A survey on the Virasoro constraints in moduli spaces of sheaves and quiver representations

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Resumo:

As restrições de Virasoro para os invariantes de Gromov-Witten são um tópico que tem estado ligado desde o início ao desenvolvimento da teoria de Gromov-Witten. Neste artigo vamos sintetizar alguns dos desenvolvimentos recentes no estudo das restrições de Virasoro no mundo paralelo das teorias de contagem de feixes e representações de quivers. Damos uma visão histórica do tópico e explicaremos os resultados precisos no contexto de espaços moduli de representações de quivers. Vamos ainda discutir a vertex algebra que controla fenómenos de wall-crossing, que é a principal ferramenta na prova dos resultados existentes.

Abstract

Virasoro constraints for Gromov-Witten invariants is a topic that has been tied together with the development of Gromov-Witten theory. In this paper we survey the recent developments in analogous Virasoro constraints in the parallel world of sheaf and quiver representations counting theories. We give a historical overview of the subject and explain the precise statements in the setting of moduli spaces of representations of a quiver. We discuss the wall-crossing vertex algebra, which is the main tool in the existing proofs.

palavras-chave: Geometria enumerativa, espaços moduli, restrições de Virasoro.

keywords: Enumerative geometry, moduli spaces, Virasoro constraints.

1 Introduction

1.1 Enumerative geometry and moduli spaces

Enumerative geometry is a very classical subject and it has been greatly influential in the development of Algebraic Geometry since the 19th century. In its classical form, the goal is to count the number of geometric objects with given properties. One beautiful theorem from the early days of enumerative geometry is the following:

Theorem 1.1 (Cayley-Salmon)**.** Let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface. Then X contains exactly 27 lines.

In the early 90s, the field was revolutionized by the introduction of ideas coming from string theory; physicists Candelas-Ossa-Green-Parks [CdlOGP] were able to predict the answer to a classical problem (counting rational curves on a quintic 3-fold) using the idea of mirror symmetry in string theory. Their prediction was then proved mathematically by Givental [Giv1]. Witten's conjecture [Wit], which we will discuss in Section 2, was another striking mathematical conjecture inspired by physical ideas, and it is the beginning of the story of Virasoro constraints in enumerative geometry.

Along with the new physical input, mathematicians developed powerful tools to define and study new enumerative invariants. In the modern approach to enumerative geometry, moduli spaces play a crucial role. A moduli space M is simply a space that parametrizes some sort of geometric objects. Enumerative invariants are typically numbers that one can extract from a moduli space. This may be achieved using intersection theory, by integrating naturally defined cohomology[1] classes $D \in H^\bullet(M)$; it is often the case that the moduli space is not smooth but we can still integrate using a virtual fundamental class $[M]^{\mathrm{vir}} \in H_\bullet(M)$, constructed by Behrend–Fantechi [BF], and define numbers

$$\int_{[M]^{\mathrm{vir}}} D \coloneqq \deg\left(D \cap [M]^{\mathrm{vir}}\right) \in \mathbb{Q}\,.$$

The Virasoro constraints are explicit and universal relations among all these numbers when we vary D.

Example 1.2. Let $M = \mathsf{Gr}(\mathbb{C}^4, 2)$ be the Grassmannian, which parametrizes 2-dimensional subspaces of $\mathbb{C}^4$ or, equivalently, lines on $\mathbb{P}^3$. The Grassmannian has a rank 2 tautological bundle $\mathcal{F} \subseteq \mathbb{C}^4 \otimes \mathcal{O}_M$ whose fiber over a point is identified with the corresponding subspace of $\mathbb{C}^4$. Given a cubic surface $X \subseteq \mathbb{P}^3$, the loci of points in M corresponding to lines contained in X is the vanishing loci of a section on the rank 4 bundle $\mathrm{Sym}^3(\mathcal{F}^\vee)$. Hence, for a generic X, we expect that the number of lines in X is

$$\int_{\mathsf{Gr}(\mathbb{C}^4,2)} c_4(\mathrm{Sym}^3(\mathcal{F}^\vee)) = 27\,.$$

[1] Throughout this paper, cohomology is always with $\mathbb{Q}$ coefficients, i.e. $H^\bullet(M) = H^\bullet(M, \mathbb{Q})$.

The number can be calculated via Schubert calculus. The special feature of Theorem 1.1 is that it holds for every smooth X, and not only for a generic one.

1.2 Outline of the paper

This paper is a survey of the recent advances in the series of papers [MOOP, Mor, vB, BLM, Boj, LM] in conjecturing and proving Virasoro constraints for moduli spaces of sheaves and representations of quivers. In Section 2 we will discuss the origin of Virasoro constraints in enumerative geometry, which is Witten's conjecture and its generalization to Gromov-Witten theory. In Section 3 we start discussing the more recent developments in sheaf theory, the relation with Gromov-Witten invariants via the Gromov-Witten/Donaldson-Thomas correspondence, and we list the families of moduli spaces for which the Virasoro constraints have been proven.

In the last two sections we specialize the discussion to the case of moduli spaces of representations of quivers (without relations). These are a simpler analogue of moduli spaces of sheaves, and most of the features of Virasoro constraints can be explained in this setting. In Section 4 we give a precise formulation of the constraints proven in [Boj, LM].

Finally, Section 5 is about wall-crossing and Joyce's vertex algebra [Joy2, Joy1, GJT]. It was discovered in [BLM] that Joyce's vertex algebra is closely related to the Virasoro constraints. The wall-crossing formulas proven by Joyce are the main tool in proving most of the known cases of Virasoro constraints in sheaf/quiver theories. We explain the idea of the proof in the quiver setting.

2 Witten's conjecture

In 1990, Witten [Wit] made a striking prediction about integration on the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. His idea came from two dimensional quantum gravity, which roughly can be thought as a theory of integration over the (infinitely dimensional) space of Riemannian metrics on a surface. He argued that this physical theory could be modeled mathematically in two ways. On one hand, it could be modeled by approaching the space of Riemannian metrics by a space of triangulations (a triangulation gives a metric which is flat in the interior of the triangles and singular along the edges); this idea establishes a connection to matrix models. These matrix models were known at the time to produce a solution to the Korteweg–de Vries (KdV) hierarchy or, equivalently, to the Virasoro constraints.

