzschild

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Abstract: We review the proof of energy boundeness and decay for solutions of the Teukolsky equation on the Schwarzschild geometry. This result was first shown by Dafermos, Holzegel and Rodniasnki (Acta. Math. 22(1):1–214, 2019). The proof here is based on an analysis of a transformed system of wave equations obtained by appropriately differentiating the Teukolsky equation. This approach was developed in collaboration with Shlapentokh-Rothman in a series of joint works (arXiv: 2007.07211, 2302.08916) concerning the Teukolsky equation on the more general Kerr family of rotating black holes.

Contents

1 Introduction

The stability of the Schwarzschild black hole,

$$g_M = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma, \quad M>0,$$

as a solution to the Einstein vacuum equations, has been a subject of great interest to the Mathematics and Physics communities since the pioneering work of Regge and Wheeler [RW57]. In this survey article, we review the proof of boundedness and decay estimates for the Teukolsky equation [BP73, Teu73]

$$\left(\Box^{[s]}_{g_M} + \frac{2s}{r}\left(1-\frac{M}{r}\right)\partial_r - \frac{2s}{r}\left(1-\frac{3M}{r}\right)\partial_t\right)\alpha^{[s]} = 0 \tag{1.1}$$

with $s=\pm 2$, which is one of the fundamental equations governing linearized perturbations of the Schwarzschild geometry. Here, $\Box^{[s]}_{g_M}$ is the spin-weighted d'Alembertian associated to the Schwarzschild metric g_M, see already Section 2.1.2 for a definition. Though we defer to Theorems 3.1 and 4.1 for the precise statements, a rough version of our main result is:

Theorem 1.1. *Fix $s\in\{\pm1,\pm2\}$ and $M>0$. On the Schwarzschild manifold, general solutions to the Teukolsky equation (1.1) arising from sufficiently regular initial data on a Cauchy surface ($\{\tau=0\}$)*

(i) satisfy a suitable version of "energy boundedness" without derivative loss: schematically,

$$\mathbb{E}(\tau_2) \leq B(M,s)\mathbb{E}(\tau_1)\,, \qquad \forall \tau_2\geq\tau_1\geq 0\,;$$

(ii) satisfy a suitable version of "integrated local energy decay" with loss of one derivative at trapping ($r=3M$): schematically,

$$\mathbb{I}^{\mathrm{deg}}(\tau_1,\tau_2) \leq B(M,s)\mathbb{E}(\tau_1)\,, \qquad \forall \tau_2\geq\tau_1\geq 0\,;$$

(iii) satisfy suitable, inverse-polynomial, energy and pointwise decay estimates with derivative loss.

The approach presented in this article is largely based on the approach in recent joint work with Shlapentokh-Rothman [SRTdC20, SRTdC23], where Theorem 1.1 is established in the more general class of Kerr black holes rotating at any speed below the maximal threshold[1]. In the Schwarzschild class considered in this work, boundedness and decay for (1.1) was first shown by Dafermos, Holzegel and Rodnianski [DHR19b] for $s = \pm 2$. Their analysis was later extended to the $s = \pm 1$ case, where (1.1) is an important component of the Maxwell system on Schwarzschild, by Pasqualotto [Pas19]. The case $s = 0$, the scalar wave equation on Schwarzschild, is by now classical, and the reader may find a proof and references to the original works in the lecture notes [DR13]. For sharp decay results, we direct the reader to [AAG23, MZ22] and the references therein.

The proof of boundedness and decay for (1.1) in [DHR19b] was a cornerstone result in the understanding of the stability properties of Schwarzschild spacetimes. Not long after its original proof [DHR19b] appeared, it was used by Klainerman and Szeftel [KS20] to establish nonlinear stability of Schwarzschild under polarized axisymmetric perturbations. The definitive result on stability of Schwarzschild, which makes no symmetry assumptions[2], was very recently obtained by Dafermos, Holzegel, Rodnianski and Taylor [DHRT21] again building on the (proof of) [DHR19b].

Having motivated the study of (1.1) and Theorem 1.1, let us turn to a discussion of its proof. As in [DHR19b, Pas19], our approach is to consider specific differential transformations of $\alpha^{[s]}$, introduced in [DHR19b] and inspired by Chandrasekhar's transformations [Cha75]. This allows one to, in particular, replace the Teukolsky equation (1.1) with a spin-weighted wave equation

$$\Box^{[s]}_{g_M} \Phi^{[s]} = 0 \tag{1.2}$$

for a new variable $\Phi^{[s]}$; for $s = \pm 2$, this equation is sometimes called Regge–Wheeler equation after [RW57]. Equation (1.2) can be treated using scalar wave methods which, as we mentioned, are by now classical (see e.g. the aforementioned lecture notes [DR13]). Thus, one can obtain energy boundedness, integrated local energy decay, and then decay for its solutions. To conclude the proof of Theorem 1.1, one must then upgrade these estimates for $\Phi^{[s]}$ to suitable estimates for $\alpha^{[s]}$.

[1]We direct the reader to these works for further details and references concerning the stability problem for rotating Kerr black holes, and for a more exaustive account of the literature on perturbations of Schwarzschild.

[2]As Schwarzschild is a member of the larger Kerr family of black holes, stability can only hold up to a co-dimension 3 "submanifold" of the moduli space of initial data for the Einstein vacuum equations, corresponding to perturbations which eventually radiate away all angular momentum. In this work, the authors teleologically identify this set of data and thus show the nonlinear stability of the Schwarzschild subfamily in full codimension.

It is in this recovery procedure that the proof of Theorem 1.1 differs from the boundedness and decay results of [DHR19b] for (1.1). In the former, estimates for $\alpha^{[s]}$ are obtained by integrating the transport equations which define $\Phi^{[s]}$; indeed,

$$\Phi^{[s]} \doteq \phi^{[s]}_{(|s|)}, \qquad \phi^{[s]}_{(k)} \doteq j_k(j_0\mathcal{L})^k(j_{-1}\alpha^{[s]}) \text{ with } k\in\{0,\dots,|s|\},$$

for some appropriate choices of weights $j_i = j_i(r)$, $i=-1,\dots,|s|$, and with $\mathcal{L}$ a particular null vector field. In the present article, as in [SRTdC20, SRTdC23], we instead rely primarily on the system of wave equations for the new variables $\phi^{[s]}_{(k)}$ that can be derived from (1.1). For instance, in the case $s=\pm 2$, this system takes the form

$$\begin{aligned}
&\Box^{[2]}_{g_M}\phi^{[\pm 2]}_{(2)} - 4\left(1-\frac{2M}{r}\right)\phi^{[\pm 2]}_{(2)} = 0,\\
&\Box^{[2]}_{g_M}\phi^{[\pm 2]}_{(1)} - 4\left(1-\frac{2M}{r}\right)\phi^{[\pm 2]}_{(1)} = \pm\frac{2(r-3M)}{r^2}\phi^{[\pm 2]}_{(2)} \mp 6M\phi^{[\pm 2]}_{(0)},\\
&\Box^{[2]}_{g_M}\phi^{[\pm 2]}_{(0)} - 2\left(1+\frac{2M}{r}\right)\phi^{[\pm 2]}_{(0)} = \pm\frac{4(r-3M)}{r^2}\phi^{[\pm 2]}_{(1)},
\end{aligned}$$

see already Remark 2.6 for more examples. With this approach, our Theorem 1.1 obtains slight improvements over [DHR19b, Theorem 2, Propositions 12.3.1-12.3.2]:

- In Theorem 4.1, we close energy boundedness estimates without derivative loss, and integrated estimates with 1 derivative loss only at trapping, for *all* derivatives of $\alpha^{[s]}$ at or below a certain level. In [DHR19b], derivatives in certain directions are not explicitly controlled.
- In Theorem 4.1, the estimates are at the level of $|s|+1$ derivatives and below, while in [DHR19b] the authors require $|s|+3$ derivatives in their energy boundedness statement to avoid derivative loss.
- There is an ϵ loss in the r-weights of the energy norms used in [DHR19b] compared to those in Theorem 1.1.

Finally, let us note that both the approach of this article, and that of the aforementioned [DHR19b, Pas19] to the Teukolsky equation (1.1) can, in principle, be extended to other $s\in\mathbb{Z}$ not contained in the statement of Theorem 1.1, see Remark 3.3 below. However, since only $s\in\{0,\pm 1,\pm 2\}$ are expected to be physically meaningful, we have chosen to exclude other values of s from our main result.

We conclude the introduction with an outline of the rest of the paper:

- Section 2 is a preliminary section introducing the Schwarzschild manifold, the relevant PDE, the spin-weighted function space on which they live, and the norms we will employ in our analysis.
- Section 3 considers the decoupled equation (1.2), and establishes energy boundedness, integrated local energy decay, energy decay and pointwise decay for its solutions.
- Section 4 then establishes similar results for all transformed variables $\upphi^{[s]}_{(k)}$ with $k < |s|$, thus concluding the proof of Theorem 1.1.

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2 Preliminaries

2.1 Geometric preliminaries

2.1.1 The Schwarzschild family

As is usual, we introduce the manifold structure of Schwarzschild independently of the black hole parameter M. To this end, consider a manifold-with-boundary $\mathcal{M} = \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{S}^2$. Coordinates $t^* \in \mathbb{R}$, $y^* \in \mathbb{R}_0^+$ and (θ, ϕ) the usual polar coordinates on $\mathbb{S}^2$ induce a global (modulo the usual degeneration of polar coordinates) differentiable structure on $\mathcal{M}$. The boundary, $\partial\mathcal{M} = \{y^* = 0\}$, is called the event horizon, $\mathcal{H}^+ \doteq \partial\mathcal{M}$. The interior, $\mathrm{int}(\mathcal{M})$, is called the domain of outer communications. We can also introduce the smooth vector fields $T \doteq \partial_{t^*}$ and $Z \doteq \partial_\phi$.

Let $r = r(y)$ be a new coordinate, smoothly depending on y, whose range is $r \in [2M, \infty)$. For $M > 0$, the Schwarzschild family is the (1-parameter) family of Lorenztian manifolds $(\mathcal{M}, g_M)$ with

$$g_M = -\left[1 - \mu_M(r)\right](\mathrm{d}t^*)^2 + 2\mu_M(r)\mathrm{d}t^*\mathrm{d}r + (1 + \mu_M(r))\, dr^2 + r^2 \mathring{\not{g}}_{\mathbb{S}^2},$$
$$\mu_M(r) \doteq \frac{2M}{r},$$

where $\mathring{\not{g}}_{\mathbb{S}^2} = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ is the usual metric on the round sphere with volume form $\mathrm{d}\sigma \doteq \sin\theta \mathrm{d}\theta\mathrm{d}\phi$. The coordinates (t^*, r, θ, ϕ) are called ingoing Eddington–Finkelstein coordinates. We can use these to define a hyperboloidal foliation of interest for $\mathcal{M}$: following [SRTdC23], we take

$$\Sigma_\tau \doteq \{\tilde{t}^* = \tau\},$$

$$\tilde{t}^* \doteq t^* + \tilde{\zeta}_1(r)\left(r - \frac{M}{2}\log r\right) - \tilde{\zeta}_2(r)\left(r + 2M\log r - \frac{3M^2}{r}\right),$$

with $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ smooth cutoff functions satisfying: $\tilde{\zeta}_1 = 1$ for $r_+ \leq r \leq 3M/2$ and $\tilde{\zeta}_1 = 0$ for $r \geq 3M$; $\tilde{\zeta}_2 = 0$ for $r \leq 3M$ and $\tilde{\zeta}_2 = 1$ for $r \geq 4M$.

In the domain of outer communications alone, instead of Eddinton–Finkelstein coordinates, we can consider other coordinates t and r, as well as t and r^*, which are induced by the transformations

$$\frac{\mathrm{d}r^*}{dr} = \frac{1}{1-\mu_M(r)}, \quad r^*(3M) = 0; \qquad t = t^* - r^* + r - 3M + 2M\log M.$$

In coordinates (t, r, θ, ϕ) or (t, r^*, θ, ϕ), called Schwarzschild coordinates, the induced metric on $\mathrm{int}(\mathcal{M})$ is given by

$$\begin{aligned} g_M &= -\left[1-\mu_M(r)\right]\left[\mathrm{d}t^2 - (\mathrm{d}r^*)^2\right] + r^2 \mathring{\not{g}}_{\mathbb{S}^2} \\ &= -\left[1-\mu_M(r)\right]\mathrm{d}t^2 + \frac{1}{1-\mu_M(r)}\mathrm{d}r^2 + r^2 \mathring{\not{g}}_{\mathbb{S}^2}. \end{aligned}$$

Schwarzschild coordinates are used predominantly in this work. We also note the often-used function

$$w \doteq \frac{1-\mu_M(r)}{r^2}.$$

We often drop the subscript on the function μ_M for readability.

Finally, of interest to us are the M-dependent vector fields

$$L \doteq \partial_{r^*} + T\,, \quad \underline{L} \doteq -\partial_{r^*} + T\,, \qquad g(L, \underline{L}) = -2\,,$$

defining two principal null directions on $\mathrm{int}(\mathcal{M})$. Note that L and $(1-\mu_M)^{-1}\underline{L}$ can be extended to $\mathcal{M}$ using the coordinate transformations above.

2.1.2 The spin-weighted structure

In this section, we introduce a spin-weighted structure on $\mathcal{M}$.

To start with, let us define spin-weighted functions and operators on the round sphere $\mathbb{S}^2$. Letting (θ, ϕ) denote standard spherical coordinates in $\mathbb{S}^2$, consider the operators[3]

$$\begin{aligned} \tilde{Z}_1 &= -\sin\phi\partial_\theta + \cos\phi\left(-is\csc\theta - \cot\theta\partial_\phi\right)\,, \\ \tilde{Z}_2 &= -\cos\phi\partial_\theta - \sin\phi\left(-is\csc\theta - \cot\theta\partial_\phi\right)\,, \quad \tilde{Z}_3 = \partial_\phi\,. \end{aligned} \tag{2.1}$$

[3]The operators $\tilde{Z}_1$, $\tilde{Z}_2$ and $\tilde{Z}_3$ arise from the action of the canonical orthonormal frame on $\mathbb{S}^3$, viewed as the Hopf bundle over $\mathbb{S}^2$, on complex-valued functions with a particular s-dependence along the $\mathbb{S}^1$ fibers, see for instance [?] for more details.

Definition 2.1 (Smooth spin-weighted functions on $\mathbb{S}^2$)**.** Fix some $s \in \frac{1}{2}\mathbb{Z}$. Let f be a complex-valued function of $(\theta,\phi) \in \mathbb{S}^2$. We say f is a smooth s-spin-weighted function on $\mathbb{S}^2$, and write $f \in \mathscr{S}_\infty^{[s]}$, if for any $k_1, k_2, k_3 \in \mathbb{N}_0$,

$$(\tilde{Z}_1)^{k_1}(\tilde{Z}_2)^{k_2}(\tilde{Z}_3)^{k_3} f$$

is a function of (θ,ϕ) which is smooth for $\theta \neq 0, \pi$ and such that

$$e^{is\phi}(\tilde{Z}_1)^{k_1}(\tilde{Z}_2)^{k_2}(\tilde{Z}_3)^{k_3} f \text{ and } e^{-is\phi}(\tilde{Z}_1)^{k_1}(\tilde{Z}_2)^{k_2}(\tilde{Z}_3)^{k_3} f$$

extend continuously to, respectively, $\theta = 0$ and $\theta = \pi$.

For any $s \in \frac{1}{2}\mathbb{Z}$, the *spin-weighted laplacian*

$$\begin{aligned} \mathring{\triangle}^{[s]} &= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\partial_\phi^2 - 2is\frac{\cos\theta}{\sin^2\theta}\partial_\phi + s^2\cot^2\theta - s \\ &= -\tilde{Z}_1^2 - \tilde{Z}_2^2 - \tilde{Z}_3^2 - s - s^2 = \mathring{\triangle} - 2is\frac{\cos\theta}{\sin^2\theta}\partial_\phi + s^2\cot^2\theta - s \end{aligned} \tag{2.2}$$

where $\tilde{Z}_1$, $\tilde{Z}_2$ and $\tilde{Z}_3$ are defined in (2.1) and $\mathring{\triangle}$ is the usual laplacian on the round sphere is a example of a smooth differential operator on $\mathscr{S}_\infty^{[s]}$. Another example is provided by the *spinorial gradient*

$$\mathring{\nabla}^{[s]} = (\partial_\theta\,,\ \partial_\phi + is\cos\theta)\,, \tag{2.3}$$

which arises naturally in connection with the spin-weighted laplacian: for $\Xi\,, \Pi \in \mathscr{S}_\infty^{[s]}$,

$$\int_{\mathbb{S}^2}\left(\mathring{\triangle}^{[s]}\Xi\right)\overline{\Pi}\mathrm{d}\sigma = \int_{\mathbb{S}^2}\left[\mathring{\nabla}^{[s]}\Xi\cdot\overline{\mathring{\nabla}^{[s]}\Pi}\right]_{\mathbb{S}^2}\mathrm{d}\sigma. \tag{2.4}$$

Let us state two useful results concerning these important operators:

Lemma 2.2 (Spin-weighted spherical harmonics)**.** *Fix $s \in \frac{1}{2}\mathbb{Z}$. On $\mathscr{S}_\infty^{[s]}$, the operator $\mathring{\triangle}^{[s]}$ has a countable set of eigenfunctions referred to as spin-weighted spherical harmonics, forming a complete orthogonal basis of $\mathscr{S}_\infty^{[s]}$.*

As is standard, we index the spin-weighted spherical harmonics and their eigenvalues by parameters m and ℓ chosen to satisfy $m - s \in \mathbb{Z}$ and $\ell \geq \max\{|s|, |m|\}$ such that:

- *the (m,ℓ) spin-weighted spherical harmonic is denoted by $S_{m\ell}^{[s]}(\theta,\phi)$;*
- *the corresponding eigenvalue is $\lambda_{m\ell}^{[s]} = \ell(\ell+1) - s^2$.*

Lemma 2.3 (Spinorial gradient)**.** *Let $s \in \mathbb{Z}$ and $\mathring{\nabla}^{[s]}$ be as defined in (2.3). Then, for any $\Xi \in \mathscr{S}_\infty^{[s]}$, one has the Poincaré inequality*

$$\int_{\mathbb{S}^2} \left|\mathring{\nabla}^{[s]}\Xi\right|^2 \mathrm{d}\sigma \geq |s| \int_{\mathbb{S}^2} |\Xi|^2 \,\mathrm{d}\sigma. \tag{2.5}$$

As the Schwarzschild manifold $\mathcal{M}$ is spherically symmetric, it inherits the spin-weighted structure over $\mathbb{S}^2$:

Definition 2.4 (Smooth spin-weighted functions on $\mathcal{M}$)**.** Fix some $s \in \frac{1}{2}\mathbb{Z}$. We say f is a smooth s-spin-weighted function on $\mathcal{M}$, and write $f \in \mathscr{S}_\infty^{[s]}(\mathcal{M})$, if f is smooth in the Eddington–Finkelstein t^* and r, and its restriction to constant (t^*, r) yields a smooth s-spin-weighted function on $\mathbb{S}^2$.

