Topology of vortices with toric targets

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Resumo: Equações de vórtices são EDPs que descrevem configurações de campos BPS para modelos sigma de gauge em fibrados. No trabalho que aqui expomos, consideramos a situação em que a base é uma superfície de Riemann e a fibra (ou alvo) é uma variedade de Kähler tórica, ambas compactas. Os espaços de módulo que parametrizam as soluções (a menos de transformações de gauge unitárias), neste caso, são espaços de configurações generalizados associados a um complexo simplicial que pode ser extraído dos dados combinatórios do alvo. Os grupos fundamentais destes espaços podem ser interpretados como um novo tipo de grupos de tranças (coloridas) em superfícies, cuja descrição apresentamos de forma bastante concreta.

Abstract: The vortex equations are PDEs describing BPS field configurations for gauged sigma models in fibre bundles. In the work surveyed here, we focus on the situation where the base is a Riemann surface and the fibre/target is a toric Kähler manifold, both assumed compact. Then the moduli spaces of solutions (up to unitary gauge) are generalised configuration spaces associated to a certain simplicial complex that can be extracted from the toric data. The fundamental groups of such spaces can be understood as a novel type of surface braid groups with coloured strands, which we shall describe in very concrete terms.

palavras-chave: teoria de gauge, vórtice, espaços de módulo, geometria tórica, grupos de tranças, grafo, sistemas diofantinos

keywords: gauge theory, vortex, moduli spaces, toric geometry, braid groups, graph, Diophantine systems

1 Introduction

In elementary particle physics and condensed-matter theory, quantum field theories (as well as approximations or extensions to them) are studied with a wide variety of tools. The concept of *particle*, which is sometimes rather elusive, is captured by 'charges' modelled on invariants provided e.g. via algebraic topology or algebraic geometry, or from the representation theory of symmetry groups. Such invariants appear naturally in classification problems in pure mathematics, where in many situations one needs to supplement them by constructing *moduli spaces* parametrising different isomorphism classes of objects with the same invariants arising in families. Prime examples of such spaces feature in the surveys by Margarida Melo [26] and Miguel Moreira [27] also included in this Special Issue.

Moduli spaces keep fascinating mathematicians [30] from a variety of backgrounds. They have been used crucially to tackle (sometimes unexpectedly) very hard problems — from the geometry of 4-manifolds to quantum gravity. In this review, we explore moduli spaces parametrising solitonic configurations called *vortices* [24] in gauge theory. Vortices are like particles in the sense of having a point core, though their energy density is extended in space, forming peaks around the cores that superpose nonlinearly. In Fig. 1 we plot the energy density distribution of two vortex configurations of topological charge 2 on the hyperbolic disc. Here the charge corresponds to the degree of a map $S^1 \to S^1$ from the boundary of the disc to a circle in the target $\mathbb{C}$ parametrising the degenerate minima of a Higgs potential. One reason we chose the hyperbolic disc is that it leads to the integrability of the vortex equations [41]; but the vortices we will concentrate on live on compact Riemann surfaces and almost never enjoy this property. Vortex moduli spaces (with their intrinsic geometry [3]) can been employed to study vortex dynamics in the classical field theory [37, 33, 31, 38]. Quantisation of moduli spaces is a more ambitious venture [33, 18, 40], but one may hope to uncover nonperturbative aspects of quantum field theories in this way.

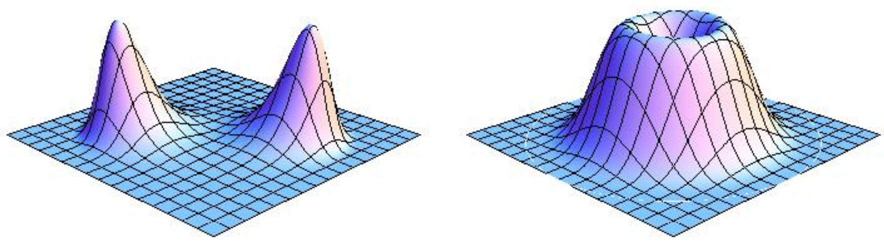

Figure 1: Plots of energy density functions for configurations of 2-vortices on the hyperbolic disc, with well-separated cores (left) and coalescing cores (right)

The topology of vortices advertised in our title refers concretely to the moduli spaces — more specifically, we shall look at their fundamental groups. It turns out that they are computable and interesting objects (providing examples of a novel type of braid group); in addition, they hint at how the solitonic particles interact in the quantum theory.

2 Gauged sigma models, the vortex equations and their moduli

We start by defining vortices more precisely and in higher generality. For that, one needs to choose:

- $(\Sigma, j_\Sigma, \omega_\Sigma)$ a riemannian, oriented surface (the *base*)
- (X, j_X, ω_X) another Kähler manifold (the *target*)

The word *gauged* refers to an internal symmetry that corresponds to a Hamiltonian and holomorphic action of a compact Lie group G on X. Letting $\mathfrak{g} := \mathrm{Lie}(G)$, we denote by $\mu^\sharp$ the composition of a moment map $\mu : X \to \mathfrak{g}^*$ for this action (satisfying $\mathrm{d}\mu(\xi) = \iota_\xi \omega_X$ for all $\xi \in \mathfrak{g}$) with the 'musical' isomorphism $\sharp : \mathfrak{g}^* \to \mathfrak{g}$ associated to an invariant metric on G.

We shall assign a distribution of energy to pairs of *fields*

$$(A, \phi) \in \mathcal{A}(P) \times \Gamma(\Sigma, P^X)$$

where A is a connection on a principal G-bundle $P \to \Sigma$ and ϕ a section of the associated bundle $P^X := P \times_G X \to \Sigma$. To each such pair we associate a topological charge

$$[\phi]_2^G := ((\tilde{f} \times \phi)/G)_*[\Sigma] \ \in \ H_2^G(X; \mathbb{Z})$$

in the G-equivariant 2-homology of X; here, $\tilde{f} : P \to \mathrm{E}G$ is the lift of a classifying map $f : \Sigma \to \mathrm{B}G$ for P with $P = f^*\mathrm{E}G$.

Given all these ingredients, the potential energy of a *gauged sigma model* is specified by the Yang–Mills–Higgs functional

$$E(A, \phi) := \frac{1}{2} \int_\Sigma \left(|F_A|^2 + |\mathrm{d}^A \phi|^2 + |\mu \circ \phi|^2 \right).$$

To find its minima, one could in principle try to solve its Euler–Lagrange equations (second-order PDEs on Σ). A more convenient strategy is to perform the so-called Bogomol'nyĭ's trick (taking advantage of the decomposition $\mathrm{d}^A = \partial^A + \bar{\partial}^A$ for the covariant derivative; see e.g. [29] for more details)

$$E(A,\phi) = \langle [\omega_X - \mu]^2_G, [\phi]^G_2 \rangle + \frac{1}{2}\int_\Sigma \left(\left| *F_A + \mu^\sharp \circ \phi \right|^2 + 2|\bar{\partial}^A \phi|^2 \right) \qquad (1)$$

and observe that the minima in each topological class, whenever the first term (which is constant) is nonnegative, satisfy the first-order PDEs

$$\bar{\partial}^A \phi = 0 \qquad (2)$$

$$*F_A + \mu^\sharp \circ \phi = 0 \qquad (3)$$

known as the *vortex equations*; solutions to these are examples of what are called (classical) BPS configurations in field theory. Equation (2) says that ϕ is a holomorphic section, whereas (3) relates the curvature F_A of A to the moment map evaluated at ϕ. Note that the metric on Σ intervenes in equation (3) via its Hodge operator $*$. Both equations are invariant under the infinite dimensional group $\mathcal{G}(P) := \mathrm{Aut}_\Sigma(P)$ of gauge transformations.

