

TROPICALIZING MODULI SPACES AND APPLICATIONS

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Resumo:

Descrevemos relações intrínsecas entre o espaço de moduli de curvas projetivas, a sua compactificação através de curvas de Deligne-Mumford estáveis, e o espaço de moduli de curvas tropicais. O ponto chave consiste na identificação de uma categoria de objetos combinatoriais que governa a estratificação por tipo topológico em ambos os espaços de moduli. Descrevemos também algumas aplicações do uso destes instrumentos ao estudo da geometria do espaço de moduli de curvas e aplicações semelhantes a outros espaços de moduli.

Abstract

We describe intrinsic relations between the moduli space of smooth projective curves, its compactification via Deligne-Mumford stable curves, and the moduli space of tropical curves. The key point lies in describing a category of combinatorial objects governing the topological type stratification of both moduli spaces. We also illustrate remarkable applications of this interplay on understanding the geometry of the original space and explain similar applications to other moduli spaces.

palavras-chave: espaço de moduli, curvas suaves, curvas estáveis, tropicalização, estratificação topológica, curvas tropicais, cohomologia racional.

keywords: moduli spaces, smooth curves, stable curves, tropicalization, topological stratification, tropical curves, rational cohomology.

1 Introduction

Moduli spaces are parameter spaces for algebro-geometric objects with certain fixed geometric invariants. In algebraic geometry, it is often the case that such spaces are also endowed with the structure of algebraic variety, algebraic scheme or algebraic stack. Therefore, one can use again algebro-geometric tools to study such spaces and, using their modular interpretation, give answers to classification problems for the original varieties and their deformations.

The most studied moduli space in algebraic geometry is certainly the moduli space of curves of given genus. Indeed, it was already considered by Riemann, who introduced the term *Moduli* and considered the set of isomorphism classes of complex structures definable over a compact, connected, topological surface of genus $g \geq 2$ as a complex variety of dimension $3g - 3$, denoted by \mathcal{M}_g . Along the second half of the 20'th century, the moduli space of curves was then extensively studied by algebraic geometers and generalized in many ways. For instance, one can fix a number n of distinct and ordered points on the curves: such a space is known as the moduli space of pointed/marked curves and is denoted by $\mathcal{M}_{g,n}$; or even add the information of a map from the curve to a given space with geometric constraints on the image of the fixed points, the moduli space of stable maps: this space is particularly relevant for applications to enumerative geometry and the development of Gromov-Witten theory.

An important limitation of the moduli space of curves is the fact that it is not compact, so it cannot be used properly to classify important phenomena regarding curves from the point of view of enumerative geometry, which requires techniques usually available for compact spaces. This limitation was completely solved after the breakthrough work by Deligne and Mumford, followed by Knudsen, who constructed a remarkable modular compactification for $\mathcal{M}_{g,n}$: the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ in [DM69] and [Knu83]. This compactification has many important geometrical properties, such as the fact that it is smooth (viewed as an algebraic stack), normal crossings (i.e., the boundary locus looks like the intersection of coordinate hyperplanes locally at any point). More importantly, its strength relies on its modular properties: $\overline{\mathcal{M}}_{g,n}$ itself is a moduli space parametrizing pointed stable curves, i.e., curves admitting nodal singularities and satisfying a stability condition, which ensures that there are only finitely many topological types of such curves.

Even if the moduli space of curves is one of the most studied and fascinating spaces in algebraic geometry, there are still many mysterious aspects of its geometry that keep attracting the attention of mathematicians with different points of view. A remarkable example of success is the structure of the stable cohomology ring of \mathcal{M}_g , which is a polynomial ring in the so-called kappa classes as conjectured by Mumford and later proved by Madsen and Weiss in [MW07]. On the other hand, the rational cohomology ring of $\mathcal{M}_{g,n}$ is completely understood only for low values of g and n , and recently there have been lots of efforts trying to understand more deeply its topology.