For some $s \in \frac{1}{2}\mathbb{Z}$, the spin-weighted d'Alembertian

$$\Box_{g_M}^{[s]} = \Box_{g_M} + \frac{2is}{r^2}\frac{\cos\theta}{\sin^2\theta}\partial_\phi + \frac{1}{r^2}(s - s^2\cot^2\theta)$$

is an example of a smooth differential operator on $\mathscr{S}_\infty^{[s]}(\mathcal{M})$. In Schwarzschild coordinates it is given by

$$\Box_{g_M}^{[s]} = \frac{1}{r^2}\partial_r\left(r^2(1-\mu_M)\partial_r\right) - \frac{1}{1-\mu_M}\partial_t^2 - \frac{1}{r^2}\mathring{\triangle}^{[s]}.$$

2.2 Analytical preliminaries

2.2.1 The Teukolsky equation

We say that a function $\alpha^{[s]}$ such that $(1-\mu)^s\alpha^{[s]} \in \mathscr{S}_\infty^{[s]}(\mathcal{M})$ satisfies the Teukolsky equation if

$$\Box_{g_M}^{[s]}\alpha^{[s]} - \frac{2s}{r}\left(1 - \frac{3M}{r}\right)T\alpha^{[s]} + \frac{2s}{r}\left(1 - \frac{M}{r}\right)\partial_r\alpha^{[s]} = 0, \tag{2.6}$$

or, in Schwarzschild coordinates,

$$\frac{1}{r^{2(1+s)}(1-\mu_M)^s}\partial_r\left(r^{2(s+1)}(1-\mu_M)^{s+1}\partial_r\right)\alpha^{[s]} \\ - \frac{1}{r^2(1-\mu_M)}\left(T + \frac{2s}{r}\left(1 - \frac{3M}{r}\right)\right)T\alpha^{[s]} - \mathring{\triangle}^{[s]}\alpha^{[s]} = 0.$$

2.2.2 The DHR transformed system

In this section, inspired by the transformations of Dafermos, Holzegel and Rodnianski in [DHR19b, DHR19a] and of Pasqualotto [Pas19], we introduce a system of equations derived from the Teukolsky equation (2.6) which was obtained with Shlapentokh-Rothman [SRTdC20, SRTdC23].

Fix $s \in \mathbb{Z}$, and consider the following rescaling of the Teukolsky variable

$$\psi^{[s]}_{(0)} = w^{|s|(1+\operatorname{sgn} s)/2} r^{1-|s|(1-\operatorname{sgn} s)} \alpha^{[s]}. \tag{2.7}$$

For $k = 1, \ldots, |s|$, we then define $\psi^{[s]}_{(k)}$ by the system of transport equations

$$\psi^{[s]}_{(k)} = w^{-1}\mathcal{L}\psi^{[s]}_{(k-1)}, \qquad k = 1, ..., |s|, \tag{2.8}$$

with $\mathcal{L} = L$ if $s < 0$, $\mathcal{L} = \underline{L}$ if $s > 0$ and $\mathcal{L}$ being the identity if $s = 0$. We will sometimes use the notation $\Psi^{[s]} \doteq \psi^{[s]}_{(|s|)}$. It will also be useful to introduce a further rescaling

$$\phi^{[s]}_{(k)} \doteq r^{-1}\psi^{[s]}_{(k)}, \quad \Phi^{[s]} \doteq \phi^{[s]}_{(|s|)} = r^{-1}\Psi^{[s]}.$$

Proposition 2.5 (DHR transformed system). *Fix $s \in \mathbb{Z}$ and let $\alpha^{[s]}$ solve the Teukolsky equation (2.6). Then, for $k = 0, ..., |s|$, $\psi^{[s]}_{(k)}$ given in (2.7)–(2.8) satisfy the following system of wave equations*

$$\Box^{[s]}_g \phi^{[s]}_{(k)} - U^{[s]}_{(k)}(r)\phi^{[s]}_{(k)} = -\operatorname{sgn} s(|s|-k)\frac{w'}{w}\phi^{[s]}_{(k+1)} - \sum_{j=0}^{k-1}\mathfrak{J}^{[s]}_{(k),(j)}[\psi^{[s]}_{(j)}], \tag{2.9}$$

where

$$U^{[s]}_{(k)} \doteq |s| + k(2|s|-k-1) - \frac{2M}{r}\Big(3|s| - 2s^2 + 3k(2|s|-k-1)\Big), \tag{2.10}$$

$$\mathfrak{J}^{[s]}_{(k),(j)}[\psi^{[s]}_{(j)}] \doteq c^{\mathrm{id}}_{s,k,j}(r)\psi^{[s]}_{(j)}, \tag{2.11}$$

and $c^{\mathrm{id}}_{s,k,j}$, $j = 0, ..., k-1$, denote functions of r which can be explicitly computed for through a recursive relation (2.16) initialized by (2.17), and which have the following properties:

- *if $0 < k < |s|$ and $j = k-1$, $c^{\mathrm{id}}_{s,k,j}$ is a nonzero constant;*
- *otherwise, $c^{\mathrm{id}}_{s,k,j}$ vanishes;*

Equivalently, $\psi^{[s]}_{(k)}$ *satisfy the equations*

$$\mathfrak{R}^{[s]}_{(k)}\psi^{[s]}_{(k)} = \operatorname{sgn} s(|s|-k)w'\psi^{[s]}_{(k+1)} + w\sum_{j=0}^{k-1}\mathfrak{I}^{[s]}_{(k),(j)}[\psi^{[s]}_{(j)}]\,, \tag{2.12}$$

where

$$\mathfrak{R}^{[s]}_{(k)} \doteq \frac{1}{2}\left(L\underline{L}+\underline{L}L\right) + \frac{2Mw}{r} + w\mathring{\triangle}^{[s]} + wU^{[s]}_{(k)}. \tag{2.13}$$

In particular, for $k=|s|$*,* $\Phi^{[s]}$ *(equivalently* $\Psi^{[s]}$*) solves a decoupled wave equation*

$$\Box^{[s]}_g\Phi^{[s]} = U^{[s]}(r)\Phi^{[s]}, \qquad \mathfrak{R}^{[s]}\Psi^{[s]} = 0, \tag{2.14}$$

with $\mathfrak{R}^{[s]} \doteq \mathfrak{R}^{[s]}_{(|s|)}$ *and* $U^{[s]} \doteq U^{[s]}_{(|s|)} = s^2\mu(r)$*. For* $k<|s|$*, using* (2.8)*, we can recast* (2.12) *as the constraint equation*

$$\begin{aligned}&\underline{\mathcal{L}}\psi^{[s]}_{(k+1)} - \operatorname{sgn} s(|s|-k-1)\frac{w'}{w}\psi^{[s]}_{(k+1)}\\ &\quad= -\mathring{\triangle}^{[s]}\psi^{[s]}_{(k)} - \left[U^{[s]}_{(k)}(r) + \frac{2M}{r}\right]\psi^{[s]}_{(k)} + \sum_{j=0}^{k-1}\mathfrak{I}^{[s]}_{(k),(j)}[\psi^{[s]}_{(j)}].\end{aligned} \tag{2.15}$$

Proof. This proof was given in [SRTdC20] for general rotating Kerr black holes; here we repeat the main points for Schwarzschild black holes.

The result follows by recursion in k. To obtain the wave equation (2.12) at kth level, we start from the $(k-1)$th wave equation (2.12). Noting $\underline{L}L + L\underline{L} = 2\underline{\mathcal{L}}\mathcal{L}$, we use the definition of $\psi^{[s]}_{(k)}$ in (2.8); we thus obtain the $(k-1)$th transport relation (2.15). Then, we divide by w and apply $\mathcal{L}$. We once again simplify terms using the definitions of $\psi^{[s]}_{(j)}$, $j=0,\dots,k$, except for the term $\mathcal{L}\underline{\mathcal{L}}\psi^{[s]}_{(k)}$.

With our choice of weights, the PDEs for the transformed quantities take the form

$$\mathcal{L}\underline{\mathcal{L}}\psi^{[s]}_{(k)} - \operatorname{sgn} s(|s|-k)\frac{w'}{w}\mathcal{L}\psi^{[s]}_{(k)} + \sum_{X\in\mathfrak{X}}\sum_{j=0}^{k} c^X_{s,k,j}\psi^{[s]}_{(j)} = 0\,,$$

where $\mathfrak{X} = \{\mathring{\triangle}^{[s]}, \mathrm{id}\}$ and coefficients $c^X_{s,k,j}$ satisfy the recursive relation

$$\begin{cases} c^X_{s,k,0} &= -\operatorname{sgn} s\left(\dfrac{c^X_{s,k-1,0}}{w}\right)' \\ c^X_{s,k,j} &= -\operatorname{sgn} s\left(\dfrac{c^X_{s,k-1,j}}{w}\right)' + c^X_{s,k-1,j-1} \qquad \text{for} \quad j=1,\dots,k-1\end{cases}, \tag{2.16}$$

initialized by the relations, for $j = k = 0, ..., |s|$,

$$
\begin{aligned}
c^{\mathring{\triangle}}_{s,k,k} &= w, \\
c^{\mathrm{id}}_{s,k,k} &= w\left(|s| + \frac{2M(1-3|s|+2s^2)}{r}\right) + \frac{k(2|s|-k-1)}{2}\left(\frac{w'}{w}\right)' \qquad (2.17)\\
&= w\left[|s| + k(2|s|-k-1)\right] + \frac{2Mw}{r}\left[1-3|s|+2s^2-3k(2|s|-k-1)\right].
\end{aligned}
$$

Since $(1/r)' = -w$, we can deduce the properties of $c^{\mathrm{id}}_{s,k,j}$, for $j = 0, \ldots, k-1$, claimed in the statement. □

Remark 2.6 (Full computation of the wave systems in particular cases)**.** We carry out the computations of (the proof of) Proposition 2.5 for particular cases $s \in \mathbb{Z}$. For $|s| = 1$, (2.9) become

$$
\begin{aligned}
\Box^{[1]}_g \phi^{[\pm 1]}_{(1)} - \left(1 - \frac{2M}{r}\right)\phi^{[\pm 1]}_{(1)} &= 0\\
\Box^{[1]}_g \phi^{[\pm 1]}_{(0)} - \left(1 - \frac{2M}{r}\right)\phi^{[\pm 1]}_{(0)} &= \pm\frac{2(r-3M)}{r^2}\phi^{[\pm 1]}_{(1)}.
\end{aligned}
$$

For $|s| = 2$, (2.9) become

$$
\begin{aligned}
\Box^{[2]}_g \phi^{[\pm 2]}_{(2)} - 4\left(1 - \frac{2M}{r}\right)\phi^{[\pm 2]}_{(2)} &= 0,\\
\Box^{[2]}_g \phi^{[\pm 2]}_{(1)} - 4\left(1 - \frac{2M}{r}\right)\phi^{[\pm 2]}_{(1)} &= \pm\frac{2(r-3M)}{r^2}\phi^{[\pm 2]}_{(2)} \mp 6M\phi^{[\pm 2]}_{(0)},\\
\Box^{[2]}_g \phi^{[\pm 2]}_{(0)} - 2\left(1 + \frac{2M}{r}\right)\phi^{[\pm 2]}_{(0)} &= \pm\frac{4(r-3M)}{r^2}\phi^{[\pm 2]}_{(1)}.
\end{aligned}
$$

For $|s| = 3$, (2.9) become

$$
\begin{aligned}
\Box^{[3]}_g \phi^{[\pm 3]}_{(3)} - 9\left(1 - \frac{2M}{r}\right)\phi^{[\pm 3]}_{(3)} &= 0,\\
\Box^{[3]}_g \phi^{[\pm 3]}_{(2)} - 9\left(1 - \frac{2M}{r}\right)\phi^{[\pm 3]}_{(2)} &= \pm\frac{2(r-3M)}{r^2}\phi^{[\pm 2]}_{(3)} \mp 16M\phi^{[\pm 3]}_{(1)},\\
\Box^{[3]}_g \phi^{[\pm 3]}_{(1)} - \left(7 - \frac{6M}{r}\right)\phi^{[\pm 3]}_{(1)} &= \pm\frac{4(r-3M)}{r^2}\phi^{[\pm 3]}_{(2)} \mp 20M\phi^{[\pm 3]}_{(0)},\\
\Box^{[3]}_g \phi^{[\pm 3]}_{(0)} - 3\left(1 + \frac{6M}{r}\right)\phi^{[\pm 3]}_{(0)} &= \pm\frac{4(r-3M)}{r^2}\phi^{[\pm 3]}_{(2)},
\end{aligned}
$$

Notice that there is no smallness parameter in the coupling of the kth equation to any of the $j = 0, \ldots, k-1$ equations nor to the $(k+1)$th equation.

Remark 2.7 (Rescaling the transformed variables)**.** In what follows, it will be useful to consider rescalings of the unknowns in (2.8). If we let $\psi^{[s]}_{(k)} = c_k(r)\tilde{\psi}^{[s]}_{(k)}$, then (2.8) becomes

$$\mathcal{L}\tilde{\psi}^{[s]}_{(k)} = \frac{wc_{k+1}}{c_k}\tilde{\psi}^{[s]}_{(k+1)} + \operatorname{sgn} s\frac{c_k'}{c_k}\tilde{\psi}^{[s]}_{(k)}, \tag{2.18}$$

and the PDEs (2.12) become

$$\begin{aligned}&\left[\mathfrak{R}^{[s]}_{(k)} - \left(\frac{c_k'}{c_k}\right)' - \operatorname{sgn} s\frac{c_k'}{c_k}\underline{\mathcal{L}}\right]\tilde{\psi}^{[s]}_{(k)} \\ &\quad = -\operatorname{sgn} s\left(\frac{c_k'}{c_k} - (|s|-k)\frac{w'}{w}\right)\frac{wc_{k+1}}{c_k}\tilde{\psi}^{[s]}_{(k+1)} + w\frac{c_j}{c_k}\mathfrak{J}^{[s]}_{(k),(j)}[\tilde{\psi}^{[s]}_{(j)}].\end{aligned} \tag{2.19}$$

2.2.3 Energy norms

In this section, we introduce the definitions of the energy norms we will use throughout the rest of the paper. Fix $s \in \mathbb{Z}$, $-\infty \leq \tau_1 < \tau < \tau_2 \leq \infty$ and recall the definitions (2.7) and (2.8). Then, set

$$\tilde{\psi}^{[s]}_{(k)} \doteq c_k^{-1}(r)\psi^{[s]}_{(k)}, \quad c_k(r) = \begin{cases} r^{-(|s|-k)}, & s > 0 \\ (1-\mu)^{|s|-k}, & s < 0 \end{cases}.$$

First order energy norms. For $k \in \{0, \ldots, |s|\}$, $p \in (-1, 2]$, and $q \in [0, 1]$, we define energy fluxes

$$\begin{aligned}\mathbb{E}^{[s]}_{(k),p,q}(\tau) &\doteq \int_{\Sigma_\tau}\left(r^p|L\tilde{\psi}^{[s]}_{(k)}|^2 + \frac{r^{-2}}{(1-\mu)^{1+q}}|\underline{L}\tilde{\psi}^{[s]}_{(k)}|^2\right)\mathrm{d}r\mathrm{d}\sigma \\ &\quad + \int_{\Sigma_\tau} r^{-2}|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2\mathrm{d}r\mathrm{d}\sigma, \\ \mathbb{E}^{[s]}_{(k),\mathcal{H}^+,q}(\tau_1,\tau_2) &\doteq \int_{\mathcal{H}^+_{(\tau_1,\tau_2)}}\left(|\underline{L}\tilde{\psi}^{[s]}_{(k)}|^2 + \mathbb{1}_{\{q=1\}}|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2\right)\mathrm{d}\sigma\mathrm{d}\tau', \\ \mathbb{E}^{[s]}_{(k),\mathcal{I}^+,p}(\tau_1,\tau_2) &\doteq \lim_{v\to\infty}\int_{S_{(\tau_1,\tau_2)}(v)}\left(r^p|L\tilde{\psi}^{[s]}_{(k)}|^2 + \mathbb{1}_{\{p=2\}}|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2\right)\mathrm{d}\sigma\mathrm{d}\tau',\end{aligned}$$

where if $p = q = 0$ we drop these superscripts, and $\mathbb{1}_{\{x=y\}}$ is 1 if $x = y$ and 0 otherwise. Here, $S_{(\tau_1,\tau_2)}(v)$ denote null hypersurfaces which approach $\mathcal{I}^+_{(\tau_1,\tau_2)}$ as $v \to \infty$, and $\mathcal{H}^+_{(\tau_1,\tau_2)} = \mathcal{H}^+ \cap \left(\cup_{\tau'\in[\tau_1,\tau_2]}\Sigma_\tau\right)$.