Remark 2.1. There is a completely analogous manipulation for the case $\langle [\omega_X - \mu]^2_G, [\phi]^G_2 \rangle \leq 0$ yielding *antivortex equations*, where $\bar{\partial}^A$ is replaced by ∂^A in (2) and the $(+)$ sign flips to $(-)$ in (3); in this sense, it is clear that a pair (A,ϕ) on a connected surface Σ cannot be of both vortex and antivortex type unless A is flat (i.e. $F_A = 0$) and ϕ covariantly constant, which corresponds to a *vacuum* configuration. We will see in a moment (cf. Example 2.1) how to implement coexistence of vortices and antivortices within a BPS solution in another sense.

Vortex *moduli spaces* are defined by fixing $\mathbf{h} \in H^G_2(X;\mathbb{Z})$ and taking

$$\mathcal{M}^X_{\mathbf{h}}(\Sigma) := \left\{ (A,\phi) \;\middle|\; \begin{array}{c} (2),\,(3) \text{ are satisfied} \\ \text{and } [\phi]^G_2 = \mathbf{h} \end{array} \right\} \Big/ \mathcal{G}(P)$$

(and similarly for antivortices). Such a space (of gauge orbits) can be formally understood as a Kähler quotient [28], and more rigorously one can show [29] that it receives a Kähler structure ω_{L^2} from the natural L^2 inner product on the space of fields $\mathcal{A}(P) \times \Gamma(\Sigma, P^X)$. Usually it is straightforward to understand the underlying complex structure, but it is much harder to describe or compute the Kähler form ω_{L^2} (or the corresponding Kähler metric g_{L^2}) [3, 6].

Though we have sketched how vortex moduli spaces are motivated by physics, we would like to point out that these objects have also found many applications in pure mathematics. Just to mention a few: they have been useful to compute Gromov–Witten invariants of symplectic quotients [4], to define invariants for Hamiltonian actions in analogy to Gromov–Witten theory [29, 25, 14], as a tool to study the topology of other interesting moduli spaces [21], or in a proof of the celebrated Verlinde formula [39].

In what follows, we shall restrict attention to gauged sigma models where

- X is a Kähler toric manifold (see [12, 1]);
- $G = T \subset \mathbb{T} := T^{\mathbb{C}} \subset X$ is its (real) torus.

Assuming (for simplicity) that X, Σ are compact, we obtain a very neat description of $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$, which we now explain.

Toric manifolds can be described via certain combinatorial data. One possibility (favoured by algebraic geometers [15]) is in terms of a *fan* F consisting of simplicial real *cones* in all dimensions from 0 up to $\dim_{\mathbb{C}} X$. In particular, the 1-dimensional cones $\rho \in F(1)$ are called *rays* and determine T-invariant divisors D_ρ in X. Another possibility, with more of a symplectic flavour, is through the *Delzant polytope* [11]

$$\Delta = \mu(X) \subset \mathfrak{g}^*$$

(the image of the moment map for the T-action). This Δ determines a *normal fan* $F = \mathrm{Fan}_\Delta$, whose rays consist of the inner normals to the facets of Δ.

The following notation will be very useful. We denote by

$$S^k\Sigma \equiv \mathrm{Sym}^k(\Sigma) := \Sigma^k/\mathfrak{S}_k$$

the k-th symmetric product of the Riemann surface Σ; this is a smooth complex manifold, whose points are interpreted as effective divisors in Σ in classical algebraic geometry of curves. Let us now suppose that we are given a simplicial complex Λ together with a function $\mathbf{k} : \mathrm{Sk}^0(\Lambda) \to \mathbb{N}$ on its 0-skeleton (whose points/vertices we shall sometimes refer to as *colours*). For convenience, we set

$$S^{\mathbf{k}}\Sigma := \prod_{\lambda \in \mathrm{Sk}^0(\Lambda)} S^{\mathbf{k}(\lambda)}\Sigma.$$

Let $[\lambda_i, \cdots, \lambda_j]$ denote the simplex with vertices $\lambda_i, \ldots, \lambda_j \in \mathrm{Sk}^0(\Lambda)$. We consider the following space of effective divisors in Σ *braided* by Λ:

$$\mathrm{Div}_+^{\mathbf{k}}(\Sigma;\Lambda) := \left\{ \mathbf{d} \in S^{\mathbf{k}}\Sigma : [\lambda_0, \cdots, \lambda_\ell] \notin \Lambda \Rightarrow \bigcap_{i=0}^{\ell} \mathrm{supp}(d_{\lambda_i}) = \emptyset \right\} \tag{4}$$

We can think of each component d_λ of $\mathbf{d}$ as a set of $\mathbf{k}(\lambda)$ points (counting multiplicities) of colour λ in Σ. The space (4) is a considerable generalisation of the usual notion of *configuration space* for Σ; we shall refer to particular realisations of it in what follows.

Now we go back to our toric targets. In that context, we shall take as simplicial complex

$$\Lambda = (\partial\Delta)^\vee. \tag{5}$$

when a Delzant polytope $\Delta = \mu(X)$ is given for a target X. Let us spell out the notation in (5): by $\partial\Delta$ we mean its boundary, which we may interpret as spherical polytope, and $(\cdot)^\vee$ means the dual polytope in that sense. For instance: if $\partial\Delta$ is the (outer shell of a) cube, then its dual $(\partial\Delta)^\vee$ is an octahedron; this example corresponds to $X = (\mathbb{P}^1)^3$.

Under an appropriate (open) stability condition, the moduli spaces $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$ can be described quite neatly. To spell out this condition precisely, we need a little more detail (see [9] for a full discussion, in a more general setting). In the commutative diagram of Abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\alpha} & \mathrm{Div}_{\mathbb{T}}(X) & \xrightarrow{\beta} & \mathrm{Cl}(X) & \longrightarrow & 0 \\ & & \cong \downarrow c & & \cong \downarrow c_1^{\mathbb{T}} & & \cong \downarrow c_1 & & \\ 0 & \longrightarrow & H^2(\mathrm{B}\mathbb{T};\mathbb{Z}) & \xrightarrow{a} & H^2_{\mathbb{T}}(X;\mathbb{Z}) & \xrightarrow{b} & H^2(X;\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

the top row is the standard short exact sequence computing the divisor class group $\mathrm{Cl}(X)$ of a toric variety X, M denoting the character lattice of $\mathbb{T}$ (see [15]); we interpret the map $c_1^{\mathbb{T}}$ as constructing equivariant first Chern classes [22] from $\mathbb{T}$-equivariant divisors on X. Let $a^*_{\mathbb{R}}$ be the dual map to the extension of the morphism a to real coefficients, and $\mathrm{Vol}(\Sigma) = \int_\Sigma \omega_\Sigma$. Assume that $\mathbf{h} \in H_2^T(X;\mathbb{Z})$ is a BPS charge [9]: in our setting, this simply means that it is an element of the lattice cone dual to the image of the cone of equivariant effective divisors $\mathrm{Div}^+_{\mathbb{T}}(X)$ under $c_1^{\mathbb{T}}$. Then the stability condition reads

$$\frac{a^*_{\mathbb{R}}(\mathbf{h})}{\mathrm{Vol}(\Sigma)} \in \mathrm{int}\,\Delta$$

under identifications $H_2(\mathrm{B}\mathbb{T};\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(M,\mathbb{R}) = \mathrm{Lie}(T)$ (see [15], pp. 597 and 574, respectively). We have the following result.