On the other hand, supersymmetry indicates that the integral over the space of all metrics should localize to an integral over the space of conformal metrics, which is finite dimensional. Mathematically, this can be made precise by considering integration over the Deligne-Mumford moduli space of stable curves

$$\overline{\mathcal{M}}_{g,n} = \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \begin{array}{l} C \text{ nodal curve of genus } g, \\ p_1, \ldots, p_n \in C^{\text{smooth}},\ \#\mathrm{Aut}(C, p_1, \ldots, p_n) < \infty \end{array} \right\}.$$

These are smooth and projective Deligne-Mumford stacks (or orbifolds). Over the moduli of stable curves there are line bundles $\mathbb{L}_1, \ldots, \mathbb{L}_n$ defined by

$$(\mathbb{L}_i)_{(C,p_1,\ldots,p_n)} = T_{p_i}^{\vee} C\,.$$

The first Chern class of these line bundles $\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$ are called the psi classes. We can integrate them to define numbers:

$$\langle \tau_{k_1} \ldots \tau_{k_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \ldots \psi_n^{k_n} \in \mathbb{Q}\,. \tag{1}$$

These are the Gromov-Witten invariants of a point. Now, Witten's prediction was that if one organizes these numbers correctly we should also get a solution to the KdV hierarchy, or to the Virasoro constraints! Define the generating function

$$F = \sum_{g,n \geq 0} u^{2g-2} \sum_{k_1,\ldots,k_n \geq 0} \frac{t_{k_1} \ldots t_{k_n}}{n!} \langle \tau_{k_1} \ldots \tau_{k_n} \rangle_g$$

and let $Z = \exp(F)$. Both F and Z are generating series in the formal variables u and $t_1, t_2, \ldots$; Z is called the partition function. For $n \geq -1$, let L_n be the differential operator

$$L_n = \frac{1}{4} \sum_{k+l=2n} \frac{\partial^2}{\partial T_k \partial T_l} + \frac{1}{2} \sum_{k \geq 0} (2k+1) T_{2k+1} \frac{\partial}{\partial T_{2k+2n+1}} - \frac{1}{2u^2} \frac{\partial}{\partial T_{2n+3}} + \frac{\delta_{n,-1} T_1^2}{4} + \frac{\delta_{n,0}}{16}$$

where $T_{2k+1} = t_k/(2k+1)!!$.

Remark 2.1. These operators satisfy the Lie bracket relation

$$[L_n, L_m] = (n-m) L_{n+m} \text{ for } n, m \geq -1\,.$$

The Virasoro Lie algebra Vir is the Lie algebra spanned by $\{L_n\}_{n\in\mathbb{Z}}$ and a central element C with Lie bracket given by

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m=0}\frac{n^3-n}{12}C\,.$$

Thus, the operators L_n define a representation of the Lie subalgebra $\mathrm{Vir}_{\geq -1} \subseteq \mathrm{Vir}$ spanned by $\{L_n\}_{n\geq -1}$. It was observed in [Get] that one may extend this to a full representation of the entire Virasoro algebra with central charge 1 (i.e. C acting as the identity). However, the Virasoro operators L_n with $n < -1$ do not impose new constraints.

Witten's conjecture was proven by Kontsevich, and it says the following:

Theorem 2.2 ([Kon])**.** For every $n \geq -1$ one has

$$L_n(Z) = 0\,.$$

2.1 The string equation

The case $n = -1$ of Theorem 2.2 is known as the string equation and it was proved by Witten in [Wit]. By taking the $t_{k_1}\dots t_{k_n}$ coefficient, it is equivalent to the statement that

$$\left\langle \tau_0\tau_{k_1}\dots\tau_{k_n}\right\rangle_g = \sum_{i=1}^{n}\left\langle \tau_{k_1}\dots\tau_{k_i-1}\dots\tau_{k_n}\right\rangle_g \tag{2}$$

for $2g-2+n>0$, together with the exceptional case

$$\left\langle \tau_0\tau_0\tau_0\right\rangle_0 = 1\,.$$

This equation is much easier to prove than the general case of Witten's conjecture, and it follows from an analysis of the geometry of the map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ which forgets one of the marked points and stabilizes the resulting curve.

To make a connection with how we will later write the Virasoro constraints for sheaves and quiver representations, let us write down the string equation in the language of the descendent algebra. Define the descendent algebra $\mathbb{D}^{\mathrm{Wit}}$ to be the polynomial algebra in infinitely many variables

$$\mathbb{D}^{\mathrm{Wit}} = \mathbb{Q}[\tau_0, \tau_1, \tau_2, \dots]\,.$$

For each genus g, we have a linear map $\langle \cdot \rangle_g \colon \mathbb{D}^{\mathrm{Wit}} \to \mathbb{Q}$ defined on monomials $\tau_{k_1} \dots \tau_{k_n}$ by integration over $\overline{M}_{g,n}$, cf. (1). Then the string equation (for $g > 0$, so that we can ignore the exceptional case) can be written as

$$\big\langle (R_{-1} - \tau_0) D \big\rangle_g = 0 \text{ for every } D \in \mathbb{D}^{\mathrm{Wit}},\ g > 0\,.$$

It is possible to formulate Witten's conjecture in the same spirit for every n, see [MOOP, Proposition 5].

2.2 Generalization to Gromov-Witten theory

In 1997, Eguchi-Hori-Xiong [EHX] proposed a generalization of Witten's conjecture to the Gromov-Witten theory of an arbitrary (smooth, projective) target variety X. In Gromov-Witten theory, one considers the moduli spaces of stable maps

$$\overline{\mathcal{M}}_{g,n}(X, \beta)\,.$$

Its points parametrize (marked) curves together with a map $f \colon C \to X$ in curve class β, i.e. with $f_*[C] = \beta$ in $H_2(X; \mathbb{Z})$. Gromov-Witten invariants are obtained by integrating psi classes together with pull-backs of cohomology classes from X via the evaluation maps $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$. The moduli spaces of stable maps are often not smooth, so the integration needs to be done in a virtual sense. The Gromov-Witten invariants are

$$\Big\langle \tau_{k_1}(\gamma_1) \dots \tau_{k_n}(\gamma_n) \Big\rangle_{g,\beta}^{\mathrm{GW}} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \in \mathbb{Q} \tag{3}$$

where $k_i \geq 0$ and $\gamma_i \in H^\bullet(X)$. When X is a point we recover the integrals over the moduli of stable curves that we discussed before. We refer the reader to Chapters 25 and 26 in [HKK$^+$] for a gentle and more detailed introduction to Gromov-Witten invariants.