We also define bulk energies as follows. For $p \in [0,2]$ and $q \in [0,1]$,

$$\begin{aligned}
\mathbb{I}^{[s]}_{\delta,p,q}(\tau_1,\tau_2) \doteq &\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{-3+p}(2-p+r^{-1})|\mathring{\slashed{\nabla}}\tilde{\uppsi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{p-1}(p+r^{-1})|L\tilde{\uppsi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{r^{-1-\delta}}{(1-\mu)^{1+q}}|\underline{L}\tilde{\uppsi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau', \\
\mathbb{I}^{[s],\,\mathrm{deg}}_{\delta,p,q}(\tau_1,\tau_2) \doteq &\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{-3+p}(2-p+r^{-1})\left(1-\frac{3M}{r}\right)^2|\mathring{\slashed{\nabla}}\tilde{\uppsi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{p-1}(p+r^{-1})\left(1-\frac{3M}{r}\right)^2|L\tilde{\uppsi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{r^{-1-\delta}}{(1-\mu)^{2+q}}(1+q(1-\mu))\left(1-\frac{3M}{r}\right)^2|\underline{L}\tilde{\uppsi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{-1-\delta}|(L-\underline{L})\tilde{\uppsi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau',
\end{aligned}$$

where if $\delta=1$, $p=q=0$ we may drop the subscripts. Note that these two definitions differ by the fact that in the second one the r-weights on some of the derivative degenerates (hence the notation "deg") at $r=3M$.

Now take $k\in\{0,\dots,|s|-1\}$. For $s>0$, $p\in[0,2)$ and $q\in[0,1]$, define

$$\begin{aligned}
\mathbb{I}^{[s]}_{(k),p,q}(0,\tau) \doteq &\int_0^\tau\int_{\Sigma_{\tau'}} r^{-1}\left(r^p|L\tilde{\uppsi}^{[s]}_{(k)}|^2+\frac{1}{(1-\mu)^2}|\underline{L}\tilde{\uppsi}^{[s]}_{(k)}|^2\right)\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+\int_0^\tau\int_{\Sigma_{\tau'}} r^{p-2}|\mathring{\slashed{\nabla}}\tilde{\uppsi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau'.
\end{aligned}$$

We can may also denote this same norm by $\mathbb{I}^{[s]}_{(k),\delta,p,q}(0,\tau)$ for convenience, even though it does not depend on the parameter δ. For $s<0$, $q\in[0,1]$, $\delta\in(0,1]$ and $p\in[0,2]$, define

$$\begin{aligned}
\mathbb{I}^{[s]}_{(k),\delta,p,q}(0,\tau) \doteq &\int_0^\tau\int_{\Sigma_{\tau'}} \left(r^{\max\{p,1\}}|L\tilde{\uppsi}^{[s]}_{(k)}|^2+\frac{r^{-1-\delta}}{(1-\mu)^{2+q}}(1+q(1-\mu))|\underline{L}\tilde{\uppsi}^{[s]}_{(k)}|^2\right)\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+\int_0^\tau\int_{\Sigma_{\tau'}} r^{\max\{p,1\}-2}(2-p+r^{-1})|\mathring{\slashed{\nabla}}\tilde{\uppsi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau'.
\end{aligned}$$

Higher order energy norms. If X is a vector field on $\mathcal{M}$ or $\mathrm{int}(\mathcal{M})$, adding a superscript X to any of the previous norms denotes the same quantity with $\tilde{\psi}^{[s]}_{(k)}$ replaced by $X\tilde{\psi}^{[s]}_{(k)}$. We also set

$$\mathbb{E}^{[s],\,J}_{(k),p,q}(\tau) \doteq \sum_{X\in\{\mathrm{id},T,r^{-1}\mathring{\slashed{\nabla}}^{[s]}\}} \mathbb{E}^{[s],\,J-1,\,X}_{(k),p,q}(\tau),$$

with $\mathbb{E}^{[s],\,0}_{(k),p,q} = \mathbb{E}^{[s]}_{(k),p,q}$ defined above. We take definitions for the fluxes $\mathbb{E}^{[s],\,J}_{(k),\mathcal{H}^+,q}(0,\tau)$ and $\mathbb{E}^{[s],\,J}_{(k),\mathcal{I}^+,p}(0,\tau)$, as well as for the non-degenerate bulk term $\mathbb{I}^{[s],\,J}_{(k),\delta,p,q}(0,\tau)$. Finally, for $k<|s|$ and $J=|s|$, we define

$$\mathbb{I}^{[s],\,\mathrm{deg},\,|s|-k}_{(k),\delta,p,q}(0,\tau) \doteq \mathbb{I}^{[s],\,|s|-k-1}_{(k),p,q}(0,\tau) + \sum_{X\in\{T,r^{-1}\mathring{\slashed{\nabla}}^{[s]}\}} \mathbb{I}^{[s],\,\mathrm{deg},\,|s|-k-1,\,X}_{(k),p,q}(0,\tau).$$

2.3 Notational conventions

In our estimates throughout this article, we use B to denote possibly large positive constants and b to denote possibly small positive constants depending only on $M>0$ and $s\in\mathbb{Z}$. Whenever the constant depends, additionally, on another parameter that has not (yet) been fixed, say x, we write $B(x)$ or $b(x)$. We rarely keep track of changes in such constants, and thus by convention we have

$$B+B=BB=B, \qquad b+b=bb=b, \qquad B+b=B,$$
$$Bb=B, \qquad b^{-1}=B, \qquad \text{etc.}$$

3 Estimates for the top level wave equation

In this section, we prove Theorem 1.1 for the transformed wave equation (2.14). To be precise, we will show:

Theorem 3.1 (EB and ILED for $k=|s|$)**.** *Fix $M>0$, and $s\in\mathbb{Z}$ with $|s|\leq 4$. For any $p\in[0,2]$, $q\in\{0,1\}$, and $\delta\in(0,1]$, and all $\tau>0$, we have the following uniform-in-τ estimates:*

- *energy boundedness without derivative loss:*

$$\mathbb{E}^{[s]}_{(|s|),p,q}(\tau) \leq B\,\mathbb{E}^{[s]}_{(|s|),p,q}(0); \tag{3.1}$$

- *integrated local energy decay with loss of one derivative at trapping:*

$$\mathbb{I}^{[s],\,\mathrm{deg}}_{(|s|),\delta,p,q}(0,\tau) \leq B(\delta)\,\mathbb{E}^{[s]}_{(|s|),p,q}(0). \tag{3.2}$$

Corollary 3.2 (Decay for $k = |s|$)**.** *Fix $M > 0$, and $s \in \mathbb{Z}$ with $|s| \leq 4$. We have the following uniform-in-τ estimates:*

- *energy decay with derivative loss:*
$$\mathbb{E}^{[s]}_{(|s|),0,1}(\tau) \leq (1+\tau)^{-2} B \mathbb{E}^{[s],\,2}_{(|s|),2,1}(0); \tag{3.3}$$
- *pointwise decay with derivative loss: for any $\delta > 0$,*
$$\sup_{\Sigma_\tau} \left|\Psi^{[s]}\right|^2 \leq (1+\tau)^{-2} B \mathbb{E}^{[s],\,4}_{(|s|),2,1}(0). \tag{3.4}$$

Remark 3.3 (Restrictions on s)**.** While the restriction to $s \in \mathbb{Z}$ is important to obtain the DHR transformed equation (2.14) in Proposition 2.5, we expect the restriction to $|s| \leq 4$ to be merely technical. This restriction is introduced so that a particular choice of s-independent multiplier estimate goes through, see the proof of Proposition 3.4 below. It seems plausible that a more refined, possibly s-dependent, choice of multiplier would allows us to obtain the same statement for all $s \in \mathbb{Z}$. However, in view of the fact that, among integer s, only the cases $s \in \{0, \pm 1, \pm 2\}$ are of physical significance, we have chosen not to attempt this optimization.

3.1 Energy boundedness and integrated local energy decay

In this section, we prove Theorem 3.1. We note that the proof given here is by no means novel: Theorem 3.1 is a classical result that can be obtained by following the methods of the lecture notes [DR13] or the more recent [DHR19b, Section 11]. We start with a slightly less sharp version:

Proposition 3.4 (Basic EB and ILED)**.** *Fix $s \in \mathbb{Z}$ with $|s| \leq 4$. The following estimates hold:*

$$\mathbb{E}^{[s]}_{(|s|)}(\tau) + \mathbb{E}^{[s]}_{|s|,\mathcal{H}^+}(0,\tau) + \mathbb{E}^{[s]}_{(|s|),\mathcal{I}^+}(0,\tau) \leq B \mathbb{E}^{[s]}_{(|s|)}(0), \tag{3.5}$$
$$\mathbb{I}^{[s],\,\deg}_{(|s|)}(0,\tau) \leq B \mathbb{E}^{[s]}_{(|s|)}(0). \tag{3.6}$$

Proof. Estimate (3.5) is obtained by using T as a multiplier for (2.14): i.e. we multiply (2.14) by $\overline{T\Phi}$, take the real part, and integrate by parts over $\mathrm{int}(\mathcal{M}) \cap \{0 \leq t^* \leq \tau\}$. Estimate (3.6) requires more work, and we will split its proof into several steps.

Step 0: decomposition in angular modes. To simplify the proof of (3.6), we start by expanding Ψ in spin-weighted spherical harmonics using Lemma 2.2:

$$\Psi^{[s]}_\ell(t,r) \doteq \int_{\mathbb{S}^2} \Psi^{[s]}(t,r,\theta,\phi) \sum_{m \leq \ell} S^{[s]}_{m\ell}(\theta,\phi) \mathrm{d}\sigma, \quad \ell \geq |s|,$$

satisfy the PDE

$$(\Psi_\ell^{[s]})'' - T^2\Psi_\ell^{[s]} - \mathcal{V}_\ell^{[s]}(r)\Psi_\ell^{[s]} = 0, \tag{3.7}$$
$$\mathcal{V}_\ell^{[s]}(r) = \frac{1-\mu}{r^2}\left(\ell(\ell+1) + \frac{2M}{r}(1-s^2)\right).$$

In what follows, we drop the superscript $[s]$ for readability, both from the transformed variable and its norms. We will add a superscript ℓ to the energy norms of Section 2.2.3 to indicate norms where $\Psi^{[s]}$ has been replaced by $\Psi_\ell^{[s]}$, and we will drop the subscript $(|s|)$.

Step 1: Morawetz estimate. Now we apply multiplier estimates to the ℓ-dependent PDE (3.7). We set

$$f(r) \doteq \begin{cases} \left(1-\frac{3M}{r}\right)\left(1+\frac{M}{r}\right) & \text{if } \ell \geq \max\{1,|s|\} \\ 1 & \text{if } s=\ell=0 \end{cases},$$
$$y(r) \doteq \left(1-\frac{3M}{r}\right)^3.$$

For the first multiplier estimate, we multiply (3.7) by $\overline{f'\Psi_\ell + 2f\Psi_\ell'}$, take the real part, and integrate by parts in $\{0 \leq t^* \leq \tau\}$ to obtain

$$\begin{aligned}
&b\int_0^\tau \int_{\Sigma_{\tau'}} \left[\frac{1-\mu}{r^2}|\Psi_\ell'|^2 + \frac{(1-\mu)}{r^3}\left[\ell(\ell+1)\left(1-\frac{3M}{r}\right)+r^{-1}\right]|\Psi_\ell|^2\right] \mathrm{d}r^*\mathrm{d}\tau' \\
&\quad\leq \int_0^\tau \int_{\Sigma_{\tau'}} \left[2f'|\Psi_\ell'|^2 - f\mathcal{V}_\ell'|\Psi_\ell|^2 - \frac{1}{2}f'''|\Psi_\ell|^2\right] \mathrm{d}r^*\mathrm{d}\tau' \\
&\quad= \left(\int_{\Sigma_\tau} - \int_{\Sigma_0}\right)\left(\mathrm{Re}[(f'\Psi_\ell + 2f\Psi_\ell')\overline{T\Psi_\ell}]\right)\mathrm{d}r^*\mathrm{d}\sigma \\
&\qquad + \int_0^\tau \int_{\Sigma_{\tau'}} \left[f|\Psi_\ell'|^2 - f|T\Psi_\ell|^2\right]' \mathrm{d}r^*\mathrm{d}\tau' \\
&\quad\leq B\mathbb{E}^\ell(\tau) + B\mathbb{E}^\ell(0) + B\mathbb{E}^\ell_{\mathcal{H}^+}(0,\tau) + B\mathbb{E}^\ell_{\mathcal{I}^+}(0,\tau),
\end{aligned}$$

as long as[4] $|s| \leq 4$. Then, we multiply (3.7) by $\overline{2y\Psi_\ell'}$, take the real part, and integrate by parts in $\{0 \leq t^* \leq \tau\}$ to obtain, in an entirely analogous manner,

$$b\int_0^\tau \int_{\Sigma_{\tau'}} \frac{(1-\mu)}{r^2}\left(1-\frac{3M}{r}\right)|T\Psi_\ell|^2 \mathrm{d}r^*\mathrm{d}\tau'$$

[4]The restriction here arises from our choice of factor $(1+M/r)$ in the definition of f. It is conceivable that other choices, perhaps depending on $|s|$, would allow us to drop this requirement. See Remark 3.3.

$$
\begin{aligned}
&\leq \int_0^\tau \int_{\Sigma_{\tau'}} \left[y'|\Psi_\ell'|^2 + y'|T\Psi_\ell|^2\right] \mathrm{d}r^*\mathrm{d}\tau' \\
&= \int_0^\tau \int_{\Sigma_{\tau'}} (y\mathcal{V}_\ell)'|\Psi_\ell|^2 \mathrm{d}r^*\mathrm{d}\tau' + \left(\int_{\Sigma_\tau} - \int_{\Sigma_0}\right) \left(2y\,\mathrm{Re}[\Psi_\ell'\overline{T\Psi_\ell}]\right) \mathrm{d}r^*\mathrm{d}\sigma \\
&\quad + \int_0^\tau \int_{\Sigma_{\tau'}} \left[y|\Psi_\ell'|^2 - y|T\Psi_\ell|^2\right]' \mathrm{d}r^*\mathrm{d}\tau' \\
&\leq B\int_0^\tau \int_{\Sigma_{\tau'}} \frac{(1-\mu)}{r^3}\left(\ell(\ell+1) + r^{-1}\right)|\Psi_\ell|^2 \mathrm{d}r^*\mathrm{d}\tau' \\
&\quad + B\mathbb{E}^\ell(\tau) + B\mathbb{E}^\ell(0) + B\mathbb{E}^\ell_{\mathcal{H}^+}(0,\tau) + B\mathbb{E}^\ell_{\mathcal{I}^+}(0,\tau).
\end{aligned}
$$

By combining the two previous estimates, we have

$$
\mathbb{I}^{\deg,\ell}(0,\tau) \leq B\mathbb{E}^\ell(\tau) + B\mathbb{E}^\ell(0) + B\mathbb{E}^\ell_{\mathcal{H}^+}(0,\tau) + B\mathbb{E}^\ell_{\mathcal{I}^+}(0,\tau).
$$

Step 2: sum over angular modes. Making use of Lemma 2.2, we can use the L^2 orthogonality of the angular modes to conclude to sum the previous estimate for all $\ell \geq |s|$: we have

$$
\mathbb{I}^{\deg}(0,\tau) \leq B\mathbb{E}(\tau) + B\mathbb{E}(0) + B\mathbb{E}_{\mathcal{H}^+}(0,\tau) + B\mathbb{E}_{\mathcal{I}^+}(0,\tau),
$$

and thus (3.6) follows after applying (3.5). □

Having obtained an energy boundedness statement and an integrated energy decay statement, we can now improve their r weights in the two limits $r \to \infty$ and $r \to 2M$:

Proof of Theorem 3.1. In Proposition 3.4, we have already shown a version of the statement with $p = q = 0$ and $\delta = 1$. To conclude the proof, we just need to show that the large r weights can be improved using parameters $p \in (0, 2]$ and $\delta \in (0, 1]$ and that the r weights close to $r = 2M$ can be improved using a parameter $q \in (0, 1]$.