Theorem 2.2. *Suppose that X is constructed from a Delzant polytope Δ and that*

$$k_\rho = \langle c_1^T(D_\rho), \mathbf{h}\rangle \quad \text{for } \rho \in \mathrm{Fan}_\Delta(1).$$

Then $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$ is nonempty and there is a diffeomorphism

$$\mathcal{M}_{\mathbf{h}}^X(\Sigma) \cong \mathrm{Div}_+^{\mathbf{k}}(\Sigma;(\partial\Delta)^\vee) \subset \prod_{\rho\in\mathrm{Fan}_\Delta(1)} S^{k_\rho}\Sigma. \tag{6}$$

Proof. The result (6) is a particular case of a more general theorem in [9] for vortices in toric fibre bundles over Kähler manifolds of any dimension. The statement $\mathcal{M}_{\mathbf{h}}^X(\Sigma) \neq \emptyset$ follows from the projectivity of Σ (implying that positive line bundles admit holomorphic sections). □

Example 2.1 (The gauged $\mathbb{P}^1$-model)**.** The simplest type of nonlinear vortices (with X compact) are obtained for $X = \mathbb{P}^1 \cong S^2$ and $T = \mathrm{U}(1) \cong S^1$, with a fan of two rays $\rho_\pm$ corresponding to the North and South poles. Essentially, the case $\Sigma = S^2$ was first considered in [34]. Theorem 2.2 gives the description

$$\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma) = S^{k_+}\Sigma \times S^{k_-}\Sigma \setminus D_{(k_+,k_-)},$$

where $D_{(k_+,k_-)}$ is the diagonal in the Cartesian product; this agrees with previous work in the literature [35, 29, 2]. In this simple example we see that the solutions of the vortex equations (2) and (3), up to gauge equivalence, are completely specified by the positions of zeros and poles of the section ϕ (taking multiplicities into account). These "vortex cores" can be totally arbitrary on the surface provided zeroes and poles do not coalesce — even though zeroes and poles can coalesce among themselves. One such configuration is suggested in Fig 2, where the larger dots convey zeroes and poles of higher multiplicity, around which the energy density will have a shape somewhat similar to the graph on right half of Fig 1. The main difference between zeroes and poles is that the magnetic field F_A has (respectively) the same or opposite sign to the orientation of Σ; thus the shape of the fields near a pole is rather similar to that of an antivortex in the sense of Remark 2.1.

We note that the moduli spaces $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$ are complex manifolds (their natural complex structure being induced on $S^{\mathbf{k}}\Sigma$ from j_Σ) with boundary normal-crossing divisors. For instance, in Example 2.1 we have

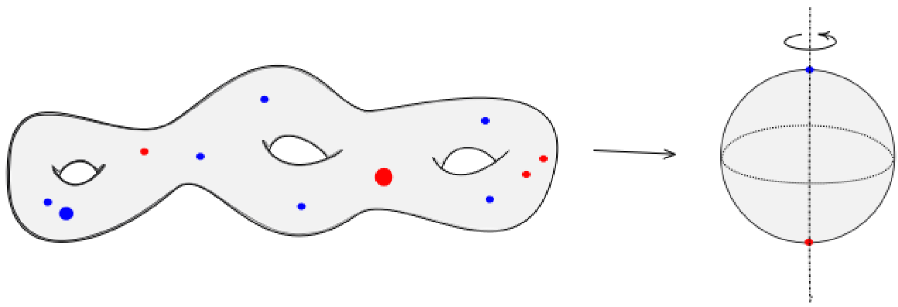

Figure 2: A configuration of vortices and antivortices in the gauged $\mathbb{P}^1$-model

$\partial\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma) = D_{(k_+,k_-)}$. The Kähler metric g_{L^2} has been studied close to this boundary in [32].

3 Fundamental groups of vortex moduli spaces with toric targets

Very concretely, the topology of vortices (in the title) that we propose to discuss is the fundamental group of the moduli spaces $\mathcal{M}^X_{\mathbf{h}}(\Sigma)$ in Theorem 2.2. Before we get there, we make a short interlude to advertise a useful variation on the braid group on surfaces [5].

Let Σ be an orientable surface. We introduce groups $DB_{\mathbf{k}}(\Sigma, \Gamma)$ of *divisor braids* on Σ that depend on a graph Γ, undirected and not necessarily connected, whose vertices are decorated with integers through a map $\mathbf{k} : \mathrm{Sk}^0(\Gamma) \to \mathbb{N}$. Here is an informal definition:

- Divisor braids have *coloured* strands, and the colours used are in bijection with the vertices $\mathrm{Sk}^0(\Gamma)$ of Γ. To allow for composition, we must demand that they are *colour-pure*.
- We extend the set of isotopies to allow strands of the same colour to pass through each other transversally, unlike ordinary braids.
- Strands of different colour are also allowed to pass through each other *unless* the vertices corresponding to their colours are connected by an edge in Γ. Clearly, we can assume that Γ has no multiple edges (connecting two given vertices), and we forbid self-loops (i.e. edges starting and ending at the same vertex).

- As subscript we use a map $\mathbf{k}:\mathrm{Sk}^0(\Gamma)\to\mathbb{N}$ (or decoration of vertices) recording how many strands $\mathbf{k}(\lambda)=k_\lambda$ there are of a given colour λ.