After one organizes the Gromov-Witten invariants into a generating series Z^X, which is called the partition function, the general Virasoro constraints are the conjecture that for $n \geq -1$

$$L_n^X(Z^X) = 0$$

where L_n^X are some explicit differential operators. The reader can find the precise form of the operators in [Get].

There are two large families of varieties X for which the Gromov-Witten Virasoro conjecture is known: X with semisimple quantum cohomology (this includes any toric variety and Grassmannians, for instance) [Giv2, Tel] and curves [OP]. Apart from those, it is a widely open problem.

3 Virasoro constraints in sheaf and quiver theories: overview

Recently, there has been a proposal of Virasoro constraints in a new setting: moduli spaces of sheaves and moduli spaces of quiver representations. For now, we want to explain how the constraints were first discovered in this setting, give a rough idea of what they look like, and list the cases in which the constraints are proven.

3.1 Gromov-Witten/Donaldson-Thomas correspondence

Maulik, Nekrasov, Okounkov and Pandharipande [MNOP1, MNOP2] conjectured that the Gromov-Witten invariants of a 3-fold X should be related to the Donaldson-Thomas invariants of the same 3-fold; we refer to this as the GW/DT correspondence. Donaldson-Thomas invariants are a different kind of enumerative invariants, defined by integration over moduli spaces of ideal sheaves, instead of stable maps. Given a curve $C \subseteq X$, we denote by I_C its ideal sheaf. For each $m \in \mathbb{Z}$ and $\beta \in H_2(X;\mathbb{Z})$ we have a moduli space

$$I_m(X,\beta) = \{I_C \colon C \subseteq X \text{ a 1-dimensional subscheme}, [C] = \beta, \chi(\mathcal{O}_C) = m\}\,.$$

There is a universal curve $\mathcal{C} \subseteq I_m(X,\beta) \times X$ and a universal ideal sheaf $I_\mathcal{C}$ on $I_m(X,\beta) \times X$. The universal ideal sheaf may be used to construct natural cohomology classes

$$\mathrm{ch}_k(\gamma) = \pi_{I*}(\mathrm{ch}_k(I_\mathcal{C})\pi_X^*\gamma) \in H^\bullet(I_m(X,\beta))$$

where $k \geq 0$ is an integer, $\mathrm{ch}_k(I_\mathcal{C}) \in H^\bullet(I_m(X,\beta) \times X)$ denotes the k-th Chern character of the sheaf $I_\mathcal{C}$, $\gamma \in H^\bullet(X)$ and π_I, π_X are the projections of $I_m(X,\beta) \times X$ onto $I_m(X,\beta)$ and X, respectively. The moduli spaces $I_m(X,\beta)$ have a virtual fundamental class and the Donaldson-Thomas invariants are defined to be

$$\int_{[I_m(X,\beta)]^{\mathrm{vir}}} \mathrm{ch}_{k_1}(\gamma_1)\ldots\mathrm{ch}_{k_n}(\gamma_n) \in \mathbb{Q}\,. \tag{4}$$

The correspondence proposed in [MNOP1, MNOP2] states that there is a universal way to determine all the Gromov-Witten invariants (3) from the Donaldson-Thomas invariants (4), and vice-versa. The conjectured correspondence is quite complicated. In involves taking a fairly strange change of variables $q = e^{-iu}$ to relate two generating series and it involves a complicated transformation to convert the Gromov-Witten descendents $\tau_{k_1}(\gamma_1)\ldots\tau_{k_n}(\gamma_n)$ into Donaldson-Thomas descendents $\mathrm{ch}_{k_1}(\gamma_1)\ldots\mathrm{ch}_{k_n}(\gamma_n)$.

In light of this, there is a very natural question: can the Gromov-Witten Virasoro constraints be moved to the Donaldson-Thomas side? What do they say? Oblomkov, Okounkov and Pandharipande were able to guess a precise conjecture that first appeared[2] in [Pan].

The understanding of the GW/DT correspondence has improved a lot since it was first proposed, and this allowed to partially establish the connection between Virasoro constraints for Gromov-Witten and for Donaldson-Thomas invariants.

Theorem 3.1 ([MOOP, Theorem 5])**.** Suppose that the GW/DT correspondence holds. Then the Gromov-Witten Virasoro constraints in the stationary regime[3] and the Donaldson-Thomas constraints in the stationary regime are equivalent.[4]

A consequence of this are the stationary Virasoro constraints for Donaldson-Thomas invariants of toric 3-folds, since both the GW/DT correspondence and the Virasoro on the Gromov-Witten side are known.

3.2 Universal constraints

It turns out that Virasoro constraints are present (or at least expected to be) in a much more general family of moduli spaces, which includes the moduli of ideal sheaves used to define Donaldson-Thomas invariants. It was noted in [MOOP, Mor] that the Virasoro constraints for Pandharipande-Thomas invariants (a closely related cousin to Donaldson-Thomas invariants) imply similar constraints for Hilbert schemes of points on surfaces. Work of van Bree [vB] conjectured and gave strong numerical evidence that moduli spaces of stable torsion-free sheaves on surfaces should also be constrained.

This lead to the general conjectures appearing in [BLM], which apply to a large family of moduli spaces. It applies to moduli spaces of sheaves (on curves, surfaces, ideal sheaves on 3-folds, etc.) but also to many variations:

[2] The author explains that they made this conjecture roughly 10 years before the paper, so around 2007.

[3] Stationary regime means essentially that we only consider invariants (3) and (4) with $\gamma_i \in H^{\geq 2}(X)$. Away from the stationary regime, the GW/DT correspondence is poorly understood.

[4] The results in [MOOP] are stated for Pandharipande-Thomas invariants rather than Donaldson-Thomas invariants, but in the stationary case these are essentially the same by [OOP, Theorem 22].

sheaves with or without fixed determinant, moduli spaces of pairs, moduli spaces of quiver representations, moduli spaces of sheaves with Oh-Thomas virtual fundamental classes, moduli spaces of Bridgeland stable objects, and moduli spaces of parabolic bundles on curves. We remark that this general setting does not apply to Gromov-Witten theory, in which Virasoro constraints have a fundamentally different flavour.