Weights at $r \to \infty$. For the improvement in the bulk term only, related to the new δ constant, we employ a large r Morawetz multiplier: in the language of the proof of Proposition 3.4, we choose $y(r) = (1 - r^{-\delta})\chi$ for χ supported at sufficiently large r, so that the error is contained in a bounded $|r|$ region where Proposition 3.4 can be used. For the improvement in the bulk term and energy fluxes related to the p constant, we rely on the r^p-weighted estimates of [DR10]: we multiply (2.14) by $r^p(1 + 4Mr^{-1})\chi\overline{L\Psi}$ for χ supported at sufficiently large r, take the real part and integrate by parts. It is clear from, for instance, the identities in [DHR19a, Page 46] or [SRTdC23, Section 4.1.4] that the errors produced can be controlled by Proposition 3.4.

Weights at $r \approx 2M$. This improvement can be done through the redshift multiplier introduced in [DR09]: we multiply (2.14) by $(1-\mu)^{-q} r^4 \chi \overline{\underline{L}\Psi}$ for χ supported at sufficiently small $r-2M$, take the real part and integrate by parts. It is clear from, for instance, the identities in [DHR19a, Page 92] or [SRTdC23, Section 4.1.4] that the cost to obtain the improvement can be controlled by Proposition 3.4. □

We conclude the section with a trivial corollary of the previous results that will be useful in the next section:

Lemma 3.5 (Nondegenerate ILED)**.** *Fix $s \in \mathbb{Z}$ with $|s| \leq 4$. For any $p \in [0,2]$ and $q \in [0,1]$, the following estimate holds:*

$$\mathbb{I}^{[s]}_{(|s|),1,p,q}(0,\tau) \leq B\mathbb{E}^{[s]}_{|s|,p,q}(0) + B\mathbb{E}^{[s],T}_{|s|,0,0}(0).$$

3.2 Decay of the energy and the solution

In this section, we prove Corollary 3.2. The proof given here is, again, not novel: it is based on Dafermos and Rodnianski's r^p method [DR10].

Proof of Corollary 3.2. In this proof, we will lighten the notation as follows: we drop the superscript $[s]$ and the subscript $(|s|)$ from our definition of norms; we take $\delta = 1$ and drop this subscript.

Take $p \in [1,2]$, $0 < \tau_A < \tau_B < \infty$. By direct inspection of the definitions of the norms,

$$\int_{\tau_A}^{\tau_B} \mathbb{E}_{p-1,1}(\tau) \mathrm{d}\tau \leq B\mathbb{I}_{p,1}(\tau_A,\tau_B)$$

and thus, by Lemma 3.5, we have

$$\int_{\tau_A}^{\tau_B} \mathbb{E}_{p-1,1}(\tau) \mathrm{d}\tau \leq B\mathbb{E}_{p,1}(\tau_A) + B\mathbb{E}^{T}_{0,0}(\tau_A).$$

Step 1: decay along dyadic sequences. Fix some $0 < \tau_0 < \infty$. Taking $p = 1$ in the above, we have

$$\int_{\tau_0}^{\infty} \left[\mathbb{E}^{T}_{1}(\tau) + \mathbb{E}^{0}_{1,1}(\tau)\right] \mathrm{d}\tau \leq B\mathbb{E}_{2,1}(\tau_0) + B\mathbb{E}^{T}_{2,1}(\tau_0) + B\mathbb{E}^{TT}_{0,0}(\tau_0)$$

$$\implies \begin{cases} \mathbb{E}^T_1(\tau_n) \leq \frac{B}{\tau_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}^T_{2,1}(\tau_0)+\mathbb{E}^{TT}_{0,0}(\tau_0)\right), \\ \mathbb{E}^0_1(\tau_n) \leq \frac{B}{\tau_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}^T_{2,1}(\tau_0)+\mathbb{E}^{TT}_{0,0}(\tau_0)\right), \end{cases}$$

along a dyadic sequence such as $\tau_n = 2^n\tau_0$.

We now state two estimates for $p=0$. Firstly, we note

$$\begin{aligned} \int_{\tau_n}^{\tau_{n+1}} \mathbb{E}^T_{0,1}(\tau)\mathrm{d}\tau &\leq B\mathbb{E}^T_{1,1}(\tau_n) + B\mathbb{E}^{TT}_{0,0}(\tau_n) \\ &\leq \frac{B}{\tau_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}^T_{2,1}(\tau_0)\right)+\mathbb{E}^{TT}_{0,0}(\tau_0) \\ \implies \mathbb{E}^T_{0,1}(\overline{\tau}_n) &\leq \frac{B}{\overline{\tau}_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}^T_{2,1}(\tau_0)+\mathbb{E}^{TT}_{0,0}(\tau_0)\right), \end{aligned}$$

along another dyadic sequence $\overline{\tau}_n \in [\frac{3}{4}\tau_n, \frac{5}{4}\tau_n]$. The last inequality in the first line follows by the first estimate in this step, which is used to control $\mathbb{E}^T_{1,1}(\tau_n)$, and energy boundedness (Proposition 3.4), which is used to control $\mathbb{E}^{TT}_{0,1}(\tau_n)$. Secondly, we have

$$\begin{aligned} \int_{\overline{\tau}_n}^{\overline{\tau}_{n+1}} \mathbb{E}_{0,1}(\tau)\mathrm{d}\tau &\leq B\mathbb{E}_{1,1}(\overline{\tau}_n) + B\mathbb{E}^T_{0,0}(\overline{\tau}_n) \leq B\mathbb{E}_{1,1}(\tau_n) + B\mathbb{E}^T_{0,0}(\overline{\tau}_n) \\ &\leq \frac{B}{\overline{\tau}_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}^T_{2,1}(\tau_0)+\mathbb{E}^{TT}_{0,0}(\tau_0)\right) \\ \implies E_{0,1}(\overline{\overline{\tau}}_n) &\leq \frac{B}{\overline{\overline{\tau}}^2_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}^T_{2,1}(\tau_0)+\mathbb{E}^{TT}_{0,0}(\tau_0)\right), \end{aligned}$$

along another dyadic sequence $\overline{\overline{\tau}}_n \in [\frac{3}{4}\overline{\tau}_n, \frac{5}{4}\overline{\tau}_n]$. In the first line, we have used energy boundedness (Theorem 3.1) to control $\mathbb{E}_{1,1}(\overline{\tau}_n)$ and then the previous estimates.

Step 2: decay along foliation. By combining the decay of $\mathbb{E}_{0,1}$ along a dyadic sequence established in Step 1 with the energy boundedness of Proposition 3.4, we deduce that $\mathbb{E}_{0,1}(\tau)$ decays in τ. Sobolev estimates using commutation with angular symmetries then imply pointwise decay for the field Ψ away from $r=2M$. □

4 Estimates for the lower level wave equations

In this section, we prove Theorem 1.1 for the transformed wave equation (2.14). To be precise, we will show:

Theorem 4.1 (EB and ILED for $k \leq |s|$). *Fix $M > 0$, $s \in \mathbb{Z}$ with $|s| \leq 4$, and $k \in \{0, \dots, |s|-1\}$. For any $p \in [0,2)$, $q \in [0,1]$, and $\delta \in (0,1]$, and all $\tau > 0$, we have the following uniform-in-τ estimates:*

- *energy boundedness without derivative loss:*

$$\sum_{k=0}^{|s|} \mathbb{E}^{[s],|s|-k}_{(k),p,q}(\tau) \leq B \sum_{k=0}^{|s|} \mathbb{E}^{[s],\,|s|-k}_{(k),p,q}(0); \tag{4.1}$$

- *integrated local energy decay with loss of one derivative at trapping:*

$$\sum_{k=0}^{|s|} \mathbb{I}^{[s],\,\mathrm{deg},\,|s|-k}_{(k),\delta,p,q}(0,\tau) \leq B(\delta) \sum_{k=0}^{|s|} \mathbb{E}^{[s],\,|s|-k}_{(k),p,q}(0). \tag{4.2}$$

Corollary 4.2 (Decay for $k \leq |s|$). *Fix $M > 0$, and $s \in \mathbb{Z}$ with $|s| \leq 4$. We have the following uniform-in-τ estimates:*

- *energy decay with derivative loss: for any $\eta \in (0,1)$,*

$$\sum_{k=0}^{|s|} \mathbb{E}^{[s],|s|-k}_{(j),0,1}(\tau) \leq (1+\tau)^{-2+\eta} B(\eta) \sum_{k=0}^{|s|} \mathbb{E}^{[s],\,|s|-k+2}_{(|s|),2-\eta,1}(0); \tag{4.3}$$

- *pointwise decay with derivative loss: for any $\delta > 0$ and $\eta \in (0,1)$,*

$$\sup_{k\leq|s|} \sup_{\Sigma_\tau} |\tilde{\psi}^{[s]}_{(k)}|^2 \leq (1+\tau)^{-2+\eta} B(\eta) \sum_{k=0}^{|s|} \mathbb{E}^{[s],\,|s|-k+4}_{(|s|),2-\eta,1}(0). \tag{4.4}$$

If $s < 0$, the above holds with $\eta = 0$ as well.

In these statements, as in those of Section 3, we expect the condition that $|s| \leq 4$ to be suboptimal, see Remark 3.3 for details.

4.1 First order energy boundedness and integrated local energy decay

In this section, we obtain a weak form of Theorem 4.1. Namely, we will show that all first order energy norms of $\psi^{[s]}_{(k)}$ are controlled up to zeroth order energy norms involving $\psi^{[s]}_{(k+1)}$:

Proposition 4.3 (First order EB and ILED). *Take $s \in \pm\mathbb{Z}_{\leq 4}$. If $s < 0$, for all $\delta \in (0,1]$, $p \in [0,2]$ and $q \in [0,1]$, we have*

$$\mathbb{E}^{[s]}_{(k),p,q}(\tau) + \mathbb{E}^{[s]}_{(k),\mathcal{H}^+,q}(0,\tau) + \mathbb{E}^{[s]}_{(k),\mathcal{I}^+,p}(0,\tau) + b(\delta)\mathbb{I}^{[s]}_{(k),\delta,p,q}(0,\tau)$$

$$\leq B\sum_{j=0}^{k}\mathbb{E}^{[s]}_{(j),p,q}(0)+B\left(\int_{\Sigma_\tau}+\int_{\Sigma_0}\right)w|\tilde{\psi}^{[s]}_{(k+1)}|\mathrm{d}r^*\mathrm{d}\sigma$$
$$+B\int_0^\tau\int_{\Sigma_{\tau'}}\frac{w}{r}(1+r\mathbb{1}_{\{p=2\}})|\tilde{\psi}^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'.$$

If $s>0$, for all $p\in[0,2)$ and $q\in[0,1]$, we have

$$\mathbb{E}^{[s]}_{(k),p,q}(\tau)+\mathbb{E}^{[s]}_{(k),\mathcal{H}^+,q}(0,\tau)+\mathbb{E}^{[s]}_{(k),\mathcal{I}^+,p}(0,\tau)+\mathbb{I}^{[s]}_{(k),p,q}(0,\tau)$$
$$\leq B\sum_{j=0}^{k}\mathbb{E}^{[s]}_{(j),p,q}(0)+B\left(\int_{\Sigma_\tau}+\int_{\Sigma_0}\right)w|\tilde{\psi}^{[s]}_{(k+1)}|\mathrm{d}r^*\mathrm{d}\sigma$$
$$+B\int_0^\tau\int_{\Sigma_{\tau'}}wr^{p-1}|\tilde{\psi}^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'.$$

The proof of Proposition 4.3 follows by combining the results of the subsections below: Lemma 4.4 with either Lemmas 4.7 and 4.8 (if $s<0$) or Lemmas 4.6 and 4.9 (if $s>0$). The last four lemmas, though nontrivial, can be viewed as mostly technical. It is Lemma 4.4 which is the heart of the proof of Proposition 4.3 and, indeed, Theorem 4.1.

4.1.1 Estimates with suboptimal weights

We begin by closing estimates for the transformed variables with weaker weights as $r\to\infty$ and $r\to 2M$ than those in Proposition 4.3. This will be loosely based on the insights and, to large extent, approach of [SRTdC20], though see Remark 4.5 for a comparison. Our result of the section is:

Lemma 4.4 (EB and ILED with suboptimal weights)**.** *Fix $s\in\mathbb{Z}$ with $1\leq|s|\leq 4$. Then, for all $k\in\{0,\dots,|s|-1\}$, we have the estimate*

$$\int_0^\tau\int_{\Sigma_{\tau'}}w\left[|\partial_{r^*}\psi^{[s]}_{(k)}|^2+|T\psi^{[s]}_{(k)}|^2+r^{-1}|\mathring{\nabla}^{[s]}\psi^{[s]}_{(k)}|^2+r^{-1}|\psi^{[s]}_{(k)}|^2\right]\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+\int_{\Sigma_\tau}\left[r^{-2}|\underline{L}\psi^{[s]}_{(k)}|^2+(1-\mu)|L\psi^{[s]}_{(k)}|^2+w|\mathring{\nabla}^{[s]}\psi^{[s]}_{(k)}|^2\right]dr^*d\sigma$$
$$\leq B\int_0^\tau\int_{\Sigma_{\tau'}}wr^{-1}|\tilde{\psi}^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'+B\int_{\Sigma_\tau}w|\psi^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma$$
$$+B\int_{\Sigma_0}w|\psi^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma+B\sum_{j=0}^{k}\mathbb{E}_j(0).$$

Let us start with the easier case $k=0$. Note that, for $|s|=1$, this is the only case we need to study.

Proof of Lemma 4.4 if $k=0$. Consider the wave equation (2.12) with $k=0$; in particular notice that, in this case, $\mathfrak{J}^{[s]}_{(k),(j)}\equiv 0$. The proof follows in two steps. First, in Step 1, we repeat the proof of Proposition 3.4, now applied to (2.12) with $k=0$, to obtain a (conditional) energy boundedness statement and a degenerate integrated local energy decay statement. Then, in Step 2, we will remove the degeneracy, thus recovering integrated (conditional) control over all derivatives of the $k=0$ transformed variable.

In this proof, and all other proofs of Section 4 from this point onwards, we will drop the superscripts $[s]$ and the parenthesis in the subscripts for readibility.

Step 1: energy estimate and degenerate bulk estimate. One can check easily that the same choices of multiplier currents from the proof of Proposition 3.4 lead, for (2.12) with $k=0$, to the estimate

$$\begin{aligned}&\int_0^\tau\int_{\Sigma_{\tau'}} w\left[|\psi_0'|^2+\left(1-\frac{3M}{r}\right)\left(|T\psi_0|^2+\frac{1}{r}|\mathring{\slashed{\nabla}}\psi_0|^2\right)+\frac{1}{r}|\psi_0|^2\right]\mathrm{d}r^*\mathrm{d}\tau'\\&\quad\leq B\mathbb{E}_0(0)+B\int_0^\tau\int_{\Sigma_{\tau'}} w\left(r^{-1}|\psi_1\psi_0'|+rw|\psi_1\psi_0|+rw|\psi_1|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau\\&\quad\leq B\mathbb{E}_0(0)+B\int_0^\tau\int_{\Sigma_{\tau'}} wr^{-2}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau,\end{aligned}$$

at least for $|s|\leq 6$, and

$$\begin{aligned}&\int_{\Sigma_\tau}\left[r^{-2}|\underline{L}\psi_0|^2+(1-\mu)|L\psi_0|^2+w|\mathring{\slashed{\nabla}}\psi_0|^2\right]dr^*d\sigma\\&\quad\leq B\mathbb{E}_0(0)+B\int_0^\tau\int_{\Sigma_{\tau'}} wr^{-2}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau.\end{aligned}$$

We have used here that (2.8) $\Leftrightarrow T\psi_0=w\psi_1+\operatorname{sgn}s\,\psi_0'$ to simplify the error terms arising from application of the T multiplier.

