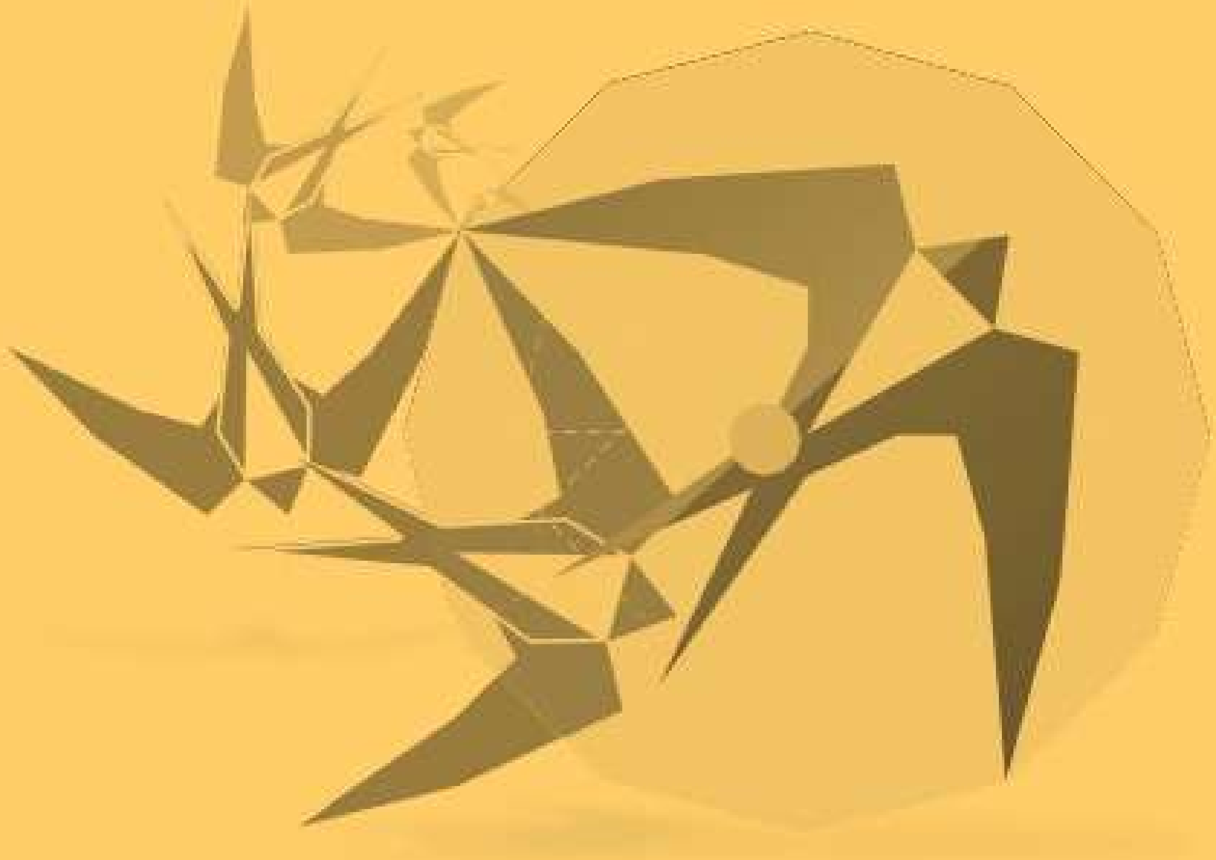
Boletim

da Sociedade Portuguesa de Matemática



Matemáticos Portugueses pelo Mundo

83
Número
DEZEMBRO 2025

Universidade de Coimbra, 2023

4 euros
[anual]

Boletim
da Sociedade Portuguesa de Matemática

Propriedade, Sede de Redacção e Edição
Sociedade Portuguesa de Matemática, Av. da República, 45–3º E, 1050–187 Lisboa
☎ 217939785 FAX 217952349
Internet: http://www.spm.pt
Número de registo na ERC 106 437
NIPC 501 065 792