It turns out that, by applying Deligne's theory of weights, part of the

cohomology of $\mathcal{M}_{g,n}$ is governed by the topology of any of its normal crossings compactifications. It is then natural to expect that combinatorial methods come into the picture in order to shed some light on the understanding of the geometry of such boundaries.

Tropical geometry is a new area within algebraic geometry whose geometric objects are piecewise linear, and therefore much more combinatorial in nature. It has experienced an extraordinary development in the last 20 years and has shown to be an important tool as well when applied to describe degenerations of algebro-geometric objects. The spirit of tropicalization is that one can zoom in locally at a given point of a variety and extract the geometrical behaviour of its nearby points by representing them within a fan of polyhedral cones. When applied to normal crossings compactifications of moduli spaces, this acquires a modular meaning: one can classify all possible degenerations of a given object by looking at the *tropicalized moduli space*, which is a (generalized) cone complex parametrizing the analogous tropical objects up to tropical modification.

The construction of tropical moduli spaces was inspired by the pioneering work of Mikhalkin-Zharkov in [MZ07], who described analogous tropical notions of abstract curves, divisors, sheaves, abelian varieties, and so on. Based on these notions, one can construct tropical moduli spaces as generalized cone complexes, as in the case of curves, Jacobians and abelian varieties in [BMV11]. These spaces have also been intrinsically realized as tropicalization of both the non-archimedean analytification and the logarithmic version of the original compactified spaces in [ACP15] and in [CCUW20].

At this point, the role of tropical geometry becomes clear as it provides a modular description for the combinatorial data used to compactify the original objects. These modular descriptions are indeed quite helpful and have been recently applied with great success in a number of results concerning geometric properties of the original spaces. One of such achievements is the beautiful computation of the top-weight cohomology of the moduli space of curves in the breakthrough work of Chan, Galatius and Payne [CGP21]. We will try to explain this application in the remaining part of this text.

We start by introducing our main combinatorial tools: graphs and categories of graphs.

We then give a short description of the moduli space of projective curves, its Deligne-Mumford compactification, and its stratification by topological type.

We continue by introducing the moduli space of tropical curves, highlighting its analogy with the moduli space of stable curves, and by briefly

explaining the tropicalization morphism from the Berkovich analytification of the Deligne-Mumford compactification to the moduli space of curves.

Finally, we put all our spaces together and explain Chan, Galatius and Payne's computation of the top-weight rational cohomology of the moduli space of curves by computing the homology of the moduli space of curves. We conclude by briefly mentioning other remarkable applications of tropical geometry to the geometry of classical algebro-geometric varieties.

2 Preliminaries: Graphs and categories of graphs

Given a graph G , we will indicate with $V(G)$ the set of vertices of G and with $E(G)$ the set of edges of G . Given an edge $e \in E(G)$, we will write $e = uv$, for $u, v \in V(G)$ to indicate that u and v are the ends of e . Notice that we will allow for the existence of multiple edges and loop edges as well (in that case we write $e = uu$). We will consider our graphs to be connected, unless otherwise stated: in that case, we will denote with $c(G)$ the number of connected components of G .

Our graphs will often be weighted, i.e., endowed with a weight function $w : V(G) \rightarrow \mathbb{Z}_{\geq 0}$. Given a (weighted) graph $G = (G, w)$, the genus of G is set to be

$$g(G) := \sum_{v \in V(G)} w(v) + b_1(G) = g,$$

where $b_1(G) = |E(G)| - |V(G)| + c(G)$ is the first Betti number of G .

Definition 2.1. Let g, n be non-negative integers such that $2g - 2 + n > 0$.

- A (g, n) -graph is a (weighted) graph G of genus g and n legs, i.e., G is a genus g graph endowed with a leg function $\text{leg} : [n] := \{1, \dots, n\} \rightarrow V(G)$.
- The (g, n) -graph G is said to be *stable* if $\forall v \in V(G)$ with $w(v) = 0$, we have that

$$\text{val}(v) + |\text{leg}^{-1}(v)| \geq 3,$$

where $\text{val}(v)$ denotes the valence (or degree) of the vertex v .