Step 2: removing the degeneration. Now take the constraint equation (2.15) with $k=0$, multiply by $\overline{\psi_0}$ and integrate by parts to obtain

$$\begin{aligned}&\int_{\mathbb{S}^2}\frac{w}{r}\left(|\mathring{\slashed{\nabla}}\psi_0|^2+\left(|s|+\frac{2M}{r}(1-3|s|+2s^2)\right)|\psi_0|^2\right)\mathrm{d}\sigma\\&\quad=\int_{\mathbb{S}^2}\frac{w}{r}\left[w|\psi_1|^2+2\operatorname{sgn}s\operatorname{Re}[\psi_1\overline{\psi_0}']+\operatorname{sgn}s\left(|s|\frac{w'}{w}-\frac{1}{r}\right)\operatorname{Re}[\psi_1\overline{\psi_0}]\right]\mathrm{d}\sigma\\&\qquad-\int_{\mathbb{S}^2}\underline{\mathcal{L}}\left(\frac{w}{r}\operatorname{Re}[\overline{\psi_0}\psi_1]\right)\mathrm{d}\sigma.\end{aligned}$$

Thus, we have the estimate

$$\begin{aligned}
&\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-1}\left[|\mathring{\nabla}\psi_0|^2+|\psi_0|^2\right]\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau\\
&\quad\leq B\int_0^\tau\int_{\Sigma_{\tau'}}|\psi_0'|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau+B\int_0^\tau\int_{\Sigma_{\tau'}}wr^{-2}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau\\
&\qquad+B\int_{\Sigma_\tau}wr^{-1}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma+B\int_{\Sigma_0}wr^{-1}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma+B\mathbb{E}_0(0).
\end{aligned}$$

Using Step 1, and the fact that (2.8) $\Leftrightarrow T\psi_0=w\psi_1+\operatorname{sgn}s\psi_0'$ (which allows us to directly estimate the bulk term in $T\psi_0$), we obtain

$$\begin{aligned}
&\int_0^\tau\int_{\Sigma_{\tau'}} w\left[|\psi_0'|^2+|T\psi_0|^2+r^{-1}|\mathring{\nabla}\psi_0|^2+r^{-1}|\psi_0|^2\right]\mathrm{d}r^*\mathrm{d}\tau'\\
&\qquad+\int_{\Sigma_\tau}\left[r^{-2}|\underline{L}\psi_0|^2+(1-\mu)|L\psi_0|^2+w|\mathring{\nabla}\psi_0|^2\right]dr^*d\sigma\\
&\quad\leq B\int_0^\tau\int_{\Sigma_{\tau'}}wr^{-2}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau+B\int_{\Sigma_\tau}\frac{w}{r}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\\
&\qquad+B\int_{\Sigma_0}\frac{w}{r}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma+B\mathbb{E}_0(0).
\end{aligned}$$

This concludes the proof of Lemma 4.4 in the case $k=0$. □

Studying the $k=0$ case has us provided a strategy to try to address the $k<|s|$ wave equations (2.12). However, in the more involved $k\geq 1$ case, as we will see shortly, this must be supplemented with new insights:

Proof of Lemma 4.4 if $k\geq 1$. Consider the wave equation (2.12), now with $k\neq 0$. We start by trying to follow the strategy of the proof of Lemma 4.4 with $k=0$. Applying the same steps as in that case, is not hard to see that such multiplier currents and use of the constraint equation (2.15) will yield the estimate

$$\begin{aligned}
&\int_0^\tau\int_{\Sigma_{\tau'}} w\left[|\psi_k'|^2+|T\psi_k|^2+r^{-1}|\mathring{\nabla}\psi_k|^2+r^{-1}|\psi_k|^2\right]\mathrm{d}r^*\mathrm{d}\tau'\\
&\quad\leq B\int_0^\tau\int_{\Sigma_{\tau'}}wr^{-2}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'+B\int_{\Sigma_\tau}\frac{w}{r}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\\
&\qquad+B\int_{\Sigma_0}\frac{w}{r}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma+B\mathbb{E}_k(0)\\
&\qquad+B\sum_{j=0}^{k-1}\int_0^\tau\int_{\Sigma_{\tau'}}w|\psi_j|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',
\end{aligned}\tag{4.5}$$

for $|s| \leq 4$ and all $1 \leq k < |s|$, and for $|s| = 5, 6$ only if $1 \leq k < |3|$, as well as the estimate

$$\int_{\Sigma_\tau} \left[r^{-2} |\underline{L}\psi_k|^2 + (1-\mu)|L\psi_k|^2 + w|\mathring{\nabla}\psi_k|^2 \right] dr^* d\sigma$$
$$\leq B \int_0^\tau \int_{\Sigma_{\tau'}} w \Big(r^{-2} |\psi_{k+1}|^2 + \sum_{j=0}^{k-1} |\psi_j|^2 \Big) \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' + B\mathbb{E}_k(0),$$

Notice that, since $\mathfrak{J}_{k,j} \not\equiv 0$ for $k \neq 0$, we now have coupling errors involving ψ_j with $j < k$. In order to conclude the proof we must control these errors. By separately examining the $s < 0$ and $s > 0$ cases below, we will see that on Schwarzschild this can be achieved by supplementing the wave estimate (4.5) with appropriate transport estimates for (2.8).

The case $s < 0$. Let us consider the transport equation (2.8) with $s < 0$. By multiplying it by $c(r)\overline{\psi_k}$, and taking the real part, we can derive the identity

$$-c'(r)|\psi_k|^2 = -L\left(c(r)|\psi_k|^2 \right) + 2c(r)w\,\mathrm{Re}[\psi_{k+1}\overline{\psi_k}].$$

Now choose $c(r) = r^{-1}$ to obtain

$$\begin{aligned} \int_0^\tau \int_{\Sigma_{\tau'}} w|\psi_k|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' &\leq B \int_{\Sigma_0} r^{-3} |\psi_k|^2 \mathrm{d}r^* \mathrm{d}\sigma \\ &\quad + B \int_0^\tau \int_{\Sigma_{\tau'}} w r^{-2} |\psi_{k+1}|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \\ &\leq B\mathbb{E}_k(0) + B \int_0^\tau \int_{\Sigma_{\tau'}} w r^{-2} |\psi_{k+1}|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau'. \end{aligned} \tag{4.6}$$

Notice that we have used the additional decay of ψ_k compared to $\tilde{\psi}_k$ to estimate the integrals over Σ_0 by our data norms.

Using the transport estimate (4.6), we deduce that the coupling errors in the last line of (4.5) satisfy

$$\sum_{j=0}^{k-1} \int_0^\tau \int_{\Sigma_{\tau'}} w|\psi_j|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \leq B \int_0^\tau \int_{\Sigma_{\tau'}} \frac{w}{r} |\psi_k|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' + B \sum_{j=0}^{k-1} \mathbb{E}_j(0).$$

Therefore, if the derivative terms dominate the left hand side of (4.5), i.e. if

$$\int_0^\tau \int_{\Sigma_{\tau'}} w \left(|T\psi_k|^2 + r^{-1} |\mathring{\nabla}\psi_k|^2 \right) \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \geq \frac{1}{\epsilon} \int_0^\tau \int_{\Sigma_{\tau'}} w r^{-1} |\psi_k|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau',$$

for sufficiently small ϵ, the coupling errors in (4.5) can be absorbed by the left hand side; we can fix ϵ so that this is the case. Then, (4.5) holds without the last line as long as we add $B\sum_{j=0}^{k-1}\mathbb{E}_j(0)$ to the right hand side.

If, on the contrary, we have

$$\int_0^\tau\int_{\Sigma_{\tau'}} w\left(|T\psi_k|^2+r^{-1}|\mathring{\nabla}\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' < \frac{1}{\epsilon}\int_0^\tau\int_{\Sigma_{\tau'}} wr^{-1}|\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',$$

then, since we have already fixed ϵ,

$$\begin{aligned}&\int_0^\tau\int_{\Sigma_{\tau'}} w\left[|\psi_k'|^2+|T\psi_k|^2+r^{-1}|\mathring{\nabla}\psi_k|^2+r^{-1}|\psi_k|^2\right]\mathrm{d}r^*\mathrm{d}\tau'\\ &\leq B\int_0^\tau\int_{\Sigma_{\tau'}} w|\psi_k|^2\mathrm{d}r^*\mathrm{d}\tau'\leq B\int_0^\tau\int_{\Sigma_{\tau'}} wr^{-1}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\tau',\end{aligned}$$

by noting that $\psi_k'=w\psi_{k+1}-T\psi_k$ and then using the transport estimate (4.6) directly.

The case $s>0$. If we repeat the procedure from the previous case now for transport equation (2.8) with $s>0$, we can derive the identity

$$c'(r)|\psi_k|^2=-\underline{L}\left(c(r)|\psi_k|^2\right)+2c(r)w\,\mathrm{Re}[\psi_{k+1}\overline{\psi_k}].$$

A natural choice (analogous to that in the previous case) is $c(r)=1-\mu$, which leads to

$$\begin{aligned}\int_0^\tau\int_{\Sigma_{\tau'}} w|\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' &\leq B\int_{\Sigma_0}(1-\mu)^2|\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\\ &\qquad+B\int_0^\tau\int_{\Sigma_{\tau'}} w(1-\mu)|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\ &\leq B\mathbb{E}_k(0)+B\int_0^\tau\int_{\Sigma_{\tau'}} w(1-\mu)|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',\end{aligned}$$

where we have again used the additional decay of ψ_k compared to $\tilde{\psi}_k$ to estimate the integrals over Σ_0 by our data norms. In this estimate, the bulk term in ψ_{k+1} on the right hand side has too weak decay as $r\to\infty$ for us to carry on emulating the argument of the previous case case. However, it is enough to show that the coupling errors in the last line of (4.5) satisfy

$$\sum_{j=0}^{k-1}\int_0^\tau\int_{\Sigma_{\tau'}} w|\psi_j|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\leq B\int_0^\tau\int_{\Sigma_{\tau'}} w|\psi_{k-1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'+B\sum_{j=0}^{k-2}\mathbb{E}_j(0).$$

If we seek to change the r-weights in our transport estimate, a first obvious attempt is to modify the choice of $c(r)$. For instance, consider

$$c(r) = -(r^*)^{-1}\mathbb{1}_{\{r^*\geq R^*\}} - (r^* - R^* + 1)/R^*\mathbb{1}_{\{R^*-1\leq r^*\leq R^*\}}$$

for some arbitrary, but fixed, $R^* \geq 1$. Then, we have

$$\int_0^\tau \int_{\Sigma_{\tau'}} w|\psi_k|^2 \mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' \leq B\mathbb{E}_k(0) + B\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-1}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\int_0^\tau \int_{\Sigma_{\tau'}\cap\{R^*-1\leq r^*\leq R^*\}} |\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',$$

where the r-weight in the ψ_{k+1} bulk term has enough decay as $r\to\infty$; this comes at the cost of having to control a bounded r term bulk term in ψ_k. The coupling terms in (4.5) therefore are controlled by

$$\begin{aligned} &B\int_0^\tau \int_{\Sigma_{\tau'}} w|\psi_{k-1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\sum_{j=0}^{k-2}\mathbb{E}_j(0) \\ &\quad\leq B\int_0^\tau \int_{\Sigma_{\tau'}} \frac{w}{r}|\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\int_0^\tau \int_{\Sigma_{\tau'}\cap\{0\leq|r^*-R^*|\leq 1\}} |\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' \\ &\qquad + B\sum_{j=0}^{k-1}\mathbb{E}_j(0). \end{aligned}$$

As in the case $s<0$, if the terms involving T and $\mathring{\nabla}$ derivatives dominate the left hand side of (4.5), we can now conclude using their comparative largeness.

In contrast, if the terms involving T and $\mathring{\nabla}$ derivatives on the left hand side of (4.5) are not dominant, the above approach will not help us. Let us instead consider the modified transport equation (2.18). Multiplying (2.18) by $c(r)\overline{\tilde{\psi}_k}$ yields for $s>0$

$$\begin{aligned} &\left(c'(r) + 2rw(|s|-k)c(r)\right)|r^{|s|-k}\psi_k|^2 \\ &\quad = -\underline{L}\left(c(r)|\tilde{\psi}_k|^2\right) + 2c(r)r^{2(|s|-k)}w\,\mathrm{Re}[\psi_{k+1}\overline{\psi_k}]. \end{aligned}$$

Now notice that choosing $c(r) = r^{-2}$, we find that

$$\begin{aligned} &(|s|-k-1)\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-1}|r^{|s|-k}\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' \\ &\quad \leq B\mathbb{E}_k(0) + B\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-1}|\tilde{\psi}_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'. \end{aligned}$$

Unless $(s, k) = (+1, 0)$, this transport estimate represents the desired gain in r-weights as $r \to \infty$, and it is enough to conclude, arguing similarly to the $s < 0$ case, when neither T nor $\mathring{\nabla}$ derivatives are dominant. □

As required, we have obtained energy boundedness and integrated local energy decay estimates for $\psi^{[s]}_{(k)}$ conditional on lower order statements holding for $\psi^{[s]}_{(k+1)}$. However, as we have mentioned, these estimates have suboptimal r-weights, and this is easy to see: for instance, we expect $\psi^{[s]}_{(k)} \to 0$ as $r^* \to (\operatorname{sgn} s)\infty$, that is, we expect that $\psi^{[s]}_{(k)}$ are not radiation fields. In the next two sections, accordingly, we will improve the r-weights of Lemma 4.4 as $r^* \to (\operatorname{sgn} s)\infty$ and beyond.

Before doing so, and because Lemma 4.4 is the heart of the proof of Theorem 4.1, let us wrap up the section with a reflection on its proof and outlook to the rotating Kerr case in [SRTdC20]:

Remark 4.5 (From $k = 0$ to $k \geq 1$ to rotating Kerr)**.** In the above proofs of Lemma 4.4, we saw that in turning from the case of $k = 0$ to $k \geq 1$ the significant difficulty we encounter is the fact that, in the latter case, the wave equation (2.12) is coupled not only to the $(k+1)$th equation but also to the jth wave equations, with $0 \leq j < k$. Since the coupling constant is not small, for us to close any wave-type estimates, we must find smallness elsewhere.

For the Schwarzschild case considered here, we easily find smallness of this backward coupling (coupling to $j < k$ equations) in the time/angular derivative-dominated regime by making use of the constraint equation (2.15), obtained by combining the wave equation (2.12) and the transport equation (2.8). In the complementary regime, we avoid wave-type estimates altogether, using transport estimates for (2.8) instead. Crucially, we use the fact that we never need to gain any smallness in the forward coupling, i.e. the coupling to the $(k+1)$th equation; because we control the top, $k = |s|$, transformed variable *unconditionally* by the results of Section 3, our task for $k < |s|$ is simply to convert the $k < |s|$ errors into (zeroth order) bulk terms involving $k = |s|$.

Turning to the rotating Kerr case analyzed in [SRTdC20] presents a further layer of difficulties. First of all, the coupling to the $j < k$ wave equations occurs through angular derivatives as well, making the gain of smallness in the time/angular derivative-dominated regime significantly harder to achieve. Secondly, in the complementary regime, we cannot hope to rely on transport estimates alone, as the top, $k=|s|$, wave equation is coupled (with an $O(M)$ coupling constant) to all the $k<|s|$ equations.

4.1.2 Improving weights near $\mathcal{I}^+$

To improve the r-weights in the estimates of Lemma 4.4, we start with the asymptotically flat region, $r\to\infty$. We emphasize that the statements and proofs contained in this section are not new: they have appeared, in a more condensed form, in [SRTdC23]. For simplicity, we will state them (and prove them) with a simplified notation, where we drop the superscript $[s]$ and the parenthesis in the subscript in the transformed variables and their norms.

Lemma 4.6 (Improving weights for $s>0$)**.** *Fix $s>0$. Let $p\in[0,2)$ and $R^*>0$ be sufficiently large. Then, for all $k\in\{0,\dots,|s|-1\}$, we have the estimate*

$$\begin{aligned}
&\mathbb{E}_{k,p,0}(\tau)+\mathbb{E}_{k,\mathcal{I}^+,p}(0,\tau)+\mathbb{I}_{k,p,0}(0,\tau)\\
&\quad\le B\sum_{j=0}^{k}\mathbb{E}_{j,p,0}(0)+B\int_{\Sigma_0} w|\tilde{\psi}_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\\
&\qquad+B\int_{\Sigma_\tau} w|\tilde{\psi}_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma+B\int_0^\tau\int_{\Sigma_{\tau'}} wr^{p-1}|\tilde{\psi}_{k+1}|\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\qquad+B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{|r^*|\le R^*\}}\Big(|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+\sum_{j=0}^{k}|\tilde{\psi}_j|^2\Big)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\qquad+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\le -R^*\}}(1-\mu)\Big(\frac{1}{1-\mu}|\underline{L}\tilde{\psi}_k|^2+|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2\Big)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\qquad+B\int_{\Sigma_\tau\cap\{r^*\le -R^*\}}(1-\mu)\Big(\frac{1}{1-\mu}|\underline{L}\tilde{\psi}_k|^2+|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2\Big)\mathrm{d}r^*\mathrm{d}\sigma.
\end{aligned}$$

Lemma 4.7 (Improving weights for $s<0$)**.** *Fix $s<0$. Let $p\in[0,2]$ and $R^*>0$ be sufficiently large. Then, for all $k\in\{0,\dots,|s|-1\}$, we have the estimate*

$$\begin{aligned}
&\int_{\Sigma_\tau\cap\{r^*\ge R^*\}} r^{-2}\left(r^{p+2}|L\psi_k|^2+|\underline{L}\psi_k|^2+|\mathring{\nabla}\psi_k|^2+|\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\\
&\quad+\mathbb{E}_{\mathcal{I}^+,p}(0,\tau)+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\ge R^*\}}\left(r^{p-1}|L\psi_k|^2+\delta r^{-1-\delta}|\underline{L}\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\ge R^*\}} r^{p-3}\left(2-p+r^{-1}\right)\left(|\mathring{\nabla}\psi_k|^2+|\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'
\end{aligned}$$

$$
\begin{aligned}
&\leq B(R^*)\int_0^\tau \int_{\Sigma_{\tau'}\cap\{R^*-1\leq r^*\leq R^*\}} \Big(|\underline{L}\psi_k|^2+|\mathring{\nabla}\psi_k|^2+\sum_{j=0}^k|\psi_j|^2\Big)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\quad + B\int_0^\tau \int_{\Sigma_{\tau'}\cap\{r^*\geq R^*-1\}} r^{p-4}(1+r^{-1})|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\quad + B\sum_{j=0}^k E_{j,p,0}(0)+B\int_{\Sigma_0} w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma.
\end{aligned}
$$

Proof of Lemma 4.6. We consider the rescaled equation (2.19) for $s>0$, focusing especially on the case $r\gg 1$ where the rescaling weight $c_k(r)$ becomes prominent.