In the next section we will explain the precise conjecture in the setting of quivers. For now, we shall only describe its general shape. Given a moduli space M of sheaves or quiver representations, there are certain natural cohomology classes we can construct on M; for example in the Donaldson-Thomas case those were $\mathrm{ch}_k(\gamma)$. We write this in terms of a descendent algebra $\mathbb{D}$ which admits a realization map $\mathbb{D} \to H^\bullet(M)$. If M has a virtual fundamental class, we get numerical invariants $\int_{[M]^{\mathrm{vir}}} D \in \mathbb{Q}$ by realizing D and integrating it. The Virasoro constraints are universal and explicit linear relations among such numbers. We always describe them analogously to how we wrote the string equation in Section 2.1. We define some operator (or operators) $\mathsf{L}\colon \mathbb{D} \to \mathbb{D}$ and formulate the constraints as

$$\int_{[M]^{\mathrm{vir}}} \mathsf{L}(D) = 0 \quad \text{for every } D \in \mathbb{D}\,.$$

The operators L will be defined in terms of some canonical representation $\{\mathsf{L}_n\}_{n\geq -1}$ of Vir_{-1} on $\mathbb{D}$. Depending on the context, L might be precisely L_n, but it might also be a small perturbation or a combination of all the L_n operators.

Remark 3.2. The sheaf/quiver Virasoro constraints relate invariants defined by integration on a single moduli space. The same is not true for Gromov-Witten theory: for instance in the string equation (2) the left hand side is defined by integration in $\overline{\mathcal{M}}_{g,n+1}$ and the right hand side by integration in $\overline{\mathcal{M}}_{g,n}$. This is a hint that the sheaf/quiver Virasoro constraints are in some sense simpler.

We list now the cases in which the Virasoro constraints have been proven in the past few years:

1. Donaldson-Thomas and Pandharipande-Thomas invariants on toric 3-fold X in the stationary regime [MOOP].
2. Hilbert scheme of points on a surface S with $h^1(S) = 0$ [Mor].[5]

[5] Joyce announced that he is able to remove the assumption $h^1(S) = 0$ in upcoming work [Joy3].

3. Moduli of stable bundles on a curve [BLM].
4. Moduli of Bradlow pairs on a curve [BLM]. This was conjectured by van Bree in [vB].
5. Moduli of torsion free stable sheaves on a surface S with $h^{0,1}(S) = h^{0,2}(S) = 0$ [BLM].
6. Moduli of Bradlow pairs on a surface S with $h^{0,1}(S) = h^{0,2}(S) = 0$ [BLM]. In particular they hold for the nested Hilbert schemes $S_\beta^{[0,m]}$ on such surfaces.
7. Moduli of 1-dimensional sheaves on a surface S with $h^{0,1}(S) = h^{0,2}(S) = 0$ [BLM] provided the technical condition [BLM, Assumption 5.7] holds. When $S = \mathbb{P}^2, S = \mathbb{P}^1 \times \mathbb{P}^1$ or $S = \mathsf{Bl}_{\mathsf{pt}}(\mathbb{P}^2)$ this is shown unconditionally in [LM].
8. Punctual Quot schemes on a curve or on a surface with $h^{0,2}(S) = 0$ [BLM] holds.
9. Moduli spaces of quiver representations, possibly with relations [Boj, LM].

In Section 5 we will sketch the proof of (9). The ideas explained there are the main tool used in the proof of results (3) – (9).

4 Virasoro constraints for moduli of quiver representations

In this section we define moduli spaces parametrizing representations of a quiver and give a precise formulation of the Virasoro constraints in this setting.

4.1 Moduli spaces of quiver representations

A quiver is a directed graph, which we write as a 4-tuple $Q = (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are the sets of vertices and arrows, respectively, and $s, t\colon Q_1 \to Q_0$ are the functions assigning to an arrow its source and target, respectively.

A representation V of a quiver is an assignement of a vector space V_v to each node $v \in Q_0$ and linear maps $f_e\colon V_{s(e)} \to V_{t(e)}$ to each arrow $e \in Q_1$.

Given a representation V, we say that $(\dim(V_v))_{v\in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$ is the dimension vector of V. A morphism bewteen representations V and W consists in a collection of maps $V_v \to W_v$ for each $v \in Q_0$ that makes all the maps assign to the edges commute. Thus, representations of a quiver Q form an abelian category Rep_Q.

The construction of well-behaved moduli spaces of quiver representations requires the choice of a stability condition. Given a choice of weights $\theta = (\theta_v)_{v\in Q_0} \in \mathbb{R}^{Q_0}$ one defines the slope of a representation by

$$\mu_\theta(V) = \frac{\sum_{v\in Q_0} \theta_v \dim(V_v)}{\sum_{v\in Q_0} \dim(V_v)}\,.$$

Note that this only depends on the dimension vector of V, so we will also write $\mu_\theta(d)$ for $d \in \mathbb{Z}^{Q_0}$ with the obvious meaning.

Definition 4.1. A representation of a quiver V is said to be θ-semistable if

$$\mu_\theta(W) \leq \mu_\theta(V) \text{ for every } 0 \neq W \subsetneq V\,.$$

It is said to be θ-stable if the inequality above is strict for every $W \subsetneq V$.

Given a dimension vector $d \in \mathbb{Z}_{\geq 0}^{Q_0}$ let

$$Z_d^{\theta-\mathrm{st}} \subseteq Z_d^{\theta-\mathrm{ss}} \subseteq \prod_{e\in Q_1} \mathrm{Hom}(\mathbb{C}^{d_{s(e)}}, \mathbb{C}^{d_{t(e)}})$$

be the subsets of θ-stable and θ-semistable representations. Note that some of these representations are isomorphic, so we can construct a moduli space parametrizing (S-equivalence classes of) θ-semistable representations by taking the quotient (in the sense of geometric invariant theory [MFK])

$$M_d^{\theta-\mathrm{ss}} = Z_d^{\theta-\mathrm{ss}} /\!\!/ G_d$$

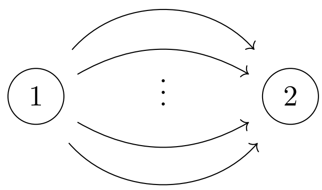
by the natural action of

$$G_d = \prod_{v\in Q_0} \mathrm{Hom}(\mathbb{C}^{d_v}, \mathbb{C}^{d_v})\,.$$

We refer to [Kin] for further details on the construction. Similarly we may define $M_d^{\theta-\mathrm{st}} \subseteq M_d^{\theta-\mathrm{ss}}$; when $M_d^{\theta-\mathrm{ss}} = M_d^{\theta-\mathrm{st}}$ we will sometimes abbreviate to M_d^θ. When the quiver Q is acyclic (i.e. it does not contain oriented cycles), the moduli spaces $M_d^{\theta-\mathrm{ss}}$ are projective. When θ and d are such that there are no strictly semistable objects (i.e. $Z_d^{\theta-\mathrm{st}} = Z_d^{\theta-\mathrm{ss}}$) the moduli space M_d^θ is smooth of dimension

$$1 + \sum_{e\in Q_1} d_{s(e)} d_{t(e)} - \sum_{v\in Q_0} d_v^2\,.$$

Example 4.2. Let K_N be the Kronecker quiver, with two vertices $\{1, 2\}$ and N arrows from 1 to 2:



Let $\theta = (\theta_1, \theta_2)$ with $\theta_1 > \theta_2$. Then the moduli of quiver representations

$$M^{\theta}_{(k,1)}(K_N) \simeq \mathsf{Gr}(\mathbb{C}^N, k)\,,$$

is the Grassmannian parametrizing k dimensional subspaces of $\mathbb{C}^N$.