For instance, we could take $\Gamma=\Gamma_{(r)}$ to be the *complete graph* with r vertices (any pair of vertices is connected by an edge) and the constant function $\mathbf{k}=1$; then it is easy to check that $DB_1(\Sigma,\Gamma_{(r)})$ is the usual pure braid group $PB_r(\Sigma)$. But we will also be interested in more general graphs with the properties listed above (see e.g. Fig. 3). Given such a graph, we denote by $\neg\Gamma$ its negative: it has the same set of vertices as Γ and the complementary set of edges. We define

$$\mathrm{Conf}_{\mathbf{k}}(\Sigma,\Gamma):=\mathrm{Div}_+^{\mathbf{k}}(\Sigma;\neg\Gamma)\quad\subset\quad S^{\mathbf{k}}\Sigma. \tag{7}$$

This generalises the usual notion of configuration space $\mathrm{Conf}_k(\Sigma)$ of k points on Σ, which is obtained as

$$\mathrm{Conf}_k(\Sigma)=\mathrm{Conf}_1(\Sigma,\Gamma_{(k)});$$

it also extends nontrivially the notion of configuration space defined from a graph without decorations in [13]. The definition (7) can be used to give an alternative (and rigorous) definition of divisor braid groups as

$$DB_{\mathbf{k}}(\Sigma,\Gamma):=\pi_1\left(\mathrm{Conf}_{\mathbf{k}}(\Sigma,\Gamma)\right).$$

Figure 3: Sightseeing in Coimbra provides opportunities for graph theorists

Let us now go back to vortices. Recall that Theorem 2.2 gave a description of the moduli spaces $\mathcal{M}_{\mathbf{h}}^{X_\Delta}(\Sigma)$ under certain assumptions on the geometric ingredients defining the gauged sigma model. These were interpreted as spaces of effective divisors $\mathrm{Div}_+^{\mathbf{k}}(\Sigma,\Lambda)$ braided by $\Lambda = (\partial\Delta)^\vee$ that are actually of a more general sort than the configuration spaces (7), since they involve conditions where multiple intersections of points in Σ may occur. However, we have the following result from [8]:

Theorem 3.1. *There is an isomorphism of fundamental groups*

$$\pi_1\mathrm{Div}_+^{\mathbf{k}}(\Sigma,\Lambda) \;\cong\; \pi_1\mathrm{Div}_+^{\mathbf{k}}(\Sigma,\mathrm{Sk}^1(\Lambda)).$$

In particular, $\pi_1\left(\mathcal{M}_{\mathbf{h}}^{X_\Delta}(\Sigma)\right)$ *is a divisor braid group.*

Thus the study of divisor braid groups is well motivated by physics. In the rest of this article, our focus will be on describing presentations for these groups.

The first obvious task is to search for a convenient set of generators. If $k = |\mathbf{k}| := \sum_{\lambda\in\mathrm{Sk}^0(\Gamma)} k_\lambda$, then $DB_{\mathbf{k}}(\Sigma,\Gamma)$ is obviously a quotient of the usual braid group $B_k(\Sigma)$ on k strands. Let

$$\mathfrak{S}_{\mathbf{k}} := \prod_{\lambda\in\mathrm{Sk}^0(\Gamma)} \mathfrak{S}_{k_\lambda}, \qquad B_{\mathbf{k}}(\Sigma) := \sigma^{-1}(\mathfrak{S}_{\mathbf{k}}) \subset B_k(\Sigma).$$

Then we have a diagram of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & PB_k(\Sigma) & \overset{\text{colouring}}{\longrightarrow} & B_{\mathbf{k}}(\Sigma) & \longrightarrow & \mathfrak{S}_{\mathbf{k}} & \longrightarrow & 1 \\ & & & & \downarrow & & & & \\ & & & & DB_{\mathbf{k}}(\Sigma) & & & & \end{array}$$

where the row is short exact, and the map $\downarrow$ is surjective. We also have

Lemma 3.2. The composite $\Psi : PB_k(\Sigma) \to B_{\mathbf{k}}(\Sigma) \to DB_{\mathbf{k}}(\Sigma)$ is surjective.

In what follows, we shall work under two simplifying assumptions:

(**A**) Σ is compact of genus g.
If $g > 0$, it is convenient to regard Σ as resulting from identifying *opposite* sides/edges of a $4g$-gon, labelled e_ℓ, and respecting mirror orientations. For $g = 0$, we may start with any such polygon and collapse its entire boundary to a point.

(**B**) $\mathbf{k}$ is such that $k_\lambda \geq 2$ for any colour λ.

Definition 3.3. For the pure braid group $PB_k(\Sigma)$, fix k distinct basepoints z_i in Σ, labelled by $1 \le i \le k$. Consider a closed path $\gamma_i : [0,1] \to \Sigma$ with $\gamma_i(0) = \gamma_i(1) = z_i$ and $\gamma_i(t) \neq z_j$ for all $j \neq i, t \in [0,1]$. Let $\Phi(\gamma_i)$ be a braid defined by a path in Σ^k with components

$$\Phi(\gamma_i)_j(t) = \begin{cases} z_j & \text{if } j \neq i, \\ \gamma_i(t) & \text{if } j = i, \end{cases} \qquad (j = 1, \dots, k).$$

A pure braid of this type is called *monic*.

Our next goal is to show that divisor braid groups are generated by (images under Ψ of) certain monic braids. For convenience, let us fix distinct basepoints z_i along a bisector of the $4g$-gon used to construct Σ. We define the following monic braids:

- $a_{i,\ell} = \Phi(\gamma_{i,\ell}), \quad 1 \le i \le k,\ 1 \le \ell \le 2g$

  The path $\gamma_{i,\ell}$ runs from z_i straight to the midpoint of edge e_ℓ, crosses to the opposite side, and finally returns straight to z_i; see Fig, 4.

- $b_{i,j} = \Phi(\gamma_{i,j}), \quad 1 \le i < j \le k$

  The path $\gamma_{i,j}$ starts at z_i, encircles the point z_j positively, and then traces back its way to point z_i; see Fig, 5 (left).

- $t_{i,j} = \Phi(\tilde{\gamma}_{i,j}), \quad 1 \le i < j \le k$

  The path $\tilde{\gamma}_{i,j}$ starts at z_i, encircles the points $z_{i+1}, \dots, z_j$, and goes back to point z_i without any further zigzagging; see Fig, 5 (right).

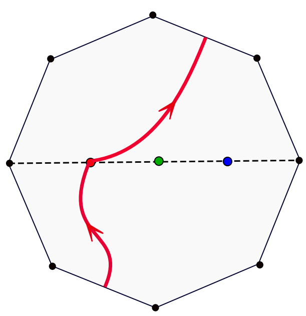

Figure 4: A divisor braid of type $a_{i,\ell}$ on a surface of genus $g > 0$

We can use these braids to construct convenient sets of generators for pure braid groups.

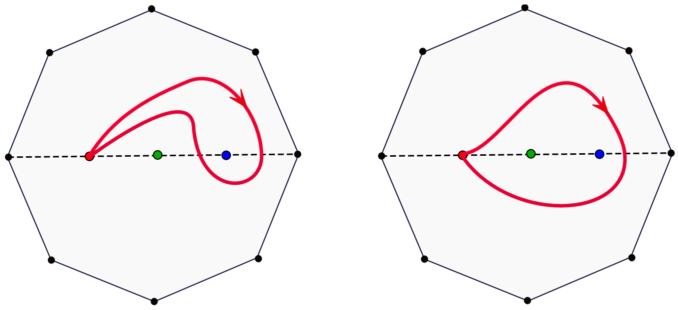

Figure 5: Divisor braids of types $b_{i,j}$ (left) and $t_{i,j}$ (right)

Lemma 3.4. Suppose Σ is a compact and oriented surface of genus g.
If $g \geq 1$, $PB_k(\Sigma)$ is generated by the classes $a_{i,\ell}$ and $b_{i,j}$.
If $g = 0$, $PB_k(\Sigma)$ is generated by the classes $t_{i,j}$.