In the proof, we will consider the cases $k=0$ (Step 1) and $k>0$ (Step 2) separately in the basic setting $p=0$, $\delta=1$. In both instances, and for all the lemmas in Sections 4.1.2 and 4.1.3, the proof follows by considering multipliers $z(r)L$ and $z(r)\underline{L}$ for some function z: that is, by considering the identities generated by multiplying (2.19) by $z(r)\overline{L\tilde{\psi}_k}$ and $z(r)\overline{\underline{L}\tilde{\psi}_k}$, respectively, taking the real part, and integrating by parts. The precise choices in integration by parts depend on the situation; thus, it will be useful to keep in mind the notation

$$
\begin{aligned}
&\left(\int_{\Sigma_\tau}-\int_{\Sigma_0}\right)\left(v_1(r)r^{-2}F^X_{\underline{L}}+v_2(r)(1-\mu)F^X_{\underline{L}}\right)\mathrm{d}r^*\mathrm{d}\sigma\\
&\qquad + \int_{\mathcal{H}^+_{(0,\tau)}} v_3(r)F^X_{\underline{L}}\mathrm{d}\sigma\mathrm{d}\tau' + \int_{\mathcal{I}^+_{(0,\tau)}} v_4(r)F^X_{L}\mathrm{d}\sigma\mathrm{d}\tau' \qquad (4.7)\\
&+\int_0^\tau\int_{\Sigma_{\tau'}} v_5(r)I^X\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'=0,
\end{aligned}
$$

for the identity induced by a given multiplier X, not just in this proof but in the following three proofs; here $b\leq v_i(r)\leq B$ only depending on M. In Step 3, we sketch the proof for general $p\in[0,2)$, $\delta\in(0,1]$. For readability, we have kept the discussion in these steps at the level of the $r\gg 1$ region only; in Step 4 we explain how to make these estimates become the stated almost global estimates.

Step 1: $k=0$. Since (2.19) with $k=0$ couples only to $\tilde{\psi}_1$, our estimates in this case are slightly easier. First, we apply the multiplier $z(r)L$ to obtain an identity of the form of (4.7) with

$$
\begin{aligned}
&F^{zL}_{\underline{L}} = z|L\tilde{\psi}_0|^2,\\
&F^{zL}_{L} = \frac{1}{2}zw|\mathring{\nabla}\tilde{\psi}_0|^2+\frac{Mzw}{r}(1-|s|+2s^2)|\tilde{\psi}_0|^2,
\end{aligned}
$$

$$I^{zL}=\left(|s|rwz+z'\right)|L\tilde{\psi}_0|^2-\frac{(zw)'}{2}|\mathring{\nabla}\tilde{\psi}_0|^2-\left(\frac{Mzw}{r}\right)'(1-3|s|+2s^2)|\tilde{\psi}_0|^2\\+z|s|w\left(1-\frac{4M}{r}\right)\mathrm{Re}\left[\overline{L\tilde{\psi}_0}\tilde{\psi}_1\right].$$

Here, we have treated the terms $(U_0+\frac{2M}{r})w-\left(\frac{c_0'}{c_0}\right)'$ in (2.19) together. Choosing z to be a cutoff function which equals 1 for $r^*\geq R^*+1$ and equals 0 for $r^*\leq R^*$, we have

$$\int_{\Sigma_\tau\cap\{r^*\geq R^*+1\}}\left(|L\tilde{\psi}_0|^2+r^{-4}|\mathring{\nabla}\tilde{\psi}_0|^2+r^{-4}|\tilde{\psi}_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\\+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*+1\}}\left(r^{-1}|L\tilde{\psi}_0|^2+r^{-3}|\mathring{\nabla}\tilde{\psi}_0|^2+r^{-3}|\tilde{\psi}_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\\leq B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\frac{w}{r}|\tilde{\psi}_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'+B\mathbb{E}_0(0)\\+B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{R^*\leq r^*\leq R^*+1\}}\left(|L\tilde{\psi}_0|^2+|\mathring{\nabla}\tilde{\psi}_0|^2+|\tilde{\psi}_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',$$

as long as R^* is chosen to be sufficiently large.

To improve the weights on Σ_τ in this estimate, we then consider an identity generated by using the $z(r)\underline{L}$ multiplier, specifically taking the form

$$F_L^{z\underline{L}}=z|\underline{L}\tilde{\psi}_0|^2-\frac{1}{2}s^2w^2r^2z|\tilde{\psi}_0|^2,\\F_{\underline{L}}^{z\underline{L}}=\frac{1}{2}zw|\mathring{\nabla}\tilde{\psi}_0|^2+\frac{Mzw}{r}(1-|s|+2s^2)\tilde{\psi}_0|^2,\\I^{z\underline{L}}=-z'|\underline{L}\tilde{\psi}_0|^2+z|s|r^2w^2\,\mathrm{Re}[\tilde{\psi}_1\overline{\underline{L}\tilde{\psi}_0}]\\+\frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_0|^2+\frac{1}{2}\left[w\left(s^2+\frac{2M}{r}(1-|s|+s^2)\right)z\right]'|\tilde{\psi}_0|^2\\+|s|z(r-4M)r^2w^3|\tilde{\psi}_1|-s^2(r-4M)r^2w^3z\,\mathrm{Re}[\tilde{\psi}_1\overline{\tilde{\psi}_0}],$$

in the notation of (4.7). To obtain these expressions, we have treated the terms $(U_0+\frac{2M}{r})w-\left(\frac{c_0'}{c_0}\right)'$ in (2.19) together again. We also used the fact that $\underline{L}\tilde{\psi}_0=wr\left(\tilde{\psi}_1-|s|\tilde{\psi}_0\right)$ to treat the other new terms in (2.19) which are absent from (2.12), and integrated by parts any terms involving $\mathrm{Re}[\overline{\tilde{\psi}_0}L\tilde{\psi}_0]$. With $z(r)$ the cutoff function from before, we deduce

$$\int_{\mathcal{I}^+_{(0,\tau)}}|\underline{L}\tilde{\psi}_0|\mathrm{d}\sigma\mathrm{d}\tau'+\int_{\Sigma_\tau\cap\{r^*\geq R^*+1\}}\left(r^{-2}|\underline{L}\tilde{\psi}_0|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_0|^2+r^{-2}|\tilde{\psi}_0|^2\right)$$

$$\leq B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}} r^{-3}\left(r^2|\underline{L}\psi_0|^2+|\mathring{\nabla}\tilde{\psi}_0|^2+|\tilde{\psi}_0|^2+|\tilde{\psi}_1|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\mathbb{E}_0(0),$$

for sufficiently large R^*. Adding a small multiple of this second estimate with the previous one, and then using the identity $\underline{L}\tilde{\psi}_0 = rw\left(\tilde{\psi}_1 - |s|\tilde{\psi}_0\right)$ to directly estimate the $\underline{L}\tilde{\psi}_0$ bulk terms, we obtain suitable flux and bulk estimates for large r:

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r^*\geq R^*+1\}}\left(|L\tilde{\psi}_0|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_0|^2+r^{-2}|\tilde{\psi}_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\\&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*+1\}}\frac{1}{r}\left(|L\tilde{\psi}_0|^2+|\underline{L}\tilde{\psi}_0|^2+\frac{1}{r^2}|\mathring{\nabla}\tilde{\psi}_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\&\quad\leq B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\frac{w}{r}|\tilde{\psi}_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'+B\mathbb{E}_0(0)\\&\qquad+B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{R^*\leq r^*\leq R^*+1\}}\left(|L\tilde{\psi}_0|^2+|\mathring{\nabla}\tilde{\psi}_0|^2+|\tilde{\psi}_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'.\end{aligned}$$

Step 2: $k>0$ and $|s|>1$. Our strategy will be similar to that in the previous step; however, because (2.19) with $0<k<|s|$ is coupled to the $(k+1)$th equation *and* all the jth equations with $j<k$, the implementation will be much more involved. The main difference is that we will need to work harder to obtain multiplier identities generated by the $z(r)L$ and $z(r)\underline{L}$ multipliers which are suitable for our intended estimates. We will use the notation

$$c(r)\doteq\frac{c_k'}{c_k}-(|s|-k)\frac{w'}{w}=\frac{(|s|-k)(r-4M)}{r^2}=(|s|-k)rw-\frac{2M}{r^2}(|s|-k).$$

It will be useful to note that for $0<k<|s|$ and $|s|\geq 2$,

$$\begin{aligned}&w\left(U_k+\frac{2M}{r}\right)-\left(\frac{c_k'}{c_k}\right)'\\&\quad=k(2|s|-k)w+\frac{2M}{r}w\left[1-|s|+2s^2+k(1+3k-6|s|)\right]\geq 2w,\\&w\left(U_k+\frac{2M}{r}\right)-\left(\frac{c_k'}{c_k}\right)'+2\left[\left(\frac{c_k'}{2c_k}\right)'-\frac{c_k'}{c_k}(|s|-k)rw\right]\\&\quad=w[2s(|s|-k)+|s|-k+k^2]+\frac{2M}{r}w[1-2k(2|s|-k-3)-3s]\geq 3w.\end{aligned}$$

Let us first state and explain the the relevant multiplier identities. Firstly, the $z\underline{L}$ multiplier applied to (2.12) yields an identity of the form

(4.7) with

$$
\begin{aligned}
F_L^{z\underline{L}} &= z|\underline{L}\tilde{\psi}_k|^2 - \frac{c_k'}{c_k} z \operatorname{Re}[\underline{L}\tilde{\psi}_k\overline{\tilde{\psi}_k}] + \frac{1}{2}\left(\frac{c_k'}{c_k}\right)^2 |\tilde{\psi}_k|^2 \\
F_{\underline{L}}^{z\underline{L}} &= \frac{1}{2} z w |\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2} z\left[w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)'\right] |\tilde{\psi}_k|^2 \\
&\quad + \left[\left(\frac{z c_k'}{2c_k}\right)' - \frac{c_k'}{c_k} z r w(|s|-k)\right] |\tilde{\psi}_k|^2 \\
I^{z\underline{L}} &= (cz - z')|\underline{L}\tilde{\psi}_k|^2 + \frac{2M}{r^2}(|s|-k)\frac{c_k'}{c_k}\operatorname{Re}[\underline{L}\tilde{\psi}_k\overline{\tilde{\psi}_k}] \\
&\quad + \frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2}\left(\left[w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)'\right] z\right)' |\tilde{\psi}_k|^2 \\
&\quad + \left[\left(\frac{z c_k'}{2c_k}\right)'' - (|s|-k)\left(\frac{z w r c_k'}{c_k}\right)' - \left(\frac{z}{2}\left(\frac{c_k'}{c_k}\right)^2\right)'\right] |\tilde{\psi}_k|^2 \\
&\quad - \frac{c_k'}{c_k} z\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)' - \frac{c_k'}{c_k} c\right) |\tilde{\psi}_k|^2 \\
&\quad + z\sum_{j=0}^{k-1} \frac{w c_j}{c_k} c_{s,k,j}^{\mathrm{id}} \left(\frac{c_k'}{c_k}\operatorname{Re}\left[\overline{\tilde{\psi}_k}\tilde{\psi}_j\right] - \operatorname{Re}\left[\overline{\underline{L}\tilde{\psi}_k}\tilde{\psi}_j\right]\right).
\end{aligned}
$$

In deriving this identity, in addition to the obvious integration by parts argument for the wave operator $\mathfrak{R}_k - \left(\frac{c_k'}{c_k}\right)'$, we have used the fact that

$$
\begin{aligned}
&\frac{w c_{k+1}}{c_k} cz \operatorname{Re}\left[\tilde{\psi}_{k+1}\overline{\underline{L}\tilde{\psi}_k}\right] - \frac{c_k'}{c_k} z \operatorname{Re}\left[\overline{L\tilde{\psi}_k}\underline{L}\tilde{\psi}_k\right] \\
&\quad = cz|\underline{L}\tilde{\psi}_k|^2 - cz\frac{c_k'}{c_k}\operatorname{Re}\left[\psi_k\overline{\underline{L}\tilde{\psi}_k}\right] - \frac{c_k'}{c_k} z \operatorname{Re}\left[\overline{L\tilde{\psi}_k}\underline{L}\tilde{\psi}_k\right],
\end{aligned}
$$

where the second term (or a subset thereof) can be integrated in $\underline{L}$ and the third term can be integrated in L; one then uses (2.12) to simplify $L\underline{L}\tilde{\psi}_k$, and integrates by parts the resulting expression. Let us remark that, as $r \to \infty$,

$$
\begin{aligned}
&\left(\frac{c_k'}{2c_k}\right)'' - (|s|-k)\left(rw\frac{c_k'}{c_k}\right)' + c\left(\frac{c_k'}{c_k}\right)^2 - \left(\frac{1}{2}\left(\frac{c_k'}{c_k}\right)^2\right)' \\
&\quad - \frac{c_k'}{c_k}\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)'\right)
\end{aligned}
$$

$$= -\frac{c_k'}{c_k}\left(s^2 - (|s| - k) - 1\right) + O(r^{-4})$$

which is either positive, since $0 < k < |s|$ and $|s| > 1$ by assumption, or $O(r^{-4})$. Secondly, the zL multiplier applied to (2.12) yields an identity of the form (4.7) where we have

$$\begin{aligned}
F_{\underline{L}}^{zL} &= z|L\tilde{\psi}_k|^2 - \frac{1}{2}(|s| - k)(zr^{-1})'|\tilde{\psi}_k|^2 + \frac{1}{2}c^2|\tilde{\psi}_k|^2, \\
F_{L}^{zL} &= \frac{1}{2}zw|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2}z\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)'\right)|\tilde{\psi}_k|^2 \\
&\quad + cz\frac{wc_{k+1}}{c_k}\operatorname{Re}[\overline{\tilde{\psi}_k}\tilde{\psi}_{k+1}] - \frac{2M}{r^2}(|s| - k)\frac{c_k'}{c_k}z|\tilde{\psi}_k|^2, \\
I^{zL} &= \left(z' - \frac{c_k'}{c_k}z\right)|L\tilde{\psi}_k|^2 - \frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_k|^2 \\
&\qquad + 2M(|s| - k)\left[\left(\frac{z}{r^2}\right)'\operatorname{Re}[\underline{L}\tilde{\psi}_k\overline{\tilde{\psi}_k}] + \frac{zc_k'}{r^2c_k}\operatorname{Re}[L\tilde{\psi}_k\overline{\tilde{\psi}_k}]\right] \\
&\quad - \frac{1}{2}\left[\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)' - c^2\right)z\right]'|\tilde{\psi}_k|^2 \\
&\qquad + czw|\mathring{\nabla}\tilde{\psi}_k|^2 + cz\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)' - \frac{c_k'}{c_k}c\right)|\tilde{\psi}_k|^2 \\
&\qquad + (|s| - k)\left[\left(\frac{c_k'}{c_k}zrw\right)' - \frac{1}{2}(zr^{-1})''\right]|\tilde{\psi}_k|^2 \\
&\qquad - z\sum_{j=0}^{k_1}\frac{wc_j}{c_k}c_{s,k,j}^{\mathrm{id}}\left(\operatorname{Re}\left[\overline{L\tilde{\psi}_k}\tilde{\psi}_j\right] - c\operatorname{Re}\left[\overline{\tilde{\psi}_k}\tilde{\psi}_j\right]\right).
\end{aligned}$$

Here, in addition to the obvious integration by parts argument for the wave operator $\mathfrak{R}_k - \left(\frac{c_k'}{c_k}\right)'$, we have used the fact that

$$\frac{wc_{k+1}}{c_k}cz\operatorname{Re}\left[\tilde{\psi}_{k+1}\overline{L\tilde{\psi}_k}\right] = zc\operatorname{Re}\left[\left(\underline{L}\tilde{\psi}_k - \frac{c_k'}{c_k}\tilde{\psi}_k\right)\overline{L\tilde{\psi}_k}\right]$$

can be integrated by parts in $\underline{L}$ (first term) and L (second term), that we can use (2.12) to simplify $\underline{L}L\tilde{\psi}_k$, and that the resulting expression can also be integrated by parts. Let us remark that

$$cw|\mathring{\nabla}\tilde{\psi}_k|^2 + \left[c\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)'\right)\right]|\tilde{\psi}_k|^2$$

$$+\left[-\frac{c_k'}{c_k}c^2z+(|s|-k)\left(\frac{c_k'}{c_k}zrw\right)'-\frac{1}{2}(|s|-k)(zr^{-1})''+\frac{1}{2}(c^2z)'\right]|\tilde{\psi}_k|^2$$
$$=cw|\mathring{\nabla}\tilde{\psi}_k|^2+cw(s^2-2(|s|-k)-1)|\tilde{\psi}_k|^2+O(r^{-4})|\tilde{\psi}_k|^2$$

as $r\to\infty$; thus, in view of the properties of the spin-weighted laplacian, this expression is either positive for $0<k<|s|-1$ and $s\geq 1$, or it is $O(r^{-4})|\tilde{\psi}_k|^2$.