A *morphism* of weighted graphs $G = (G, w)$ and $G' = (G', w')$ is a pair $\pi = (\pi_V, \pi_E)$, where $\pi_V : V(G) \rightarrow V(G')$ is surjective and $\pi_E : E(G) \rightarrow E(G') \cup V(G')$ is surjective onto $E(G')$ such that:

- π_E is compatible with the incidence function, i.e., if $e = uv \in E(G)$ and $\pi_E(e) = e' = u'v' \in E(G')$, then we must have that $\pi_V(u) = u'$ and $\pi_V(v) = v'$;
- π respects the weight function, i.e., for all $v' \in V(G')$, $\pi^{-1}(v') \subset G$ is a connected subgraph of G with $g(\pi^{-1}(v')) = w'(v')$;
- π induces an inclusion $i_\pi : E(G') \hookrightarrow E(G)$ such that $\pi_E \circ i_\pi = \text{id}_{E(G')}$.

If both G and G' are endowed with leg functions $\text{leg} : [n] \rightarrow V(G)$ and $\text{leg}' : [n] \rightarrow V(G')$, we further ask that π_V be compatible with induced markings, i.e., $\forall i \in [n]$, $\text{leg}'(i) = \pi_V(\text{leg}(i))$.

The morphism π is said to be an *isomorphism* if i_π is bijective (that is, edges of G are mapped bijectively to the edges of G'). Indeed, since we are assuming that our graphs are connected, this implies that also π_V is bijective. Given $S \subset E(G)$ inducing a connected subgraph $G[S]$ of G , the contraction of S is a particular morphism $\pi_S : G \rightarrow G/S$ which maps all the vertices and edges of S to a vertex $v_S \in G/S$ and is an isomorphism elsewhere. One can check that morphisms of weighted graphs are obtained by iterating isomorphisms and contractions.

Remark 2.2. It follows easily from the definitions that morphisms of weighted graphs preserve the genus, the markings and also the stability condition, i.e., the image of a stable graph via a morphism of graphs is still stable.

In the light of Remark 2.2, we can consider the set of (stable) (g, n) -graphs as a category, as follows.

Definition 2.3. Let g, n be non-negative integers as above. The category of stable graphs of type (g, n) is the category $\mathcal{G}_{g,n}$ such that:

- objects are stable weighted graphs of genus g and n markings;
- morphisms are morphisms of weighted stable graphs.

Remark 2.4. The stability condition implies that, for fixed g and n , there are only finitely many isomorphism classes of (g, n) -stable graphs. Moreover, it is an easy consequence of the Handshaking Lemma, that the maximal number of edges and vertices of a stable graph of genus g with n legs is, respectively, $3g - 3 + n$ and $2g - 2 + n$.

The reader might suspect that the fact that the maximum number of edges of a stable graph of given genus matches the dimension of $M_{g,n}$ is not a coincidence.

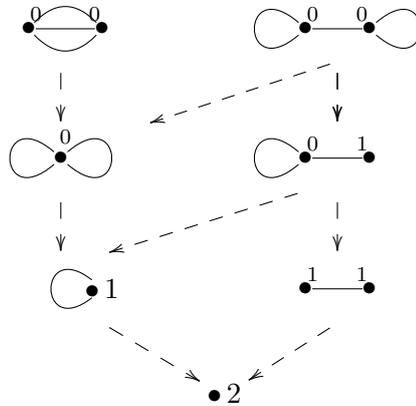


Figura 1: The category of stable graphs of genus 2 and no markings, $\mathcal{G}_{2,0}$. Dotted arrows represent morphisms in the category obtained by (weighted) contraction of edges.