Informações aos autores

O Boletim da SPM é uma publicação da Sociedade Portuguesa de Matemática, com periodicidade anual.
O Boletim constitui um espaço diversificado de informação, promove a circulação de ideias e de opiniões, bem como troca de experiências entre os que ensinam, investigam ou aplicam a Matemática.
O Boletim não publica artigos de investigação especializada. Estes trabalhos poderão ser submetidos à Portugaliæ Mathematica, revista de prestígio internacional, também propriedade da SPM e editada pela European Mathematical Society.
Os artigos dedicados a assuntos de natureza pré-universitária deverão, de preferência, ser submetidos à Gazeta de Matemática.
As actividades da SPM, nomeadamente das suas Delegações Regionais e Secções, são noticiadas com regularidade no Boletim.
As opiniões expressas pelos autores dos artigos publicados no Boletim não representam necessariamente posições da SPM.
Os Editores das Secções são os únicos responsáveis pela aceitação de artigos nas Secções que dirigem. A Secção Opinião — Cartas ao Director é da exclusiva responsabilidade da Directora do Boletim. Todos os outros trabalhos serão enviados pelos editores a *Referees* especializados para aconselharem sobre a respectiva publicação (com eventuais alterações).
Os manuscritos devem ser submetidos em `http://revistas.rcaap.pt/BoletimSPM` ou enviados por correio electrónico para um dos editores. Agradece-se o envio da versão PDF do texto. Os autores devem indicar as respectivas instituições, bem como os seus endereços de correio electrónico. Os trabalhos submetidos devem incluir um sumário em português e em inglês, e uma lista de palavras-chave nestas duas línguas. Recomenda-se vivamente que os trabalhos sejam preparados em LaTeX. A bibliografia deve seguir o padrão habitual no LaTeX.

Endereço para correspondência:

Boletim da Sociedade Portuguesa de Matemática
Av. da República, 45–3º E, 1050–187 Lisboa
Número de registo na ERC 106 437
NIPC 501 065 792

ISSN 0872–3672

SUMÁRIO

Artigos

Nota Prévia

É com enorme prazer que o Boletim da SPM se continua a associar à iniciativa da série de conferências “Matemáticos Portugueses pelo Mundo”. Neste número publicamos uma parte significativa dos trabalhos apresentados na sua terceira edição, edição essa que teve lugar na Universidade de Coimbra.

O trabalho de edição deste suplemento, obedecendo ao formato adoptado pelo Boletim, foi levado a cabo pelos colegas Ricardo Campos, Maria Manuel Clementino e João Gouveia, a quem deixo aqui um agradecimento especial.

Helena Reis

Dezembro 2025

Introduction

It is with great enthusiasm and pride that we present the proceedings of the third edition of the Global Portuguese Mathematicians Conference, held from July 26 to July 28, 2023, at the Department of Mathematics of the University of Coimbra. Following the success of previous editions in Lisbon (2017) and Porto (2019), this conference has continued to bring together Portuguese mathematicians (in the broadest sense) from all corners of the globe.

The primary aim of this event was to create a forum for sharing research, fostering collaborations, and building connections in a welcoming and stimulating environment. This year we were honored to host over 60 participants and 12 invited speakers, whose expertise spans a diverse range of mathematical disciplines. Their contributions provided valuable insights into the ongoing work of the Portuguese mathematical diaspora.

In addition to the engaging talks, the conference offered numerous opportunities for informal discussions and networking. Such interactions are the cornerstone of the mission to unite and strengthen the global Portuguese mathematical community.

We are also pleased to highlight that, in this edition of the Boletim da Sociedade Portuguesa de Matemática, we feature contributions from seven of the invited speakers, loosely based on the inspiring talks they presented during the conference. These articles offer a glimpse into the rich and diverse mathematical topics that were explored.

We would like to express our heartfelt gratitude to all participants, speakers, sponsors, and the scientific committee whose efforts made this conference a resounding success. We hope that the ideas and collaborations born during these days will flourish and inspire further advancements in mathematics.

Looking forward, we are excited to announce that the next edition of the Global Portuguese Mathematicians Conference will take place in 2025 at the University of Lisbon. We eagerly anticipate another vibrant gathering and hope to see many of you there.