We now apply the above identities in an identical fashion to Step 1. We choose $z(r^*)=\chi(r^*)$ to be a cutoff function equal to 1 for $r^*\geq R^*$ and equal to zero for $r^*\leq R^*-1$ in both the multiplier estimates above. We add a small multiple of the $z\underline{L}$ identity to the zL identity. We deduce that:

$$\begin{aligned}
&\int_{\mathcal{I}^+_{(0,\tau)}}|\underline{L}\tilde{\psi}_k|^2\mathrm{d}\sigma\mathrm{d}\tau'\\
&\quad+\int_{\Sigma_\tau\cap\{r^*\geq R^*\}}\left(|L\psi_k|^2+r^{-1}|\underline{L}\psi_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2+r^{-2}|\tilde{\psi}_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\\
&\quad+\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\frac{1}{r}\left(|L\psi_k|^2+|\underline{L}\psi_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2+r^{-2}|\tilde{\psi}_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\leq B\mathbb{E}_k(0)+B\int_{\Sigma_0}w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma+B\int_{\Sigma_\tau\cap\{r^*\geq R^*-1\}}w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\\
&\quad+\frac{B}{(R^*)^2}\sum_{j=0}^{k-1}\int_0^\tau\int_{\Sigma'_\tau\cap\{r^*\geq R^*-1\}}w'|\tilde{\psi}_j|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\quad+B(R^*)\int_0^\tau\int_{\Sigma'_\tau\cap\{0\leq R^*-r^*\leq 1\}}\Big(|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+\sum_{j=0}^{k+1}|\tilde{\psi}_j|^2\Big)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',
\end{aligned}$$

as long as R^* is chosen to be sufficiently large.

Notice that there is a small parameter multiplying the large r error involving $\tilde{\psi}_j$ for $j<k$. Thus, making R^* even larger and then fixing it, we can iterate the estimates for $j=0,\dots k$ (thus making use of Step 1) to obtain:

$$\begin{aligned}
&\int_{\mathcal{I}^+_{(0,\tau)}}|\underline{L}\tilde{\psi}_k|^2\mathrm{d}\sigma\mathrm{d}\tau'\\
&\quad+\int_{\Sigma_\tau\cap\{r^*\geq R^*\}}\left(|L\psi_k|^2+r^{-1}|\underline{L}\psi_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2+r^{-2}|\tilde{\psi}_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\\
&\quad+\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\frac{1}{r}\left(|L\psi_k|^2+|\underline{L}\psi_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2+r^{-2}|\tilde{\psi}_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\leq B\sum_{j=0}^{k}\mathbb{E}_j(0)+B\int_{\Sigma_0}w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma+B\int_{\Sigma_\tau\cap\{r^*\geq R^*-1\}}w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma
\end{aligned}$$

$$+ B\int_0^\tau \int_{\Sigma'_\tau\cap\{0\leq R^*-r^*\leq 1\}}\Big(|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+\sum_{j=0}^{k+1}|\tilde{\psi}_j|^2\Big)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'.$$

Step 3: p-weighted norms. To improve the weights further using the parameter $p\in(0,2)$, we repeat the two previous steps but now choosing $z=r^p\chi$, with χ the cutoff function from the previous step, for the $z(r)L$ multiplier only. Notice that, for $k=0$, the term

$$z|s|w\left(1-\frac{4M}{r}\right)\mathrm{Re}[L\tilde{\psi}_0\overline{\tilde{\psi}_1}],\quad z=r^p\chi,$$

prevents us from closing estimates with $p=2$.

Step 4: almost-global fluxes on Σ_τ. By combining the previous steps, we have closed estimates for $\tilde{\psi}_k$ in the large r region in terms of $\tilde{\psi}_{k+1}$ errors (and data). It is not hard to see that the energy flux estimates can be made almost global in r: indeed, if we choose $z(r)$ in the $z(r)L$ multiplier and $z(r)\underline{L}$ multipliers to be supported not just at large r but for all r away from $r=2M$, then we can control the derivatives of $\tilde{\psi}_k$ on $\Sigma_\tau\cap\{r^*\geq -R^*\}$ as long as we control $\tilde{\psi}_{k+1}$ and $\tilde{\psi}_j$, $j=0,\dots k-1$, spacetime errors away from $r\approx 2M$. This final version of our estimates is the one given in the statement. □

Proof of Lemma 4.7. Consider the wave equation (2.12) for $s<0$. As announced above, this proof will also rely on using $z(r)L$ and $z(r)\underline{L}$ multipliers for appropriate choices of functions $z(r)$.

We start by stating the forms of the multiplier identities we will consider, modeled on the notation in (4.7). For the $z(r)L$ multiplier, we have

$$\begin{aligned}
F^{zL}_{\underline{L}} &= z|L\psi_k|^2,\\
F^{zL}_{L} &= \frac{1}{2}zw|\mathring{\nabla}\psi_k|^2+\frac{1}{2}zw\left(U_k+\frac{2M}{r}\right)|\psi_k|^2 - zw\sum_{j=0}^{k-1}c^{\mathrm{id}}_{s,k,j}\,\mathrm{Re}\left[\overline{\psi_k}\psi_j\right],\\
I^{zL} &= z'|L\psi_k|^2-\frac{1}{2}(zw)'|\mathring{\nabla}\psi_k|^2-\frac{1}{2}\left[w\left(U_k+\frac{2M}{r}\right)z\right]'|\psi_k|^2\\
&\quad+(|s|-k)w'zw|\psi_{k+1}|^2\\
&\quad+\sum_{j=0}^{k-1}c^{\mathrm{id}}_{s,k,j}\left(z'w\,\mathrm{Re}\left[\overline{\psi_k}\psi_j\right]+zw^2\,\mathrm{Re}\left[\overline{\psi_k}\psi_{j+1}\right]\right),
\end{aligned}$$

which is obtained by the usual integration by parts procedure for terms generated by $\mathfrak{R}_k$, together with integration by parts of the terms $\mathrm{Re}[L\psi_k\overline{\psi_j}]$.

For the $z(r)\underline{L}$ multiplier, we have

$$F_{L}^{z\underline{L}} = z|\underline{L}\psi_k|^2, \qquad F_{\underline{L}}^{z\underline{L}} = \frac{1}{2}zw|\mathring{\nabla}\psi_k|^2 + \frac{1}{2}zw\left(U_k + \frac{2M}{r}\right)|\psi_k|^2,$$

$$I^{z\underline{L}} = -z'|\underline{L}\psi_k|^2 + \frac{1}{2}(zw)'|\mathring{\nabla}\psi_k|^2 + \frac{1}{2}\left[w\left(U_k + \frac{2M}{r}\right)z\right]'|\psi_k|^2$$
$$+ (|s|-k)w'z\operatorname{Re}[\psi_{k+1}\overline{\underline{L}\psi_k}] - z\sum_{j=0}^{k-1} wc_{s,k,j}^{\mathrm{id}}\operatorname{Re}\left[\overline{\underline{L}\psi_k}\psi_j\right].$$

Let χ be a smooth cutoff function such that $\chi = 1$ for $r^* \geq R^*$ and $\chi = 0$ for$r^* \leq R^* - 1$. Apply the $z(r)L$ identity with $z = r^p(1+4M/r)\chi$, and add a small multiple of the $z(r)\underline{L}$ identity with $z = \chi$. Since $c_k = 1 + O(r^{-1})$ as $r \to \infty$, we have

$$\int_{\mathcal{I}^+_{(0,\tau)}}\left[|L\psi_k|^2 + \mathbb{1}_{\{p=2\}}\left(|\mathring{\nabla}\psi_k|^2 + r^{p-2}|\psi_k|^2\right)\right]\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ \int_{\Sigma_\tau\cap\{r^*\geq R^*\}} r^{-2}\left(r^{p+2}|L\psi_k|^2 + |\underline{L}\psi_k|^2 + |\mathring{\nabla}\psi_k|^2 + |\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma$$
$$+ \int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\left(r^{p-1}|L\psi_k|^2 + r^{-2}|\underline{L}\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ \int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}} r^{p-3}\left(2-p+r^{-1}\right)\left(|\mathring{\nabla}\psi_k|^2 + |\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$\leq B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{R^*-1\leq r^*\leq R^*\}}\left(|\underline{L}\psi_k|^2 + |\mathring{\nabla}\psi_k|^2 + |\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*-1\}} r^{p-4}(1+r^{-1})|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ B\int_{\Sigma_0\cap\{r^*\geq R^*-1\}} r^{p-4}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma + BE_k(0)$$
$$+ B\sum_{j=0}^{k-1}\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*-1\}} r^{p-3}\left((2-p)+r^{-1}\right)|\psi_j|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ B\sum_{j=0}^{k-1}\left(\int_{\Sigma_\tau\cap\{r^*\geq R^*-1\}} + \int_{\Sigma_0\cap\{r^*\geq R^*-1\}}\right) r^{2(p-3)}|\psi_j|^2\mathrm{d}r^*\mathrm{d}\sigma.$$

Clearly, for any choice of $p \in [0,2]$, we can use the improvement in the r-weights associated to $k+1$ and iterate the above estimate for $j = 0, \ldots, k$: thus, for R^* sufficiently large, the above estimate holds without the last two lines and with $\mathbb{E}_k(0)$ replaced by $\sum_{j=0}^{k}\mathbb{E}_j(0)$.

Finally, we can improve our weights on $\underline{L}\psi_k$ bulk terms by using the $z(r)\underline{L}$ identity with $z = (1+r^{-\delta})\chi$ instead of $z = \chi$. □

4.1.3 Improving weights near $\mathcal{H}^+$

Finally, in this we turn to the task of improving Lemma 4.4 in the near horizon region. We again emphasize that the statements and proofs contained in this section are not new, as they have appeared already in [SRTdC23]. As in the previous section, we will state our results (and prove them) with a simplified notation, where we drop the superscript $[s]$ and the parenthesis in the subscript in the transformed variables and their norms.

Lemma 4.8 (Improving weights for $s<0$)**.** *Fix $s<0$. Let $q\in[0,1]$ and $R^*>0$ be sufficiently large. Then, for all $k\in\{0,\dots,|s|-1\}$, we have the estimate*

$$\begin{aligned}
&\mathbb{E}_{k,0,q}(\tau)+\mathbb{E}_{k,\mathcal{H}^+,q}(0,\tau)+\mathbb{I}_{k,1,0,q}(0,\tau)\\
&\quad\leq B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{|r^*|\leq R^*\}}\Big[|\underline{L}\tilde{\psi}_k|^2+|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+\sum_{j=0}^{k}|\tilde{\psi}_j|^2\Big]dr^*d\sigma d\tau'\\
&\qquad+B\sum_{j=0}^{k}\mathbb{E}_{j,0,q}(0)+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq -R^*\}}\frac{w}{r}|\tilde{\psi}_{k+1}|^2dr^*d\sigma d\tau'\\
&\qquad+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\Big[r^{-2}|\underline{L}\tilde{\psi}_k|^2+|L\tilde{\psi}_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2\Big]dr^*d\sigma d\tau'\\
&\qquad+B\int_{\Sigma_{\tau}\cap\{r^*\geq R^*\}}\Big[r^{-2}|\underline{L}\tilde{\psi}_k|^2+r^{-1}|L\tilde{\psi}_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2\Big]dr^*d\sigma d\tau'.
\end{aligned}$$

Lemma 4.9 (Improving weights for $s>0$)**.** *Fix $s>0$. Let $q\in[0,1]$ and $R^*<0$ be sufficiently negative. Then, for all $k\in\{0,\dots,|s|-1\}$, we have the estimate*

$$\begin{aligned}
&\mathbb{E}_{k,\mathcal{H}^+,q}(0,\tau)+\int_{\Sigma_\tau\cap\{r^*\leq R^*\}}(1-\mu)^{-q}|\underline{L}\psi_k|^2dr^*d\sigma\\
&\quad+\int_{\Sigma_\tau\cap\{r^*\leq R^*\}}(1-\mu)\Big[|L\psi_k|^2+|\mathring{\nabla}\psi_k|^2+|\psi_k|^2\Big]dr^*d\sigma\\
&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}\Big[q(1-\mu)^{-q}|\underline{L}\psi_k|^2+(1-\mu)|L\psi_k|^2\Big]dr^*d\sigma d\tau'\\
&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)^{1-q}\left((1-q)+(1-\mu)\right)\Big[|\mathring{\nabla}\psi_k|^2+|\psi_k|^2\Big]dr^*d\sigma d\tau'\\
&\leq B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*-R^*\in[0,1]\}}\Big[|L\psi_k|^2+|\mathring{\nabla}\psi_k|^2+\sum_{j=0}^{k+1}|\psi_j|^2\Big]dr^*d\sigma d\tau'\\
&\qquad+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)|\tilde{\psi}_{k+1}|^2dr^*d\sigma d\tau'
\end{aligned}$$

$$+ B \int_{\Sigma_\tau \cap \{r^* \leq R^*\}} (1-\mu)|\tilde{\psi}_{k+1}|^2 dr^* d\sigma + B \sum_{j=0}^{k} \mathbb{E}_{j,0,q}(0).$$

In view of the positive surface gravity of Schwarzschild black holes, and associated redshift effect [DR09], one can expect this region to be significantly easier to handle than the asymptotically flat $r \to \infty$ region. Indeed, this section is much less technically involved than the previous. Nevertheless, the main conceptual ideas are the same.

Proof of Lemma 4.8. Fix some $s < 0$, and consider the rescaled equation (2.19) with $k \leq |s| - 1$. This proof will follow a similar strategy to that of Lemma 4.6. To obtain the estimates, we derive suitable identities generated by multipliers $z(r)L$ and $z(r)\underline{L}$. Using the notation of (4.7), these can be stated as

$$\begin{aligned}
F_L^{z\underline{L}} &= z|\underline{L}\tilde{\psi}_k|^2, \\
F_{\underline{L}}^{z\underline{L}} &= \frac{1}{2} zw|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2} zw\left(U_k + \frac{2M}{r}\right)|\tilde{\psi}_k|^2, \\
I^{z\underline{L}} &= \left(\frac{c_k'}{c_k} z - z'\right)|\underline{L}\tilde{\psi}_k|^2 + \frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2}\left[w\left(U_k + \frac{2M}{r}\right)z\right]'|\tilde{\psi}_k|^2 \\
&\quad - z\,\mathrm{Re}\left\{\overline{\underline{L}\tilde{\psi}_k}\left[\left(\frac{c_k'}{c_k} - (|s|-k)\frac{w'}{w}\right)\frac{wc_{k+1}}{c_k}\tilde{\psi}_{k+1} + \left(\frac{c_k'}{c_k}\right)'\tilde{\psi}_k\right]\right\} \\
&\quad - z\sum_{j=0}^{k-1} \frac{wc_j}{c_k} c_{s,k,j}^{\mathrm{id}}\,\mathrm{Re}[\overline{\underline{L}\tilde{\psi}_k}\tilde{\psi}_j],
\end{aligned}$$

for the $z(r)\underline{L}\tilde{\psi}_k$ multiplier, and for the $z(r)L\tilde{\psi}_k$,

$$\begin{aligned}
F_{\underline{L}}^{zL} &= z|L\tilde{\psi}_k|^2 - \frac{1}{2} z\left(\frac{c_k'}{c_k}\right)^2|\tilde{\psi}_k|^2 \\
F_L^{zL} &= \frac{1}{2} zw|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2} zw\left(U_k + \frac{2M}{r}\right)|\tilde{\psi}_k|^2 \\
I^{zL} &= z'|L\tilde{\psi}_k|^2 - \frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_k|^2 - \frac{1}{2}\left[w\left(U_k + \frac{2M}{r}\right)z\right]'|\tilde{\psi}_k|^2 \\
&\quad - \frac{1}{2} z'\left(\frac{c_k'}{c_k}\right)^2|\tilde{\psi}_k|^2 - z\left(\frac{wc_{k+1}}{c_k}\right)^2 c(r)|\tilde{\psi}_{k+1}|^2 \\
&\quad - z\frac{wc_{k+1}}{c_k}\mathrm{Re}\left\{\overline{\tilde{\psi}_{k+1}}\left[\left(\frac{c_k'}{c_k}\right)'\tilde{\psi}_k - \frac{c_k'}{c_k}\underline{L}\tilde{\psi}_k - c(r)\frac{c_k'}{c_k}\tilde{\psi}_k\right]\right\}
\end{aligned}$$

$$- z\sum_{j=0}^{k-1}\frac{wc_j}{c_k}c^{\rm id}_{s,k,j}\,\mathrm{Re}[\overline{L\tilde{\psi}_k}\tilde{\psi}_j].$$

Here, we have used the notation

$$c(r)\doteq\frac{c_k'}{c_k}-(|s|-k)\frac{w'}{w}=\frac{2(|s|-k)w}{r},$$

and obtained the latter identity by, after the usual integration by parts procedure for the $\mathfrak{R}_k$ operator, noticing

$$\begin{aligned}
&-\mathrm{Re}\left\{\overline{L\tilde{\psi}_k}\left[c\frac{wc_{k+1}}{c_k}\tilde{\psi}_{k+1}+\left(\frac{c_k'}{c_k}\right)'\tilde{\psi}_k-\frac{c_k'}{c_k}\underline{L}\tilde{\psi}_k\right]\right\}+\left(\frac{wc_{k+1}}{c_k}\right)^2c|\tilde{\psi}_{k+1}|^2\\
&\quad=-\frac{wc_{k+1}}{c_k}\mathrm{Re}\left\{\overline{\tilde{\psi}_{k+1}}\left[\left(\frac{c_k'}{c_k}\right)'\tilde{\psi}_k-\frac{c_k'}{c_k}\underline{L}\tilde{\psi}_k-c(r)\frac{c_k'}{c_k}\tilde{\psi}_{k+1}\right]\right\}\\
&\qquad+\frac{1}{2}\left[\left(\frac{c_k'}{c_k}\right)^2\right]|\tilde{\psi}_k|^2-\frac{1}{2}\left(\frac{c_k'}{c_k}\right)^2\underline{L}|\tilde{\psi}_k|^2.
\end{aligned}$$

Similarly to Lemma 4.6, let $\chi(r^*)$ be a smooth cutoff function localized to $r\approx 2M$: $\chi=1$ for $r^*\leq R^*$ and $\chi=0$ for $r^*\geq R^*+1$, $R^*<-1$. Then, summing the $z(r)\underline{L}$ identity together with a small multiple of the $z(r)L$ identity, choosing $z=\chi$ for both, yields an estimate of the form:

$$\begin{aligned}
&\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}\left[|\underline{L}\tilde{\psi}_k|^2+(1-\mu)\left(|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+|\tilde{\psi}_k|^2\right)\right]dr^*d\sigma d\tau'\\
&\quad+\int_{\Sigma_{\tau}\cap\{r^*\leq R^*\}}\left[|\underline{L}\tilde{\psi}_k|^2+(1-\mu)\left(|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+|\tilde{\psi}_k|^2\right)\right]dr^*d\sigma\\
&\quad+\mathbb{E}_{k,\mathcal{H}^+}(0,\tau)\\
&\leq B\mathbb{E}_k(0)+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)|\tilde{\psi}_{k+1}|^2dr^*d\sigma d\tau'\\
&\quad+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*-R^*\in[0,1]\}}\left[|\underline{L}\tilde{\psi}_k|^2+|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+|\tilde{\psi}_k|^2\right]\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\quad+B\sum_{j=0}^{k-1}\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)^2|\tilde{\psi}_j|^2dr^*d\sigma d\tau',
\end{aligned}$$

for sufficiently negative R^*. By making $|R^*|$ even larger, we can use the additional factor of $(1-\mu)$ in the coupling to $\tilde{\psi}_j$ as a smallness parameter; then, iterating the estimate above for $j=0,\dots,k$, we deduce that it holds without the last line as long as $\mathbb{E}_k(0)$ is replaced by $\sum_{j=0}^k\mathbb{E}_j(0)$.