3 Moduli spaces in algebraic geometry

Let \mathcal{M}_g denote the space of isomorphism classes of smooth projective curves of genus g (remember that these correspond to compact Riemann surfaces of genus g if the curves are defined over the complex numbers). Recall that for $g = 0$, since all smooth rational curves are isomorphic to the projective line, $\mathcal{M}_0 \cong \{pt\}$; while for $g = 1$, one can see that isomorphism classes of elliptic curves are classified by an invariant, called the j -invariant, which corresponds to a point in the affine line \mathbb{A}^1 , so $\mathcal{M}_1 \cong \mathbb{A}^1$. For $g \geq 2$, in the seminal paper [DM69], Deligne and Mumford showed the existence of \mathcal{M}_g over \mathbb{Z} and proved that it is an irreducible quasiprojective variety of dimension $3g - 3$.¹

More generally, one can consider the moduli space $\mathcal{M}_{g,n}$ of smooth curves of genus g with n distinct markings. Then, if (g, n) is such that $2g - 2 + n > 0$, Knudsen in [Knu83] built on the work of Deligne-Mumford to show that $\mathcal{M}_{g,n}$ is a quasiprojective variety of dimension $3g - 3 + n$.

Since then, the moduli space of smooth curves of given genus and distinct n markings has been deeply studied. However, it is easy to see that $\mathcal{M}_{g,n}$ is not compact, as smooth curves degenerate to singular curves, and since distinct points may collide. In the above-cited papers, the authors

¹Deligne-Mumford actually constructed \mathcal{M}_g as an algebraic stack in [DM69]

constructed modular compactifications for the spaces $\mathcal{M}_{g,n}$ by means of (Deligne-Mumford) stable curves.

Definition 3.1. A stable (marked) curve is a nodal curve X , possibly with several irreducible components (and n distinct smooth points), satisfying the following stability condition: for each irreducible component C of X isomorphic to a smooth rational curve, C contains at least 3 special points of X , i.e., the number of points where C intersects the rest of X plus the number of markings on C must be at least 3.

Remark 3.2. It is easy to see that the above stability condition is equivalent to the condition that the automorphism group of X is finite. Notice also that there exist stable curves of genus g with n markings if and only if $2g - 2 + n > 0$.

Theorem 3.3. [[DM69], [Knu83]] Let g and n be non-negative integers such that $2g - 2 + n > 0$. Then there are irreducible projective varieties $\overline{\mathcal{M}}_{g,n}$ parametrizing isomorphism classes of stable curves of genus g and n markings. The compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is toroidal² and smooth if seen as an algebraic stack, and the boundary is a normal crossings divisor.

The topological type of a nodal curve X is encoded by its dual graph, defined as follows.

Definition 3.4. Let X be a nodal curve with irreducible components $C_i, i = 1, \dots, \gamma$. The dual (weighted) graph of X is the graph G_X whose vertices v_i of G_X correspond to the irreducible components C_i of X , with weight $w(v_i)$ equal to the geometric genus of C_i , and such that to each node n meeting components C_i and C_j we associate an edge e_n with ends in v_i and v_j (in particular, internal nodes correspond to loops of G_X).

Figure 1 above contains all dual graphs of stable curves of genus 2 and no markings.

Remark 3.5. It is easy to see that the arithmetic genus of a nodal curve X coincides with the genus of its dual graph G_X . Moreover, X is stable if and only if G_X is stable.

²The theory of toroidal embedding was developed in [KKMSD73] and [AMRT75] as a tool to compactify locally symmetric domains. An embedding is said to be *toroidal* if locally in the étale topology it is isomorphic to the embedding of a torus in a toric variety.

3.1 Stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type

Let $G \in \mathcal{G}_{g,n}$ be a stable graph and set

$$M_G := \{X \in \overline{\mathcal{M}}_{g,n} : G_X \cong G\},$$

i.e., M_G consists of the locus of curves in $\overline{\mathcal{M}}_{g,n}$ with fixed topological type.