Example 4.3. Let Q be an acyclic quiver. Then it is possible to choose a stability condition θ which is increasing along the edges, meaning that $\theta_{t(e)} > \theta_{s(e)}$ for any $e \in Q_1$. As shown in [GJT, Proposition 5.6], for such stability conditions, $M_d^{\theta-\mathrm{st}}$ is a point when there is a $v \in Q_0$ such that $d_v = 1$ and $d_w = 0$ for $w \neq v$; otherwise, $M_d^{\theta-\mathrm{st}}$ is the empty set.

4.2 Descendent algebra

Let $M = M_d^{\theta}$ be a moduli space of θ-stable quiver representations on Q and assume that $M_d^{\theta\text{-st}} = M_d^{\theta\text{-ss}}$. We have a universal representation $\mathcal{V}$ on M, which consists of a collection of vector bundles $\mathcal{F}_v$ of rank d_v for each $v \in Q_0$ and maps $\varphi_e\colon \mathcal{V}_{s(e)} \to \mathcal{V}_{t(e)}$ for each $e \in Q_1$. We can produce cohomology classes in $H^\bullet(M)$ by taking Chern classes of the bundles $\mathcal{V}_v$. This motivates the definition of the descendent algebra.

Definition 4.4. Let Q be a quiver. The descendent algebra of Q, denoted by $\mathbb{D}^Q$, is the free commutative $\mathbb{Q}$-algebra generated by symbols

$$\left\{\mathrm{ch}_k(v)\,\middle|\, k \in \mathbb{Z}_{\geq 0},\ v \in Q_0\right\}.$$

The universal representation $\mathcal{V}$ defines a realization homomorphism

$$\xi_{\mathcal{V}}\colon \mathbb{D}^Q \to H^\bullet(M)$$

defined on generators by

$$\mathrm{ch}_k(v) \mapsto \mathrm{ch}_k(\mathcal{V}_v)\,.$$

An important subtlety is that the universal representation $\mathcal{V}$ is not unique. If L is any line bundle on M, then $(\mathcal{V}_v \otimes L)_{v\in Q_0}$ defines a new universal representation. This issue can be addressed in two ways: either we fix a choice of universal representation in some way, or we consider the weight 0 descendent subalgebra on which $\xi_{\mathcal{V}}$ does not depend on the choice of $\mathcal{V}$. This subalgebra is defined concretely as follows:

$$\mathbb{D}^Q_{\mathrm{wt}_0} = \ker\left(\mathsf{R}_{-1}\colon \mathbb{D}^Q \to \mathbb{D}^Q\right) \subseteq \mathbb{D}^Q$$

where R_{-1} is a derivation that we define later in Definition 4.5. This subalgebra has the property that $\xi_{\mathcal{V}}(D)$ does not depend on $\mathcal{V}$ for $D \in \mathbb{D}^Q_{\mathrm{wt}_0}$. Hence we get a well-defined homomorphism

$$\xi\colon \mathbb{D}^Q_{\mathrm{wt}_0} \to H^\bullet(M)\,.$$

4.3 Virasoro constraints

To formulate the constraints we define a representation of Vir_{-1} in $\mathbb{D}^Q$ as follows:

Definition 4.5. Define the Virasoro operators $\{\mathsf{L}_n \,|\, n \geq -1\}$ on the descendent algebra $\mathbb{D}^Q$ as a sum of two operators $\mathsf{L}_n = \mathsf{R}_n + \mathsf{T}_n$. First, R_n is a derivation operator such that

$$\mathsf{R}_n(\mathrm{ch}_k(v)) = k(k+1)\cdots(k+n)\,\mathrm{ch}_{k+n}(v).$$

Second, T_n is a multiplication operator by the element

$$\mathsf{T}_n = \sum_{a+b=n} a!b!\left(\sum_{i\in Q_0}\mathrm{ch}_a(v)\mathrm{ch}_b(v) - \sum_{e\in Q_1}\mathrm{ch}_a(s(e))\mathrm{ch}_b(t(e))\right)\,.$$

We also define the weight 0 Virasoro operator

$$\mathsf{L}_{\mathrm{wt}_0} = \sum_{n\geq -1}\frac{(-1)^n}{(n+1)!}\mathsf{L}_n \circ \mathsf{R}_{-1}^{n+1}\,.$$

It is a nice exercise to check that the commutator of these operators is

$$[\mathsf{L}_n, \mathsf{L}_m] = (m-n)\mathsf{L}_{n+m}\,.$$

This is not quite a Virasoro representation as we defined earlier due to a sign difference, but it is very close; indeed, it means that the dual operators

$\mathsf{L}_n^\vee$ define a representation of Vir_{-1} on $\left(\mathbb{D}^Q\right)^\vee$. It also follows from those commutator relations that $\mathsf{R}_{-1} \circ \mathsf{L}_{\mathrm{wt}_0} = 0$; hence, the image of $\mathsf{L}_{\mathrm{wt}_0}$ is contained in $\mathbb{D}^Q_{\mathrm{wt}_0}$, so the realization $\xi(\mathsf{L}_{\mathrm{wt}_0}(D)) \in H^\bullet(M)$ is well defined. We are ready to formulate the Virasoro constraints.

Theorem 4.6 ([Boj, LM])**.** Let Q be an acyclic quiver, $d \in \mathbb{Z}_{\geq 0}^{Q_0}$ a dimension vector and $\theta \in \mathbb{R}^{Q_0}$ a stability condition such that $M_d^{\theta-\mathrm{ss}} = M_d^{\theta-\mathrm{st}}$. Then

$$\int_{M_d^\theta} \xi\big(\mathsf{L}_{\mathrm{wt}_0}(D)\big) = 0 \qquad \text{for every } D \in \mathbb{D}^Q\,.$$

When there is some vertex v_0 such that $d_{v_0} = 1$, we may give an equivalent formulation by fixing a normalized universal sheaf.