Proof. For $g \geq 1$: the $a_{i,\ell}$ and $t_{i,j}$ are well-known generators of $PB_k(\Sigma)$ (see e.g. [20] for a presentation), and one can see that the $t_{i,j}$ are products of $b_{i,j}$ for $i < j \leq k$.
For $g = 0$: the $t_{i,j}$ correspond to generators of PB_n given by Artin, as products of his generators for B_n. □

As a consequence of Lemmas 3.2 and 3.4, we obtain a first set of generators for our divisor braid groups:

Corollary 3.5. For any $(\Gamma, \mathbf{k})$ as above, the $\Psi(a_{i,\ell}), \Psi(b_{i,j})$ generate $DB_{\mathbf{k}}(\Sigma, \Gamma)$ if $g \geq 1$, and the $\Psi(t_{i,j})$ generate $DB_{\mathbf{k}}(S^2, \Gamma)$.

These generators satisfy relations that we need to understand; as it might be expected, the sets of generators themselves contain some degree of redundancy. The following result will be useful to simplify our presentation.

Lemma 3.6. Let γ, γ' be two paths in Σ such that $\gamma(0) = \gamma(1) = z_i$, $\gamma'(0) = \gamma'(1) = z_{i'}$ and $z_i \neq z_{i'}$. Suppose further that

(i) the images of γ, γ' do not intersect in Σ; or that

(ii) $z_i, z_{i'}$ belong either to strands of the same colour or of different colours not connected by an edge in Γ.

Then $\Psi(\Phi(\gamma))$ and $\Psi(\Phi(\gamma'))$ are commuting divisor braids.

An interesting consequence of Lemma 3.6, which will bring a drastic simplification to the sets of generators in Corollary 3.5, is that

$\Psi(\Phi(a_{i,\ell})), \Psi(\Phi(b_{i,j}))$ only depend on the *colour* of their basepoints; see [8] for the complete argument. So for each λ we may pick an arbitrary z_{i_λ} of this colour and restrict the set of generators to

$$\alpha_{\lambda,\ell} := \Psi(a_{i_\lambda,\ell}), \quad \beta_{\lambda,\mu} := \Psi(b_{i_\lambda,i_\mu}).$$

In fact, we can do a little better.

Lemma 3.7. The classes $\alpha_{\lambda,\ell}$ and $\beta_{\lambda,\mu}$ generate $DB_{\mathbf{k}}(\Sigma,\Gamma)$; moreover, the classes $\beta_{\lambda,\mu}$ satisfy

$$\beta_{\lambda,\lambda} = e \text{ and } \beta_{\lambda,\lambda'} = \beta_{\lambda',\lambda}.$$

Proof. The first assertion is now clear. The relation $\beta_{\lambda,\lambda} = e$ is also clear, whereas $\beta_{\lambda,\lambda'} = \beta_{\lambda',\lambda}$ is better verified by hands-on manipulation; we try to convey this in Fig. 6. □

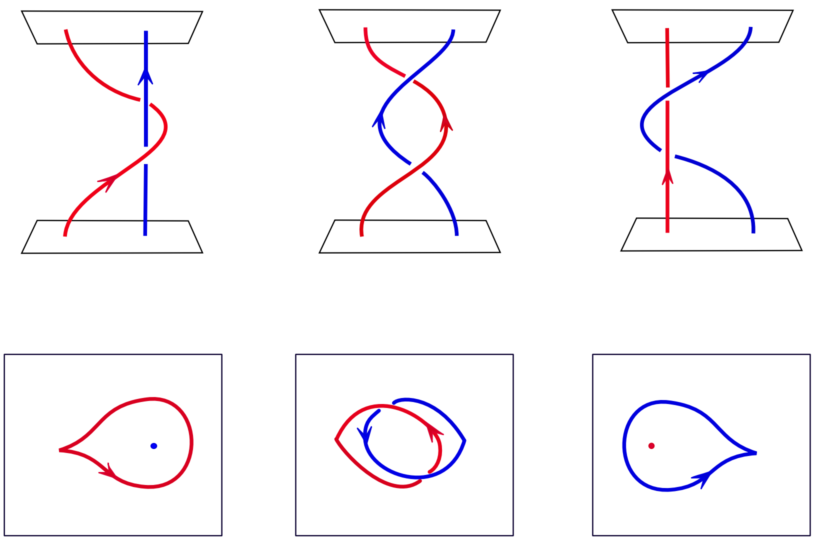

Figure 6: A pictorial check of the relation $\beta_{\lambda,\lambda'} = \beta_{\lambda',\lambda}$: all these pictures represent the same divisor braid (actually just two strands thereof, with the other strands kept straight) in two colours; the bottom row is obtained from the top row by vertical projection

We still expect further relations.

Lemma 3.8. Let λ, λ' be two different colours, and $1 \leq \ell, \ell' \leq 2g$. Then

$$[\alpha_{\lambda,\ell}, \alpha_{\lambda',\ell'}] = \beta^{-1}_{\lambda,\lambda'}.$$

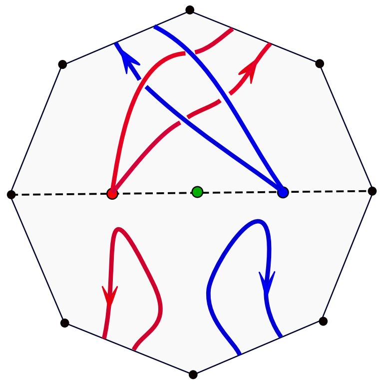

Figure 7: Checking that the relation $[\alpha_{\lambda,\ell}, \alpha_{\lambda',\ell'}] = \beta^{-1}_{\lambda,\lambda'}$ holds: one can disentangle the divisor braid depicted to obtain the inverse of Fig. 6

Proof. Again, this is best verified by drawing pictures — see Fig. 7. □

Lemmas 3.7 and 3.8 can be used to slim down further our set of generators, as follows.

Corollary 3.9. The classes $\alpha_{\lambda,\ell}$ generate $DB_{\mathbf{k}}(\Sigma, \Gamma)$ if Σ has positive genus, whereas $DB_{\mathbf{k}}(S^2, \Gamma)$ is generated by the classes $\beta_{\lambda,\lambda'}$.

Let us continue our search for relations. The following assertion depends crucially on assumption (**B**).

Lemma 3.10. The elements $\beta_{\lambda,\lambda'} \in DB_{\mathbf{k}}(\Sigma, \Gamma)$ are central.

Proof. It is required to prove that the $\alpha_{\lambda,\ell}$ commute with the $\beta_{\mu,\nu}$ (for all possible labels). Since $k_\mu \geq 2$, there is at least another basepoint $z_e \neq z_{i_\lambda}$ of colour μ. Now represent $\beta_{\mu,\nu}$ with a monic braid got from a path γ in $\Sigma \setminus \cup_{j \neq i_\mu} \{z_j\}$ starting from this z_e and avoiding the path used to define $\alpha_{\lambda,\ell}$; this commutes with $\alpha_{\lambda,\ell}$ by Lemma 3.6–(i). □

Note that the classes $\beta_{\lambda,\mu}$ were constructed from strands projecting onto a disc in Σ, so they do not depend on (the genus g of) Σ. Let $E_{\mathbf{k}}(\Gamma)$ denote the group they generate.