It is a quite important and deep fact about the geometry of $\overline{\mathcal{M}}_{g,n}$ that the category $\mathcal{G}_{g,n}$ governs the stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type, i.e. each stratum M_G is irreducible and locally closed of codimension $|E(G)|$, and its closure is made of smaller dimensional strata according to morphisms in $\mathcal{G}_{g,n}$, i.e.,

$$\overline{M_G} = \coprod_{G' \rightarrow G \in \text{Mor}(\mathcal{G}_{g,n})} M_{G'}. \quad (1)$$

So, not only do objects in the category of stable graphs $\mathcal{G}_{g,n}$ parametrize the strata of the decomposition of $\overline{\mathcal{M}}_{g,n}$ by topological type, but the morphisms encode the topology of strata: in particular, a weighted contraction $G' \rightarrow G$ indicates an algebraic degeneration from curves with dual graph G to curves with dual graph G' , while an automorphism of a graph $G \in \mathcal{G}_{g,n}$ indicates that the stratum M_G has a self-gluing.

This decomposition can be understood in terms of local affine coordinates of the variety $\overline{\mathcal{M}}_{g,n}$, showing that the boundary of the compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is a normal crossings divisor³.

4 Moduli Space of tropical curves

4.1 Tropical curves

Definition 4.1. A (g, n) tropical curve (or tropical curve of genus g with n legs, or markings), is a pair $\Gamma = (G, l)$ where G is a (g, n) stable (connected) graph, called the *combinatorial type* of Γ , and $l : E(G) \rightarrow \mathbb{R}_{>0}$ is a length function on the edge set of G .

Notice that a tropical curve is endowed with the structure of a metric space obtained by gluing intervals in Euclidean space of length $l(e)$, for all edges $e \in E(G)$, along their ends, according to the incidence rule associated with G . We say that two tropical curves are isomorphic if their combinatorial types are isomorphic via an isomorphism of graphs which induces an isometry of the associated metric spaces.

³A normal crossings divisor is a divisor that can be expressed locally as the union of coordinate hyperplanes.

The moduli space of tropical curves $M_{g,n}^{\text{trop}}$ was constructed in [BMV11] for $n = 0$ and for any n in [Cap12] following the work of Mikhalkin-Zharkov in [MZ07]. Indeed, our space is a parameter space for abstract tropical curves, i.e., tropical curves up to tropical equivalence, as discussed by Mikhalkin-Zharkov in loc.cit., as opposed to a parameter space of tropical curves embedded in a certain space.

The starting point is to associate to any stable graph $G \in \mathcal{G}_{g,n}$, a rational polyhedral cone

$$\mathcal{C}_G := \mathbb{R}_{>0}^{|E(G)|}$$

parametrizing length functions on tropical curves Γ with topological type G . Then, faces of the closed cone $\overline{\mathcal{C}}_G$ should parametrize degenerations of tropical curves in \mathcal{C}_G obtained from sending the lengths of some edges to zero. Formally, such identifications are associated to morphisms $\pi : G \rightarrow G' \in \mathcal{G}_{g,n}$, whose induced inclusions $i_\pi : E(G') \hookrightarrow E(G)$ naturally yield face morphisms

$$i_\pi : \overline{\mathcal{C}}_{G'} \hookrightarrow \overline{\mathcal{C}}_G, \quad (2)$$

which we will denote with the same symbol, by abuse of notation.

Notice that in the case when $\pi : G \rightarrow G$ is an isomorphism, the associated face morphism $i_\pi : \overline{\mathcal{C}}_G \hookrightarrow \overline{\mathcal{C}}_G$ is an automorphism of the cone $\overline{\mathcal{C}}_G$. Indeed, this corresponds to the fact that points in the moduli space of tropical curves with combinatorial type G should be in bijective correspondence with $\mathcal{C}_G/\text{Aut}(G)$.

In order to construct the moduli space $M_{g,n}^{\text{trop}}$ of tropical curves of genus g with n markings, one considers the union of all cones $\overline{\mathcal{C}}_G$, for $G \in \mathcal{G}_{g,n}$, glued together along face morphisms associated to graph morphisms $G \rightarrow G'$ in $\mathcal{G}_{g,n}$. We are therefore ready to define $M_{g,n}^{\text{trop}}$ in the category of generalized cone complexes.