Similarly to Lemma 4.6, we can now refine the estimate even further. Adding the $z(r)\underline{L}$ identity now with $z(r)=(1-\mu)^{-q}r^4\chi$, we obtain improve the left hand side of the estimate above to

$$\begin{aligned}&\int_{\Sigma_\tau\cap\{r^*\leq R^*\}}\left[(1-\mu)^{-q}|\underline{L}\tilde{\psi}_k|^2+(1-\mu)\left(|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+|\tilde{\psi}_k|^2\right)\right]dr^*d\sigma\\&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)\left[\frac{1}{(1-\mu)^{1+q}}|\underline{L}\tilde{\psi}_k|^2+|L\tilde{\psi}_k|^2\right]dr^*d\sigma d\tau'\\&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)^{1-q}[(1-q)+(1-\mu)]\left(|\mathring{\nabla}\tilde{\psi}_k|^2+|\tilde{\psi}_k|^2\right)dr^*d\sigma d\tau'\\&\quad+\mathbb{E}_{k,\mathcal{H}^+,q}(0,\tau).\end{aligned}$$

(The above estimate corresponds to $q=0$). Thus, this new estimate can again be iterated as before for $j=0,\dots k$ if R^* is chosen sufficiently negative.

Finally, to obtain the estimates in the form of the statement, we simply note that we can extend the support of χ to large, but finite, r^*. □

Proof of Lemma 4.9. Fix some $s>0$, and consider the wave equation (2.12) with $k\leq|s|-1$. Our strategy will be similar to that of Lemma 4.6 and Lemma 4.8 above: we will make use of multiplier identities for the multipliers $z(r)L$ and $z(r)\underline{L}$ acting on (2.12). With the notation from (4.7), the identities we will use are:

$$\begin{aligned}F_L^{z\underline{L}}&=z|\underline{L}\psi_k|^2,\qquad F_{\underline{L}}^{z\underline{L}}=\frac{1}{2}zw|\mathring{\nabla}\psi_k|^2+\frac{1}{2}zw\left(U_k+\frac{2M}{r}\right)|\psi_k|^2,\\I^{z\underline{L}}&=-z'|\underline{L}\psi_k|^2+\frac{1}{2}(zw)'|\mathring{\nabla}\psi_k|^2+\frac{1}{2}\left[w\left(U_k+\frac{2M}{r}\right)z\right]'|\psi_k|^2\\&\qquad-(|s|-k)w'zw|\psi_{k+1}|^2-z\sum_{j=0}^{k-1}wc_{s,k,j}^{\mathrm{id}}\operatorname{Re}\left[\overline{\underline{L}\psi_k}\psi_j\right],\end{aligned}$$

for $z(r)\underline{L}$, and for $z(r)L$

$$\begin{aligned}F_{\underline{L}}^{zL}&=z|L\psi_k|^2,\\F_L^{zL}&=\frac{1}{2}zw|\mathring{\nabla}\psi_k|^2+\frac{1}{2}zw\left(U_k+\frac{2M}{r}\right)|\psi_k|^2,\\I^{zL}&=z'|L\psi_k|^2+\frac{1}{2}(zw)'|\mathring{\nabla}\psi_k|^2+\frac{1}{2}\left[w\left(U_k+\frac{2M}{r}\right)z\right]'|\psi_k|^2\\&\qquad-(|s|-k)w'z\operatorname{Re}[\psi_{k+1}\overline{L\psi_k}]-z\sum_{j=0}^{k-1}wc_{s,k,j}^{\mathrm{id}}\operatorname{Re}\left[\overline{L\psi_k}\psi_j\right].\end{aligned}$$

Let χ be a smooth cutoff function with $\chi = 1$ for $r^* \leq R^*$ and $\chi = 0$ for $r^* \geq R^* + 1$, for $R^* < -1$. Then, we apply the $z\underline{L}$ identity with $z = (1-\mu)^{-q}\chi$ and add a small multiple of the the zL identity with $z = \chi$. We will obtain, for $q \in [0,1]$,

$$\begin{aligned}
&\mathbb{E}_{k,\mathcal{H}^+,0}(0,\tau) + \mathbb{E}_{k,\mathcal{H}^+,q}(0,\tau) + \int_{\Sigma_\tau \cap \{r^* \leq R^*\}} (1-\mu)^{-q} |\underline{L}\uppsi_k|^2 dr^* d\sigma \\
&\quad + \int_{\Sigma_\tau \cap \{r^* \leq R^*\}} (1-\mu) \left[|L\uppsi_k|^2 + |\mathring{\slashed{\nabla}}\uppsi_k|^2 + |\uppsi_k|^2 \right] dr^* d\sigma \\
&\quad + \int_0^\tau \int_{\Sigma_{\tau'} \cap \{r^* \leq R^*\}} \left[(1-\mu)^{-q} |\underline{L}\uppsi_k|^2 + (1-\mu)|L\uppsi_k|^2 \right] dr^* d\sigma d\tau' \\
&\quad + \int_0^\tau \int_{\Sigma_{\tau'} \cap \{r^* \leq R^*\}} (1-\mu)^{1-q} \left((1-q) + (1-\mu)\right) \left[|\mathring{\slashed{\nabla}}\uppsi_k|^2 + |\uppsi_k|^2 \right] dr^* d\sigma d\tau' \\
&\leq B(R^*) \int_0^\tau \int_{\Sigma_{\tau'} \cap \{r^* - R^* \in [0,1]\}} \left[|L\uppsi_k|^2 + |\mathring{\slashed{\nabla}}\uppsi_k|^2 + \sum_{j=0}^{k+1} |\uppsi_j|^2 \right] dr^* d\sigma d\tau' \\
&\qquad + B \int_0^\tau \int_{\Sigma_{\tau'} \cap \{r^* \leq R^*\}} (1-\mu) |\tilde{\uppsi}_{k+1}|^2 dr^* d\sigma d\tau' \\
&\qquad + B \int_{\Sigma_\tau \cap \{r^* \leq R^*\}} (1-\mu) |\tilde{\uppsi}_{k+1}|^2 dr^* d\sigma + B\mathbb{E}_k(0) \\
&\qquad + \frac{B}{|R^*|^q} \sum_{j=0}^{k} \int_0^\tau \int_{\Sigma_{\tau'} \cap \{r^* \leq R^*\}} (1-\mu)^{1-q} |\tilde{\uppsi}_j|^2 dr^* d\sigma d\tau',
\end{aligned}$$

for sufficiently negative R^*. In particular, choosing some $q \in (0,1)$ to start with, we can iterate the estimate in $j = 0, \ldots, k$. Making $|R^*|$ even larger so that the terms in the last line are small, we deduce that the estimate holds without the last line, for all $q \in [0,1]$, if $\mathbb{E}_k(0)$ is replaced by $\sum_{j=0}^k \mathbb{E}_j(0)$. □

4.2 Higher order energy boundedness and integrated local energy decay

In the previous section, we have shown that all first derivatives of the kth transformed variable can be controlled in terms of zeroth order quantities in the $(k+1)$th transformed variable, see Proposition 4.3. In view of Theorem 3.1, which is closed at the level of *first* order derivatives of $|s|$th transformed variable (possibly with degeneration at $r = 3M$), we should be able to upgrade Proposition 4.3 to higher order control for the $k < |s|$ variables. This is precisely the goal of the present section.

When trying to control higher order derivatives, it is common to commute with Killing fields, such as T, as we have done already in Section 3.2

to obtain Corollary 3.2. However, for the goal of obtaining Theorem 4.1, that would not be adequate: due to trapping, we cannot expect to control $T\partial\psi^{[s]}_{(k)}$ or $\mathring{\nabla}\partial\psi^{[s]}_{(k)}$ bulk terms without a degeneration. Instead, we will commute with the vector field which does not "see" trapping (cf. Theorem 3.1), that is, ∂_{r^*}.

Lemma 4.10 (Commutation with ∂_{r^*})**.** *Fix $s \in \mathbb{Z}$ and $k \in \{0, \dots, |s|\}$. We have the identity*

$$\begin{aligned}\mathfrak{R}^{[s]}_{(k)}\left[\left(\psi^{[s]}_{(k)}\right)'\right] &= w'T\psi^{[s]}_{(k+1)} + w' \operatorname{sgn} s(|s|-k+1)\left(\psi^{[s]}_{(k+1)}\right)' \\ &\quad + \operatorname{sgn} s\left((|s|-k)\left(\frac{w'}{w}\right)' + \frac{(w')^2}{w}\right)\psi^{[s]}_{(k+1)} \\ &\quad + w\left(U^{[s]}_{(k)} + \frac{2M}{r}\right)'\psi^{[s]}_{(k)} + \sum_{j=0}^{k-1} w\left(c^{\mathrm{id}}_{s,k,j}\psi^{[s]}_{(j)}\right)',\end{aligned}$$

where $\mathfrak{R}^{[s]}_{(k)}$ is the differential operator defined in (2.13).

Proof. An easy computation shows that

$$[\mathfrak{R}_k, \partial_{r^*}] = -w'\mathring{\Delta}\phi_k - \left[\left(U_k + \frac{2M}{r}\right)w\right]'\phi_k.$$

To conclude, we make use of the constraint equation (2.15). □

With this lemma, we are now ready to conclude the proof of our main result for the lower level wave equations:

Proof of Theorem 4.1. Take $s \in \mathbb{Z}$ and $k \in \{0, \dots, |s|-1\}$ We have already shown that the estimates of Theorem 4.1 holds for first order energies in the k transformed variables; thus, the we only have to show that we can bootstrap from first to second order energies.

Let us start by using Lemma 4.10. The fact that ∂_{r^*} is not a symmetry of the equation is manifest from the commutator terms of Lemma 4.10. If one repeats the arguments of the previous section, these additional terms will generate additional bulk errors. For all but the first such error, we can treat them using Cauchy–Schwarz: e.g. for the choices $p = 0 = q$ and $\delta = 1$ (and dropping those subscripts from the norms), the errors produced by all but the first term in Lemma 4.10 are controlled by

$$\epsilon B \sum_{j=0}^{k}\left(\mathbb{I}^{\partial_{r^*}}_j(0,\tau) + \mathbb{E}^{\partial_{r^*}}_j(\tau)\right)$$

$$
\begin{aligned}
&+\epsilon^{-1}B\sum_{j=0}^{k}\Big(\mathbb{I}_{j,1}(0,\tau)+\mathbb{E}_j(\tau)+\mathbb{E}_{j,\mathcal{H}^+}(0,\tau)+\mathbb{E}_{j,\mathcal{I}^+}(0,\tau)\Big)\\
&+\epsilon^{-1}B\Big(\mathbb{I}^{\mathrm{deg}}_{k+1}(0,\tau)+\mathbb{E}_{k+1}(\tau)+\mathbb{E}_{k+1,\mathcal{H}^+}(0,\tau)+\mathbb{E}_{k+1,\mathcal{I}^+}(0,\tau)\Big)\\
&+\sum_{j=0}^{k+1}\mathbb{E}_j(0)+\sum_{j=0}^{k}\mathbb{E}_j^{\partial_{r^*}}(0),
\end{aligned}
\tag{4.8}
$$

for small $\epsilon > 0$. Here, the ∂_r^* superscript represents ∂_r^*-commuted norms. For the first term in Lemma 4.10, we need extra work: repeating the steps of the previous section for the wave equation (2.12) will cause it to produce an error of the form

$$
\begin{aligned}
&w'\,\mathrm{Re}[\overline{T\psi_{k+1}}X\psi_k']\\
&\quad=\Big(w'\,\mathrm{Re}[\overline{T\psi_{k+1}}X\psi_k]\Big)'-T\,\mathrm{Re}[(w'\psi_{k+1})'\overline{X\psi_k}]+\mathrm{Re}[(w'\psi_{k+1})'\overline{XT\psi_k}]\\
&\quad=\Big(w'\,\mathrm{Re}[T\psi_{k+1}\overline{X\psi_k}]\Big)'-T\,\mathrm{Re}[(w'\psi_{k+1})'\overline{X\psi_k}]\\
&\qquad+w\,\mathrm{Re}[(w'\phi_{k+1})'\overline{X\phi_{k+1}}]+\operatorname{sgn}s\,\mathrm{Re}[(w'\phi_{k+1})'\overline{X\phi_k'}],
\end{aligned}
$$

where X denotes either zL or $z\underline{L}$ for some choice of $z(r)$ function (specified in the the relevant lemmas). After the integration by parts procedure carried out in this identity, each term can be treated using Cauchy–Schwarz and controlled by (4.8). Notice that the same reasoning holds for (2.19). Thus, making ϵ sufficiently small yields control over all appropriately r-weighted second order derivatives of $\tilde{\psi}_k$ where one of the derivatives is ∂_{r^*}.

To conclude, we need only show that we can estimate second order terms involving appropriately r-weighted angular and time derivatives:

$$
T^2\tilde{\psi}_k,\quad T\mathring{\nabla}\tilde{\psi}_k,\quad \mathring{\Delta}\tilde{\psi}_k.
$$

For the first two, we use the fact that $T\psi_k = w\psi_{k+1} + \operatorname{sgn}s\psi_k'$ to reduce it to terms that we already understand. For the last one, a similar reduction can be attained from the constraint equation (2.15).

In the above sketch, we have overlooked details regarding the precise computation of r-weights involved. In view of the level of detail of the preceding sections, the reader should be able to fill in the missing details to obtain Theorem 4.1 at last. □

4.3 Decay of the energy and the solution

In this section, we prove Corollary 4.2. The proof strategy is very similar to the that of Corollary 3.2: it is based on the r^p method introduced in [DR10].

However, because we Theorem 4.1 does not yield estimates for $p = 2$, we require an additional interpolation lemma as in [SRTdC23, Theorem 9.3].

Proof of Corollary 4.2. To lighten the notation, let us denote simply $\mathbb{E}_{p,q}(\tau)$ the sum $\sum_{k=0}^{|s|} \mathbb{E}^{|s|-k}_{(k),p,q}$ for any $q \in [0,1]$. Repeating Step 1 from the proof of Corollary 3.2 with p replaced by $p-\eta$ for $\eta \in (0,1)$, we have

$$\mathbb{E}_{-\eta,1}(\tau_n) \leq \frac{B}{\tau_n}\left(\mathbb{E}_{2-\eta,1}(\tau_0) + \mathbb{E}^T_{2-\eta,1}(\tau_0) + \mathbb{E}^{TT}_{0,0}(\tau_0)\right),$$

on a dyadic sequence $\{\tau_n\}_{n=0}^{\infty}$ satisfying $\tau_n \uparrow \infty$.

We have not shown uniform boundedness of the $\mathbb{E}_{-\eta,1}$ energy. However, with the interpolation inequality

$$\begin{aligned}\mathbb{E}_{0,1}(\tau_n) &\leq \left(\mathbb{E}_{-\eta,1}(\tau_n)\right)^{1-\eta/2}\left(\mathbb{E}_{2-\eta,1}(\tau_n)\right)^{\eta/2}\\ &\leq \frac{B}{\tau_n^{2-\eta}}\left(\mathbb{E}_{2-\eta,1}(\tau_0) + \mathbb{E}^T_{2-\eta,1}(\tau_0) + \mathbb{E}^{TT}_{0,0}(\tau_0)\right)^{1-\eta/2}\left(\mathbb{E}_{2-\eta,1}(\tau_0)\right)^{\eta/2},\end{aligned}$$

we can now easily conclude in the same fashion as Step 2 of Corollary 3.2. □

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