Theorem 3.11. $DB_{\mathbf{k}}(\Sigma, \Gamma)$ *sits in a central extension*

$$0 \longrightarrow E_{\mathbf{k}}(\Gamma) \longrightarrow DB_{\mathbf{k}}(\Sigma, \Gamma) \xrightarrow{h} H_1(\Sigma; \mathbb{Z})^{\oplus r} \longrightarrow 0 \tag{8}$$

where component λ of h sums the 1-cycles on Σ in colour λ.

Proof. Without loss of generality, we take $g \geq 1$. Certainly $E_{\mathbf{k}}(\Gamma) \subset \ker(h)$, and h factors through the quotient $DB_{\mathbf{k}}(\Sigma, \Gamma) \to DB_{\mathbf{k}}(\Sigma, \Gamma)/E_{\mathbf{k}}(\Gamma)$; so it is only required to prove that the induced map $\bar{h}$ from this quotient to $H_1(\Sigma;\mathbb{Z})^{\oplus r}$ is an isomorphism. It is clearly surjective, since the $h(\alpha_{\lambda,\ell})$ with λ fixed generate $H_1(\Sigma;\mathbb{Z})$; note also $H_1(\Sigma;\mathbb{Z})^{\oplus r}$ has rank greater or equal to that of the quotient. □

The next result summarises what we can already tell about the classes $\beta_{\lambda,\mu}$ with $1 \leq \lambda, \mu \leq k$.

Theorem 3.12. *The $\beta_{\lambda,\mu}$ satisfy:*

(i) $\beta_{\lambda,\mu} = 0$ *if there is no edge in Γ connecting λ and μ;*

(ii) $\beta_{\lambda,\lambda} = 0$*;*

(iii) $\sum_{\mu \neq \lambda} k_\mu \beta_{\lambda,\mu} = 0$.

Proof of (iii): Consider a path γ starting at z_{i_λ}, going straight to the boundary of the $4g$-gon, then around that boundary, and back to z_{i_λ}; it is a product of commutators in $\pi_1(\Sigma \setminus U_{j \neq i_\lambda}\{z_j\}, z_{i_\lambda})$, hence $\Phi(\gamma)$ is trivial. On the other hand γ encircles each z_j with $j \neq i_\lambda$ exactly once, so $\Phi(\gamma)$ represents the sum given. □

For concreteness, it is useful to consider a simple example in two colours; so we take the complete graph $\Gamma_{(2)} = \bullet\!\!-\!\!\!-\!\!\bullet$ (in other words, we go back to $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ in Example 2.1), and assume $k_+, k_- > 1$. As generators for $DB_{(k_+,k_-)}(\Sigma, \Gamma_{(2)})$, we want to streamline our previous notation and rather take

$$a_1, a_1', \ldots a_g, a_g' \qquad \text{and} \qquad a_1, a_1', \ldots a_g, a_g';$$

think of these as images of monic braids for generators of $\pi_1(\Sigma)$ based at far-enough points $*, * \in \Sigma$ around which the blue and red strand basepoints cluster. Here, subscript labels refer to *handles* of Σ, and the primed and unprimed generators for a given handle refer to symplectic conjugates in a canonical basis for $H_1(\Sigma;\mathbb{Z})$. We have also chosen to drop the colour labels and instead paint the generators explicitly. So the images of these generators under abelianisation (i.e. under the components of the map h in 8) yield generators of the two copies $H_1(\Sigma;\mathbb{Z}) \oplus H_1(\Sigma;\mathbb{Z})$. Let us now make a list of all the relations that can be written from the results obtained so far:

$$\begin{aligned}
[a_i, a_j] = [a_i, a'_j] = [a'_i, a'_j] = e \quad & \forall\, i, j \text{ by Lemma 3.6–(ii)},\\
[a_i, a_j] = [a_i, a'_j] = [a'_i, a'_j] = e \quad & \forall\, i, j \text{ by Lemma 3.6–(ii)},\\
[a_i, a_j] = [a'_i, a'_j] = e \quad & \forall\, i, j \text{ by Lemma 3.6–(i)},\\
[a_i, a'_j] = e \quad & \forall\, i \neq j \text{ by Lemma 3.6–(i)},\\
[a_i, a'_i] = c \quad & \forall\, i \text{ by Lemma 3.8}.
\end{aligned}$$

To be precise: in the last line, we labelled the commutator of pre-images of symplectic conjugates (in the singular 1-homology of Σ) of different colours by c, and noted that it does not depend on the handle label. Recall that c corresponds to the divisor braid depicted in Fig 6. What else can we say about this element c? From what we have learned until now,

- c is central, by Lemma 3.10;
- $\mathrm{ord}(c)$ divides both k_+ and k_-, by Theorem 3.12–(iii).

However, we still need to know something very crucial about c, namely:

- is $c = e$?

We invite our readers to convince themselves that the answer to this question is, in fact, impossible to deduce from the results in this section.

Remark 3.13. If assumption (**B**) is relaxed in this example, i.e. $k_\lambda = 1$ for some $\lambda \in \mathrm{Sk}^0(\Gamma_{(2)})$, then the corresponding divisor braid group is trivial — and in particular $c = e$. If $\mathbf{k}$ is the constant 1, this statement follows from $PB_2(\Sigma)$ being the trivial group; otherwise, we need to combine the usual proof of this fact with the move on two strands depicted in Fig. 6.

4 A link invariant and metabelian presentations

We will argue that $c \neq e$ is the most general answer to the question asked before Remark 3.13; in fact, we will compute the order of the element c. For this, we shall construct a link invariant for divisor braids with graph $\Gamma = \Gamma_{(2)} =$ and then calculate its value on the commutator c.

Let us consider the oriented 3-manifold $M := S^1 \times \Sigma$, with natural projections $p_1 : M \to S^1$ and $p_2 : M \to \Sigma$. Suppose that a pair (ℓ_+, ℓ_-) of closed braids of degree (k_+, k_-) is given; this means that $[\ell_\pm] \in H_1(M; \mathbb{Z})$ satisfy $p_{1*}[\ell_\pm] = k_\pm [S^1]$. Let us set

$$\bar{k} := \gcd(k_+, k_-)$$

and assume that:

- the images $\bar{\ell}_+, \bar{\ell}_- \subset M$ are disjoint;
- $p_{2*}[\ell_\pm] = 0 \in H_1(\Sigma;\mathbb{Z})$.