Definition 4.2. Let g, n be non-negative integers such that $2g - 2 + n > 0$. The moduli space of tropical curves of genus g and n markings is the generalized cone complex

$$M_{g,n}^{\text{trop}} := \varprojlim_{G \in \mathcal{G}_{g,n}} \overline{\mathcal{C}}_G$$

By its very construction, the moduli space of tropical curves is a complex of polyhedral cones of real dimension $3g - 3 + n$. Since some of the cones may have some interior points identified via self-gluing morphisms, such an object is usually called a generalized cone complex.

For the sake of applications to the topology of $\mathcal{M}_{g,n}$, it is actually more convenient to consider the link of $M_{g,n}^{\text{trop}}$, i.e., the locus of tropical curves with volume (or total length) equal to one, which we will denote by $L(M_{g,n}^{\text{trop}})$. Indeed, one can think of $L(M_{g,n}^{\text{trop}})$ as being obtained by intersecting $M_{g,n}^{\text{trop}}$ with a sphere. Indeed, while $M_{g,n}^{\text{trop}}$, being made of cones, is easily seen to be contractible, its link $L(M_{g,n}^{\text{trop}})$ is not, and its topology is much more interesting.

The reader may also notice that $L(M_{g,n}^{\text{trop}})$ is closely related to Outer Space: the classifying space for the free group on g letters constructed by Culler and Vogtmann in [CV86].

4.2 Tropicalization morphisms

The moduli space of Deligne Mumford stable curves and the moduli space of tropical curves are closely related: in fact, the category of stable graphs $\mathcal{G}_{g,n}$ governs both the toroidal stratification of the boundary $\partial(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n})$ and the topology of $M_{g,n}^{\text{trop}}$ (see (1), (2) and Definition 4.2). Indeed, in both situations, objects in $\mathcal{G}_{g,n}$ parametrize topological types of stable curves and of tropical curves, while morphisms encode the topological type of strata lying in the closure of curves of a given topological type. However, the identification is order reversing: a morphism $G \rightarrow G'$ in $\mathcal{G}_{g,n}$ indicates that the locus M_G of curves with dual graph G lies in the closure of the locus of curves with dual graph G' , and, on the other direction, that the locus of tropical curves with combinatorial type G' can be seen as a face of the cone parametrizing tropical curves of combinatorial type G (see [Cap13] for more details).

The relation between the Deligne-Mumford compactification and the moduli space of tropical curves turns out to be much deeper, as highlighted in the breakthrough work of Abramovich, Caporaso and Payne [ACP15]. Indeed, in loc. cit., the authors study the Berkovich analytification associated with the toroidal compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$, together with its intrinsic topological retraction to a generalized cone complex.⁴ The authors then show that this generalized cone complex is isomorphic to the moduli space of tropical curves $M_{g,n}^{\text{trop}}$.

As a consequence of this relation, one can also obtain the following identification:

$$\Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}) \cong L(M_{g,n}^{\text{trop}}), \quad (3)$$

⁴A tropicalization morphism for the moduli space of curves has also been constructed in the category of logarithmic geometry in [CCUW20].

where $\Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n})$ is the dual complex associated with the normal crossings compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$. Recall that the dual complex of a normal crossings compactification $M \subset \overline{M}$ is a cell complex encoding the irreducible components of the boundary divisor and the way they intersect. More precisely, if $D = \overline{M} \setminus M$ is a normal crossings divisor, the boundary complex $\Delta(D)$ associated with the embedding of D in M is a Δ -complex whose geometric realization has a vertex for each irreducible component D_i of D , an edge for each irreducible component of the intersection of two D_i 's, and so on. For instance, the dual complex of the union of three lines intersecting transversally in the projective plane is a triangle.

5 Applications to the topology of the moduli space of curves

As mentioned in the introduction, many aspects of the topology of the moduli space of curves remain unknown, despite the enormous efforts that algebraic geometers and mathematicians, in general, have been making to study this fascinating space. For instance, as mentioned in the Introduction, the rational cohomology ring of $\mathcal{M}_{g,n}$, $H^*(\mathcal{M}_{g,n}, \mathbb{Q})$, is completely known only for very small values of g and n .