Theorem 4.7. Let Q be an acyclic quiver, $d \in \mathbb{Z}_{\geq 0}^{Q_0}$ a dimension vector and $\theta \in \mathbb{R}^{Q_0}$ a stability condition such that $M_d^{\theta-\mathrm{ss}} = M_d^{\theta-\mathrm{st}}$. Suppose that there is $v_0 \in Q_0$ such that $d_{v_0} = 1$ and let $\mathcal{F}$ be the (unique) universal representation on M_d^θ such that $\mathcal{V}_{v_0}$ is the trivial line bundle.

Then

$$\int_{M_d^\theta} \xi_{\mathcal{V}}\big(\mathsf{L}_n(D)\big) = 0 \qquad \text{for every } D \in \mathbb{D}^Q \text{ and } n \geq 0\,.$$

Example 4.8. We illustrate how the Virasoro constraints look like for $M = \mathsf{Gr}(\mathbb{C}^4, 2)$, see Examples 1.2 and 4.2. Let $\mathcal{V}$ be the universal representation over $M = M^\theta_{(2,1)}(K_4)$, where K_4 is the Kronecker quiver in Example 4.2, normalized by asking that $\mathcal{V}_2$ is the trivial line bundle. Then $\mathcal{F} = \mathcal{V}_1$ is the tautological vector bundle (recall Example 1.2) and the 4 universal maps are obtained by composing $\mathcal{F} \subseteq \mathbb{C}^4 \otimes \mathcal{O}_{\mathsf{Gr}(\mathbb{C}^4,2)}$ with the 4 projections onto the coordinate axis $\mathbb{C}^4 \otimes \mathcal{O}_{\mathsf{Gr}(\mathbb{C}^4,2)} \to \mathcal{O}_{\mathsf{Gr}(\mathbb{C}^4,2)}$.

For convenience denote $p_k = k!\mathrm{ch}_k(1) \in \mathbb{D}^{K_4}$. We will omit the realization morphism $\xi_{\mathcal{V}}$, so p_k also denotes $k!\mathrm{ch}_k(\mathcal{V}_1) = k!\mathrm{ch}_k(\mathcal{F}) \in H^{2k}(\mathsf{Gr}(\mathbb{C}^4, 2))$. We have

$$\begin{aligned}
0 &= \int_{\mathsf{Gr}(\mathbb{C}^4,2)} \mathsf{L}_1(p_1^3) = 3\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^2 p_2 \\
0 &= \int_{\mathsf{Gr}(\mathbb{C}^4,2)} \mathsf{L}_2(p_1^2) = 2\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1 p_3 + \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^4 \\
0 &= \int_{\mathsf{Gr}(\mathbb{C}^4,2)} \mathsf{L}_1(p_1 p_2) = \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_2^2 + 2\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1 p_3 \\
0 &= \int_{\mathsf{Gr}(\mathbb{C}^4,2)} \mathsf{L}_3(p_1) = \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_4 + 2\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^2 p_2\,.
\end{aligned}$$

By further using that $\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^4 = 2$ [EH, Exercise 4.38] we determine all the integrals of descendents using the equations above:

$$\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^4 = 2 = \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_2^2\,, \quad \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1 p_3 = -1\,, \quad \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_4 = 0 = \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^2 p_2\,.$$

It is shown in [LM] that the Virasoro constraints for the Grassmannian recover all the descendent integrals up to a constant, as illustrated above in the case of $\mathsf{Gr}(\mathbb{C}^4, 2)$. The same is not true for other quivers. Indeed, the descendent integrals on M_d^θ will depend heavily on the choice of stability condition θ, but the Virasoro constraints are completely independent of θ!

5 Wall-crossing and the vertex algebra

5.1 Wall-crossing and flips

We have seen that the construction of moduli spaces of representations $M_d^{\theta-\mathrm{ss}}$ depends on a choice of a stability condition $\theta \in \mathbb{R}^{Q_0}$. Wall-crossing is essentially the study of how the moduli spaces and corresponding descendent integrals change when θ changes.

Example 5.1. Consider the Kronecker quiver and Examples 4.2 and 4.3: they say that $M_{(k,1)}^\theta(K_N)$ is the Grassmannian when $\theta_1 > \theta_2$ but empty if $\theta_1 < \theta_2$. There is a drastic change when we cross the wall

$$\{(\theta_1, \theta_2) \colon \theta_1 = \theta_2\} \subseteq \mathbb{R}^{Q_0}$$

in the space of stability conditions. We call the regions $\{\theta \colon \theta_1 > \theta_2\}$ and $\{\theta \colon \theta_1 < \theta_2\}$ chambers. For θ on the wall, μ_θ is constant so every representation is semistable, but only the representations with total dimension 1 are stable.

More generally, the space of stability conditions $\mathbb{R}^{Q_0}$ is divided into chambers by walls of the form

$$W(d_1, d_2) = \{\theta \in \mathbb{R}^{Q_0} \colon \mu_\theta(d_1) = \mu_\theta(d_2)\}\,.$$

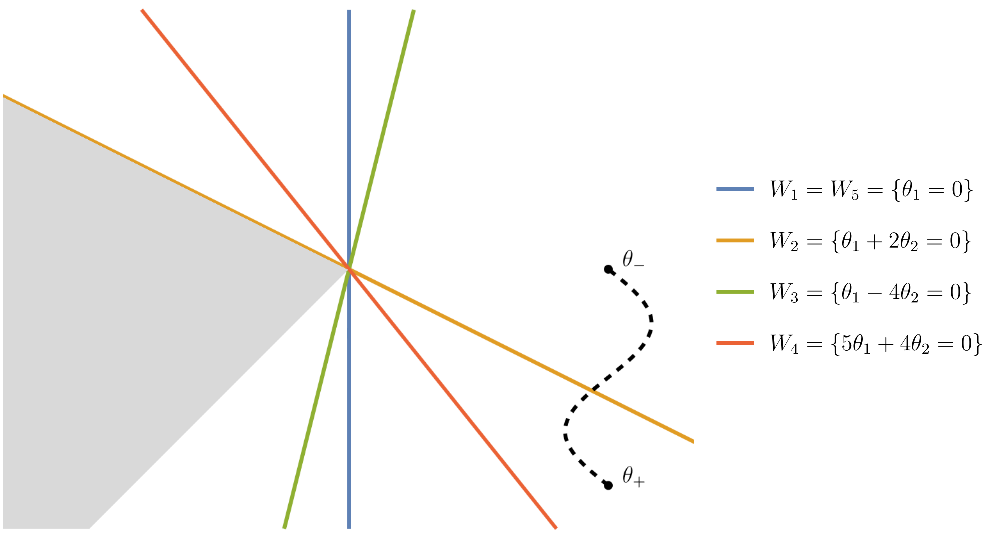
The moduli spaces M_d^θ do not change unless we cross a wall $W(d_1, d_2)$ with $d = d_1 + d_2$. Studying what changes when we cross a wall is a great tool to understand moduli spaces since sometimes we can wall-cross to a much simpler moduli space.