We consider $\bar{\ell}_- \subset M$ and the homology long exact sequence [23] for the pair $(M, M \setminus \bar{\ell}_-)$ with coefficients in $\mathbb{Z}_{\bar{k}} := \mathbb{Z}/\bar{k}\mathbb{Z}$,

$$\cdots \to H_2(M;\mathbb{Z}_{\bar{k}}) \xrightarrow{\psi} H_2(M, M\setminus\bar{\ell}_-;\mathbb{Z}_{\bar{k}}) \xrightarrow{\partial} H_1(M\setminus\bar{\ell}_-;\mathbb{Z}_{\bar{k}}) \xrightarrow{\varphi} H_1(M;\mathbb{Z}_{\bar{k}}) \to \cdots$$

Since $p_{1*}[\ell_+] = k_+[S^1]$ and $p_{2*}[\ell_+] = 0$, Künneth's formula implies that φ maps $[\ell_+] \in H_1(M \setminus \bar{\ell}_-;\mathbb{Z})$ as

$$[\ell_+] \overset{\varphi}{\longmapsto} (k_+(\mathrm{mod}\,\bar{k}), 0(\mathrm{mod}\,\bar{k})) = (0,0) \in H_1(S^1;\mathbb{Z}_{\bar{k}}) \oplus H_1(\Sigma;\mathbb{Z}_{\bar{k}}).$$

Exactness at $H_1(M \setminus \bar{\ell}_-;\mathbb{Z}_{\bar{k}})$ now ensures that

$$[\ell_+] \in \mathrm{coker}\ \psi = H_2(M, M \setminus \bar{\ell}_-;\mathbb{Z}_{\bar{k}})/\mathrm{im}(\psi). \tag{9}$$

We now invoke a somewhat exotic form of Poincaré duality known as Poincaré–Lefschetz duality, see Corollary VI.8.4 in [10]. This yields a commutative diagram

$$\begin{array}{ccccc} H^1(M;\mathbb{Z}_{\bar{k}}) & \xrightarrow{\psi'} & H^1(\bar{\ell}_-;\mathbb{Z}_{\bar{k}}) & \xrightarrow{\delta} & H^2(M,\bar{\ell}_-;\mathbb{Z}_{\bar{k}}) \\ \cong\downarrow D_M & & \cong\downarrow D_{\bar{\ell}_-} & & \cong\downarrow D_{M,\bar{\ell}_-} \\ H_2(M;\mathbb{Z}_{\bar{k}}) & \xrightarrow{\psi} & H_2(M, M\setminus\bar{\ell}_-;\mathbb{Z}_{\bar{k}}) & \xrightarrow{\partial} & H_1(M\setminus\bar{\ell}_-;\mathbb{Z}_{\bar{k}}) \end{array} \tag{10}$$

The lower row is the part of the homology long exact sequence for the pair $(M, M \setminus \bar{\ell}_-)$ above, while the upper row is part of the cohomology exact sequence [23] (cf. 3.1, p 199) corresponding to the pair $(M, \bar{\ell}_-)$. The vertical maps are isomorphisms, the leftmost being the usual Poincaré duality map.

Let us now consider the map

$$\langle\cdot, [\ell_-]\rangle : H^1(\bar{\ell}_-;\mathbb{Z}_{\bar{k}}) \to \mathbb{Z}_{\bar{k}}$$

evaluating the pairing at the generator $[\ell_-] \in H_1(\bar{\ell}_-;\mathbb{Z}_{\bar{k}})$. This vanishes on $\mathrm{im}(H^1(M;\mathbb{Z}_{\bar{k}}) \to H^1(\bar{\ell}_-;\mathbb{Z}_{\bar{k}}))$, since (once again by Künneth) it vanishes on $[S^1]$ (as $k_- \equiv 0(\mathrm{mod}\,\bar{k})$), and thus it is well defined on the cokernel of ψ'. Therefore we can evaluate it on the class $D_{\bar{\ell}_-}^{-1}[\ell_+]$ interpreted as in (9), and conclude that

$$\langle(\ell_+, \ell_-)\rangle := \langle D_{\bar{\ell}_-}^{-1}[\ell_+], [\ell_-]\rangle \in \mathbb{Z}_{\bar{k}}$$

is a well-defined *link invariant*, which is useful in our context.

Considering the link representing the divisor braid c depicted in Fig 6, and taking into account that we are evaluating at generators in each colour, we have that

$$\langle(\ell_+, \ell_-)\rangle \in \mathbb{Z}_{\bar{k}}$$

is in fact a generator of the group $\mathbb{Z}_{\bar{k}}$. It follows that c is an element of $DB_{(k_+,k_-)}(\Sigma, \Gamma_{(2)})$ of order $\gcd(k_+, k_-)$, generating its centre. More precisely, we can characterise the divisor braid group of the two-colour example at the end of section 3, and connecting to Example 2.1, as follows.

Theorem 4.1. *The divisor braid group*

$$DB_{(k_+,k_-)}(\Sigma, \Gamma_{(2)}) \cong \pi_1\left(\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)\right)$$

sits in a central extension (or abelianisation short exact sequence)

$$0 \longrightarrow \mathbb{Z}_{\gcd(k_+,k_-)} \longrightarrow DB_{(k_+,k_-)}(\Sigma, \Gamma_{(2)}) \longrightarrow H_1(\Sigma; \mathbb{Z})^{\oplus 2} \longrightarrow 0.$$

A group whose commutator is Abelian is sometimes called *metabelian*; thus a short exact sequence like the one in Theorem 3.11 is called a metabelian presentation. It is not hard to classify the representations of these groups. For instance, In the concrete example of Theorem 4.1, the representation varieties have $\gcd(k_+, k_-)$ connected components, which are copies of a $2g$-torus if g is the genus of Σ. Interestingly: in the non-coprime case, there are representations that do not factor through the abelianisation if $g > 0$. This simple fact has a remarkable physical consequence: the $\mathbb{P}^1$-model, an Abelian gauge theory which can be regarded as the simplest nonlinear extension of the familiar Abelian Higgs model, may support non-abelian anyons [36] in positive genus.

At this point one might wonder what can be said of divisor braid groups in general; by Theorem 3.11, under the simplification assumptions (**A**) and (**B**), they are still metabelian, but to which extent are their commutator subgroups nontrivial? Is it possible to construct, for more general graphs Γ, enough link invariants to rescue the elements $\beta_{\lambda,\mu}$ of the group $E_{\mathbf{k}}(\Gamma)$ in Theorem 3.11 from being trivial?

Remarkably, the answer is yes! The construction of a relevant Γ-*linking number* is more technical than the link invariant constructed above, and involves a tool from homological algebra called the Eilenberg–Zilber functor [17]. We refer the reader to [8] for an account of the general construction. The upshot is that there is a presentation of the group $E_{\mathbf{k}}(\Gamma)$ with generators $\beta_{\lambda,\mu}$ and relations

(i) $\beta_{\lambda,\mu} - \beta_{\mu,\lambda}$ for $1 \leq \lambda < \mu \leq r$;

(ii) $\beta_{\lambda,\mu}$ if there is no edge between λ and μ in Γ;

(iii) $P_\mu := \sum_{\lambda \neq \mu} k_\lambda \beta_{\lambda,\mu}$ for $1 \leq \mu \leq r$.

We conclude this section by precisely stating a presentation of divisor braid groups extending the results above, and also proved in [8]. Assuming $k_\lambda \geq 2$ for all $\lambda \in \mathrm{Sk}^0(\Gamma)$, and Σ oriented compact of genus $g \geq 0$, let us suppose that $a_{\lambda,\ell}$ are λ-coloured copies of elements a_ℓ in a basis of $H_1(\Sigma;\mathbb{Z})$.