If we see the moduli space of stable curves in the category of algebraic stacks, it turns out to be smooth. One can therefore apply Deligne's theory of weights to endow $H^*(\mathcal{M}_{g,n}, \mathbb{Q})$ with a mixed Hodge structure whose graded pieces $\mathrm{Gr}_j^W H^k(\mathcal{M}_{g,n}, \mathbb{Q})$ range between k and $\min\{2k, 2d\}$, where $d = \dim(\mathcal{M}_{g,n}) = 3g - 3 + n$. Since the graded pieces vanish for $j > 2d$, we call $\mathrm{Gr}_{2d}^W H^k(\mathcal{M}_{g,n}, \mathbb{Q})$ the top-weight rational cohomology of $\mathcal{M}_{g,n}$.

Moreover, as we discussed above, $\mathcal{M}_{g,n}$ is endowed with a normal crossings compactification: the moduli space of stable Deligne-Mumford curves $\overline{\mathcal{M}}_{g,n}$. Therefore, again by Deligne's theory of weights, the following identification holds:

$$\mathrm{Gr}_{2d}^W H^{2d-k}(\mathcal{M}_{g,n}, \mathbb{Q}) \cong H_{k-1}(\Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}), \mathbb{Q}), \quad (4)$$

where $\Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n})$ is the boundary complex of the normal crossings compactification introduced in (3).

Therefore, the identification in (4) allows us to compute part of the cohomology of $\mathcal{M}_{g,n}$ via the homology of its boundary complex, which is an object of combinatorial nature, and therefore expected to be simpler. However, as it is often the case, the combinatorial problem turned out to

be quite difficult itself, so it remained unknown until quite recently, via the insight introduced by the identification of the boundary complex with the link of the moduli space of tropical curves (3). Indeed, via this identification, the combinatorial object encoding the geometry of the boundary divisor was given a modular interpretation via tropical curves, which gave a whole new insight into the problem.

The remarkable contribution of Chan, Galatius and Payne in [CGP21] consisted in identifying a graph complex, the so-called Kontsevich commutative graph complex, which computes the cohomology of $L(M_g^{\text{trop}})$. Then, taking advantage of (3) and (4) and of previous results by Willwacher concerning the asymptotic behaviour of such complex, the authors were able to prove the following:

$$\dim H^{4g-6}(M_g, \mathbb{Q}) > \beta^g,$$

where β is a certain constant bigger than 1.

Notice that this result is contrary to expectations. In fact, the virtual cohomological dimension of \mathcal{M}_g is equal to $4g-5$ and the cohomology groups $H^{4g-4-k}(\mathcal{M}_g, \mathbb{Q})$ were expected to be non-zero for only finitely many values of g by previous conjectures by Kontsevich and Church.

5.1 Further achievements

The remarkable results obtained by Chan, Galatius and Payne motivated many similar applications. Indeed, the fact that tropical moduli spaces can give a (at least partial) modular interpretation to the combinatorial data used to compactify algebro-geometric moduli spaces has been proved to hold in other remarkable situations, as the moduli space of spin curves in [CMP20] the moduli space of admissible covers in [CMR16], Universal Jacobians in [AP20] and [MMUV22], etc.

Even if an intrinsic tropicalization result is not known in the case of the moduli space of abelian varieties of given dimension, A_g , the machinery of Chan, Galatius and Payne was nevertheless used to compute the top weight rational cohomology of A_g in [BBCMMW24] for small values of g . Similarly, it is also being applied to a number of different moduli spaces, and the community expects that several interesting results will be proved by applying closely related strategies in the near future.

There are other remarkable applications of tropical methods in approaching problems in classical algebraic geometry, as for instance the use of tropical linear series in proving results concerned with linear series in algebraic curves, by applying Baker's specialization lemma. This line of research

has been extremely successful in the last few years, starting with the tropical proof of the Brill-Noether theorem in [CDPR12] and heading to most recent achievements as the notorious computation of the Kodaira dimension of M_{22} and M_{23} in [FJP25]. We invite the reader to look at [BJ16] for a very nice survey on such applications.

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