Example 5.2. Consider a quiver with 3 vertices $\{1,2,3\}$ with any number of arrows $1 \mapsto 2$ and $2 \mapsto 3$, and let $d = (2,1,1)$. There are 5 possible walls for the moduli spaces M_d^θ:

$$W_1 = W\big((1,0,0),(1,1,1)\big)\,,\; W_2 = W\big((1,1,0),(1,0,1)\big)\,,\; W_3 = W\big((0,1,0),(2,0,1)\big)$$
$$W_4 = W\big((0,0,1),(2,1,0)\big)\,,\; W_5 = W\big((2,0,0),(0,1,1)\big)$$

In the figure below we represent the wall and chamber structure on the two dimensional subspace $\theta_1+\theta_2+\theta_3 = 0$ of the space of stability conditions $\mathbb{R}^{Q_0}$ (note that the stability of representations is unaffected by adding a constant to each entry of θ, so any stability condition is equivalent to one of these).

The shadowed region is the region of increasing stability conditions $\theta_1 < \theta_2 < \theta_3$, see Example 4.3; the moduli spaces M_d^θ are empty in this region. Note that the wall $W_1 = W_5$ is a double wall. The remaining ones are simple walls in the sense that we explain below, and the dashed path between θ_- and θ_+ is a simple wall-crossing path; as we will explain now, $M_d^{\theta_-}$ and $M_d^{\theta_+}$ are related by a flip.



We describe now a situation in which wall-crossing can be understood very geometrically, which is that of simple wall-crossing. Let $d \in \mathbb{Z}_{\geq 0}^{Q_0}$ and θ_- and θ_+ two stability conditions. We make the following assumption:

(A) θ_- and θ_+ are not on a wall, so there are no strictly $\theta_\pm$-semistable representations.

(B) There is a continuous path of stability conditions θ_t from $\theta_0 = \theta_-$ to $\theta_1 = \theta_+$ which crosses a unique wall $W(d_1, d_2)$ with $d_1 + d_2 = d$. We let θ be the stability condition at the intersection with the wall.

(C) We have

$$\mu_{\theta_-}(d_1) < \mu_{\theta_-}(d_2) \text{ and } \mu_{\theta_+}(d_1) > \mu_{\theta_+}(d_2)\,.$$

(D) The path θ_t does not cross any walls $W(d_3, d_4)$ with $d_3 + d_4 = d_1$ or $d_3 + d_4 = d_2$.

Denote

$$M_- = M_d^{\theta_-}\,,\ M_+ = M_d^{\theta_+}\,,\ M_1 = M_{d_1}^{\theta_-} \text{ and } M_2 = M_{d_1}^{\theta_-}\,.$$

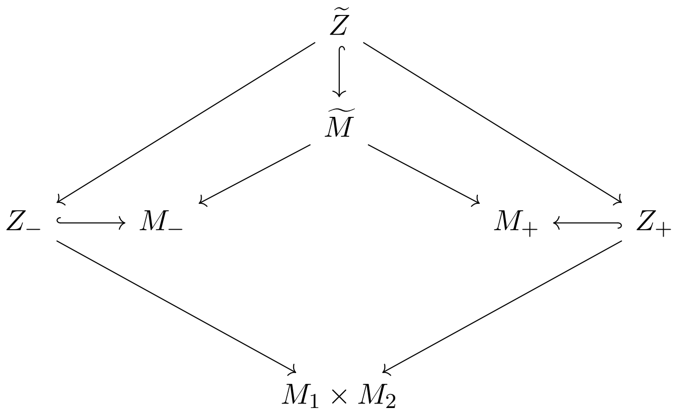
Note that M_1 and M_2 are the same whether they are defined with θ_- or θ_+ stability by condition (D). How do M_- and M_+ differ? The representations which are θ_--stable but not θ_+-stable are the non-split extensions of the form

$$0 \to V_1 \to V \to V_2 \to 0 \tag{5}$$

where V_1, V_2 are $\theta_\pm$-stable representations with dimension vectors d_1, d_2, respectively. Note that $\mu_{\theta_+}(d_1) > \mu_{\theta_+}(d_2)$ is equivalent to $\mu_{\theta_+}(d_1) > \mu_{\theta_+}(d)$, so V_1 destabilizes V with respect to θ_+-stability. On the other hand, representations which are θ_+-stable but not θ_--stable are the non-split extensions of the form

$$0 \to V_2 \to V \to V_1 \to 0\,. \tag{6}$$

Both of these families of representations are θ-stable but not θ-semistable for the stability condition θ on the wall. The representations of the form (5) and (6) define loci $Z_- \subseteq M_-$ and $Z_+ \subseteq M_+$, respectively, and $M_- \setminus Z_- = M_+ \setminus Z_+$. Both $Z_\pm$ are projective bundles over $M_1 \times M_2$; their fibers over (V_1, V_2) are $\mathbb{P}\mathrm{Ext}^1(V_2, V_1)$ and $\mathbb{P}\mathrm{Ext}^1(V_1, V_2)$, respectively. So essentially we can control the "difference" between M_- and M_+ by the "smaller" moduli spaces M_1 and M_2. Indeed, Thaddeus proves a general theorem [Tha] implying that M_- and M_+ are related by a flip. This means that the blow-ups of M_- at Z_- and M_+ at Z_+ are the same space $\widetilde{M}$, and we have a flip diagram as follows:



The curved arrows are embeddings, the arrows from $\widetilde{M}$ to $M_\pm$ are blow-ups and the exterior arrows are all projective bundles; $\widetilde{Z}$ is the common exceptional divisor of the two blow-ups.