Theorem 4.2. *The divisor braid group $DB_{\mathbf{k}}(\Sigma,\Gamma)$ is isomorphic to the group generated by $a_{\lambda,\ell}, b_{\lambda,\mu}$ ($1 \leq \ell \leq 2g$, $1 \leq \lambda, \mu \leq r$) and relations*

(i) $b_{\lambda,\mu} b_{\lambda',\mu'} = b_{\lambda',\mu'} b_{\lambda,\mu}$;

(ii) $b_{\lambda,\mu} = e$ *if no edge in* Γ *connects* λ *and* μ;

(iii) $b_{\lambda,\mu} = b_{\mu,\lambda}$;

(iv) $\prod_{\mu \neq \lambda} b_{\lambda,\mu}^{k_\mu} = e$;

(v) $b_{\lambda,\mu} a_{\nu,\ell} = a_{\nu,\ell} b_{\lambda,\mu}$;

(vi) $a_{\lambda,\ell} a_{\mu,\ell'} a_{\lambda,\ell}^{-1} a_{\mu,\ell'}^{-1} = b_{\lambda,\mu}^{\sharp(a_\ell, a_{\ell'})}$.

In (vi), $\sharp(\cdot,\cdot)$ *denotes the intersection pairing in* $H_1(\Sigma;\mathbb{Z})$.

5 Describing the centre of divisor braid groups

One can state more precisely how the finitely generated abelian group $E_{\mathbf{k}}(\Gamma)$ depends on both the graph Γ and its decoration $\mathbf{k}$. Recall that a graph is *bipartite* if its vertices can be consistently assigned opposite signs across all edges. For instance, the graph $\Gamma_{(2)}$ is bipartite but neither $\Gamma_{(r)}$ with $r \geq 3$ nor the graph illustrated by Fig. 3 (attending to the triangles formed by the edges attaching to the walls) are. The cleanest result on $E_{\mathbf{k}}(\Gamma)$ is about its rank; it depends on Γ alone.

Theorem 5.1. *Suppose Γ has connected components Γ_i, and that:*

- *r is the total number of vertices;*
- *s is the total number of edges;*

- *t is the number of components Γ_i that are bipartite.*

Then the rank of $E_{\mathbf{k}}(\Gamma)$ is $s - r + t$.

Sketch of proof. This can be neatly rephrased as a calculation of the dimension of the cokernel of a map $d_\Gamma : \mathcal{C}^0(\Gamma) \to \mathcal{C}^1(\Gamma)$ between linear spaces spanned by vertices and edges, assigning to each vertex the sum of incident edges in Γ,

$$d_\Gamma(\lambda) := \sum_{\varepsilon \text{ incident at } \lambda} \varepsilon. \tag{11}$$

See [8] for all the details. □

Can we also write down a formula for the torsion of $E_{\mathbf{k}}(\Gamma)$ in general? In other words: given a prime number p and $n \in \mathbb{N}$, can we say how many times the cyclic group $\mathbb{Z}_{p^n}$ appears in the primary decomposition of the finitely generated Abelian group $\operatorname{Tor} E_{\mathbf{k}}(\Gamma)$? As it turns out, this is a highly nontrivial problem.

One can recast this problem in purely arithmetic terms, as follows. Let us set $\mathbf{c} = (c_\lambda)_{\lambda \in \mathrm{Sk}^0(\Gamma)} = (c_1, \ldots, c_r)$ with components $c_\lambda \in \mathbb{Q}/\mathbb{Z}$, and define

$$C_{\mathbf{k}}(\Gamma) := \{\mathbf{c} \in (\mathbb{Q}/\mathbb{Z})^{\oplus r} \,|\, k_\lambda c_\mu + k_\mu c_\lambda \equiv 0 \text{ if } \exists \text{ edge between } \lambda \text{ and } \mu\}.$$

This is a group formed by the solutions of a linear system of rational congruences, one for each edge in Γ and with as many unknowns as vertices.

The linear system in the definition of $C_{\mathbf{k}}(\Gamma)$ is equivalent to the collection of (more conventional, Diophantine) linear systems

$$k_\lambda x_\mu + k_\mu x_\lambda \equiv 0 \pmod{p^n} \quad \text{if } \exists \text{ edge between } \lambda \text{ and } \mu \tag{12}$$

for all primes p and integers $n \in \mathbb{N}$. Solving (12) is an elementary problem[1], yielding a collection of Abelian groups (indexed by p and n) that can then be assembled into $C_{\mathbf{k}}(\Gamma)$ via standard homological algebra on the map (11) and the short exact sequence $\mathbb{Z} \hookrightarrow \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. For instance, let us assume that an odd prime p does not divide any of the integers k_λ. Then the solutions of the system (12) form a group $\mathbb{Z}_{p^n}^{\oplus t}$, where t is again the number of bipartite components of Γ. Each bipartite component Γ_i contributes with a copy of

[1] As historical aside, we note that linear Diophantine systems such as (12) were first discussed systematically in the mid 19th century (in work [16] published in Portuguese that was largely ignored) by Daniel Augusto da Silva, a pioneer of discrete mathematics [19] at a time when Portuguese mathematicians circulated less globally.

$\mathbb{Z}_{p^n}$ generated by the solution obtained from extending a local solution of the form

$$(x_\lambda, x_\mu) \equiv (k_\mu^{\phi(p^n)-1}, -k_\lambda^{\phi(p^n)-1}) = (k_\mu^{p^n-p^{n-1}-1}, -k_\lambda^{p^n-p^{n-1}-1}) \pmod{p^n}$$

(on the edge of Γ_i connecting the vertices λ and μ) across all edges of Γ_i using appropriate $\pm$ signs determined by the bipartitioning, and setting $x_\nu = 0$ for vertices ν in other components. When p appears as factor in one of the k_λ, the problem becomes more complicated. In particular, it is not easy to understand how the properties of the graph Γ are reflected in the order of the group of solutions to (12). An effort to address this problem (keeping Γ fixed but allowing $\mathbf{k}$ to vary) has led to a novel type of graph cohomology [7].

If Γ is bipartite and connected, we note that there is a cyclic subgroup $\Delta_{\mathbf{k}}(\Gamma) \subset C_{\mathbf{k}}(\Gamma)$ generated by a solution of the form

$$\left(\left[\pm\frac{1}{k_\lambda}\right]\right)_{\lambda\in \mathrm{Sk}^0(\Gamma)},$$

where the bipartitioning is again employed to distribute the $\pm$ signs along the whole graph. In [8] we prove

Theorem 5.2. *If Γ is connected, the linear system above determines*

$$\mathrm{Tor}\, E_{\mathbf{k}}(\Gamma) \cong \begin{cases} C_{\mathbf{k}}(\Gamma)/\Delta_{\mathbf{k}}(\Gamma) & \text{if } \Gamma \text{ is bipartite;} \\ C_{\mathbf{k}}(\Gamma) & \text{if } \Gamma \text{ is not bipartite.} \end{cases}$$

A few examples illustrating of how wildly the groups $\mathrm{Tor}\, E_{\mathbf{k}}(\Gamma)$ can vary when the charges k_λ are changed, for a given graph Γ e.g. coming from toric data as in section 2 (see Theorem 3.1), may also be found in the paper [8].

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