The flip diagram allows us to compare integrals on M_- and M_+ by pulling back to $\widetilde{M}$. The difference, of course, is related to integrals on M_1 and M_2. However, sometimes it is not possible to connect two stability conditions by simple wall-crossing paths. In Example 5.2 crossing the wall $W_1 = W_5$ is not simple wall-crossing because it is a double wall; successive extensions of V_1, V_2, V_3 with dimension vectors $(1,0,0)$, $(1,0,0)$ and $(0,1,1)$, respectively, will change stability.

5.2 The wall-crossing vertex algebra

In [Moc], Mochizuki described a way to understand the non-simple wall-crossing behavior of descendent integrals in the context of moduli spaces of sheaves on surfaces. More recently, Joyce [Joy1, Joy2, GJT] has proposed a formalism to state and prove more general wall-crossing formulas. Joyce constructs a vertex algebra to write down such formulas in a conceptual way. This vertex algebra is very closely related to the descendent algebra introduced before by results from [Gro, BLM, LM], and this is the perspective we take here.

For $d \in \mathbb{Z}^{Q_0}$ define

$$\mathbb{D}^Q_d = \mathbb{D}^Q / \langle \mathrm{ch}_0(v) = d_v \mid v \in Q_0 \rangle \, .$$

Given a moduli space M^θ_d and a fixed universal representation $\mathcal{V}$, the realization morphism $\xi_{\mathcal{V}}$ factors through $\mathbb{D}_d^Q$, so it defines a linear functional

$$\mathbb{D}_d^Q \xrightarrow{\xi_{\mathcal{V}}} H^\bullet(M_d^\theta) \xrightarrow{\int} \mathbb{Q}\,.$$

Define the space of functionals

$$V = \bigoplus_{d\in\mathbb{Z}^{Q_0}} (\mathbb{D}_d^Q)^\vee\,.$$

Joyce constructs a natural vertex algebra structure on V. We refer to [Kac] for an introduction to vertex algebras, but we can mention that a vertex algebra structure on a vector space V consists of the data

$$|0\rangle \in V\,, \quad T\colon V\to V\,, \quad Y\colon V\to \mathrm{End}(V)[\![z^{-1},z]\!]$$

satisfying some complicated axioms. In our case, $|0\rangle$ corresponds to the functional in $\mathbb{D}_0^Q$ sending the algebra unit to 1 and any (non-empty) product of descendents to 0. The operator T is the dual of the operator R_{-1} that we defined earlier. The state-field correspondence Y is the most complicated and interesting part of the construction, but we will not discuss it here.

A moduli space together with a universal representation defines an element of V. However, as explained in Section 4.2, there is no canonical choice of universal representation, so a moduli space only defines canonically a functional on

$$\mathbb{D}^Q_{d,\mathrm{wt}_0} = \ker\left(\mathsf{R}_{-1}\colon \mathbb{D}_d^Q \to \mathbb{D}_d^Q\right)\,.$$

Since T is defined to be the dual of R_{-1}, we have an isomorphism

$$\check{V} \coloneqq V/T(V) \simeq \bigoplus_{d\in\mathbb{Z}^{Q_0}} (\mathbb{D}^Q_{d,\mathrm{wt}_0})^\vee\,.$$

It is a general fact from vertex algebras, due to Borcherds [Bor], that $\check{V} = V/T(V)$ inherits a Lie algebra structure from the vertex algebra structure on V. Given a moduli space M, its class $[M]\in\check{V}$ contains information about all the descendent integrals. Joyce shows that wall-crossing formulas can be written using the Lie bracket on $\check{V}$! For example, in the setting of simple wall-crossing that we described before we have an identity

$$[M_+] = [M_-] + [[M_2],[M_1]]$$

in $\check{V}$. But Joyce's formalism is also able to deal with non-simple wall-crossing, in which case we get more complicated formulas with iterated brackets.

Example 5.3. We go back to Example 5.2 and consider $\theta_- = (1, 5, -6)$ and $\theta_+ = (-1, 5, -4)$; these are separated only by the double wall $W_1 = W_5$. Joyce's wall crossing formula in this case gives

$$[M^{\theta_+}_{(2,1,1)}] = [M^{\theta_-}_{(2,1,1)}] + [[M^{\theta_-}_{(1,0,0)}], [M^{\theta_-}_{(1,1,1)}]] + [[M^{\theta_-}_{(2,0,0)}], [M^{\theta_-}_{(0,1,1)}]] + \frac{1}{2}[M^{\theta_-}_{(1,0,0)}, [[M^{\theta_-}_{(1,0,0)}], [M^{\theta_-}_{(0,1,1)}]]] \,.$$

This can be used effectively since we understand explicitly the vertex algebra V.

Theorem 5.4. The vertex algebra V is isomorphic to the lattice vertex algebra associated to $\mathbb{Z}^{Q_0}$ and the pairing $\chi_Q^{\mathrm{sym}} \colon \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ which is the symmetrization of the Euler pairing χ_Q of Q given by

$$\chi_Q(d, d') = \sum_{v \in Q_0} d_v d'_v - \sum_{e \in Q_0} d_{s(e)} d'_{t(e)} \,.$$

5.3 Vertex algebra and Virasoro

Vertex algebras are a natural source of representations of the Virasoro algebra. Any lattice vertex algebra associated to a lattice (Λ, B) with $B \colon \Lambda \times \Lambda \to \mathbb{Z}$ a non-degenerate pairing comes equipped with a so called conformal element $\omega \in V$ which induces a representation of Vir in V. Even if B is degenerate (which is sometimes the case with χ_Q^{sym}), there is still a representation of Vir_{-1} on V.

It was shown in [BLM, Boj, LM] that this canonical representation of Vir_{-1} coming from the fact that V is a lattice vertex algebra is precisely the dual of the representation of Vir_{-1} on the descendent algebra that we defined earlier in Section 4.3. A moduli space M satisfying Virasoro constraints can be translated to a vertex algebra language: it means that the class $[M] \in \check{V}$ is a physical state, as defined in [Bor]! This is not only very interesting because it gives a conceptual meaning to the constraints, but it also has an important consequence: this point of view shows that the Virasoro constraints are compatible with wall-crossing in the sense that the subspace of physical states is a Lie subalgebra.

Proposition 5.5 ([Bor])**.** Suppose that $[M_1], [M_2] \in \check{V}$ are physical states (read as "M_1 and M_2 satisfy the Virasoro constraints"). Then $[[M_1], [M_2]] \in \check{V}$ is a physical state as well.

From this compatibility, the proof of Theorem 4.6 is straightforward. The Virasoro constraints hold trivially for the increasing stability conditions (see Example 4.3), so we can deduce them for any stability θ from the compatibility with wall-crossing!

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