Ricardo Campos

Maria Manuel Clementino

João Gouveia

Guest Editors & Organizers of the conference

Matemáticos Portugueses pelo Mundo

Departamento de Matemática, FCTUC
Universidade de Coimbra

26 a 28 de Julho de 2023

https://www.mat.uc.pt/ gpm23/

Rank functions for modules and categories

Teresa Conde
Bielefeld University
PO Box 100 131
33501 Bielefeld, Germany
e-mail: tconde@math.uni-bielefeld.de

Resumo: Em álgebra existem várias funções que medem uma certa noção de finitude ou tamanho de uma entidade álgebrica. De entre estas funções, as mais úteis tomam geralmente valores nos números reais não negativos e exibem algum tipo de aditividade. Neste texto, focamo-nos num tipo particular de funções, conhecidas como funções de dimensão, definidas em categorias trianguladas. Estas funções inspiram-se na noção clássica de dimensão de um módulo.

Abstract: In algebra one encounters various functions that measure some sort of finiteness or size of an algebraic entity. The most useful of these functions typically take values in the non-negative real numbers and exhibit some form of additivity. In this note, we focus on a particular type of functions, known as rank functions, defined on triangulated categories. These draw their inspiration from the classic notion of rank of a module.

palavras-chave: Função de dimensão; módulo; categoria triangulada.

keywords: Rank function; module; triangulated category.

1 Introduction

The notion of dimension or rank provides a measure of the 'size' of algebraic structures. In the context of modules over a ring, the rank of a module is a fundamental invariant that often signifies the maximal number of linearly independent elements. This concept extends naturally to the study of the so-called rank functions for finitely presented modules.

The study of triangulated categories, which generalise several key structures in algebraic topology, algebraic geometry, and representation theory, has brought forth the demand for analogous 'rank-like' functions. The concept of a rank function on a triangulated category was introduced by Chuang and Lazarev in [CL21], motivated by the work of Cohn, Malcolmson and Schofield on a special type of rank functions for finitely presented modules, called Sylvester rank functions. Work of Crawley-Boevey and Herzog

([Cra94a, Cra94b, Her93]) on additive functions for abelian categories inspired further recent developments on rank functions on triangulated categories ([CGMZ24]).

In this note, we provide an overview of some of the main results on rank functions on triangulated categories. We begin by revisiting rank functions for modules over rings, setting the stage for our exploration of rank functions on triangulated categories. Departing from an analogy with rank functions for modules and following the work of Chuang and Lazarev ([CL21]), we then introduce the notion of a rank function on a triangulated category and present one of the main theorems in [CL21]. Through the use of functorial methods, we demonstrate how rank functions on triangulated categories can be lifted to additive functions on certain associated abelian categories. This allows the translation of known results on the abelian context to the triangulated context and yields new results on rank functions on triangulated categories ([CGMZ24]).

2 Rank functions for modules

To motivate the concept of a rank function on a triangulated category, let us begin with the following open-ended question:

- What is the *rank* of a module?

Before refining the question above, some notation and details are needed. The term 'module' shall indicate a left module over some unitary associative (not necessarily commutative) ring A. Let A-Mod be the category of all A-modules. Recall that an A-module is ***finitely presented*** if it is the cokernel of a morphism between finitely generated free A-modules and denote the full subcategory of finitely presented A-modules by A-mod.

If A is a field (or even a division ring), then a module is simply a vector space, and by rank of a module we usually mean its dimension over A. More generally, the rank of a module is a well-defined concept whenever A is an integral domain. In this broader setting, the rank of a module X is generally understood as the maximal number of A-linear independent elements in X ([DF04]). If A satisfies the invariant basis property, the notion of rank applies to free modules: it is meant as the cardinality of an A-basis for the module. However, if A does not satisfy the invariant basis property, we encounter difficulties.

A general approach to defining a 'rank-like' function involves considering a ring morphism $h : A \to k$ from our underlying ring A to a division ring k.

In this setting, one may define the rank on an A-module X with respect to h as $\mathrm{rk}_h(X) := \dim(k \otimes_A X)$, where the right action of A on k is described via the morphism h. This yields an assignment of a non-negative integer to every finitely presented A-module. The assignment rk_h is an instance of an integral Sylvester rank function, as defined below.

Definition 2.1 ([Sch85]). A ***rank function*** ρ on A-mod assigns to each object X in A-mod an element $\rho(X) \in \mathbb{R}_{\geq 0}$ such that ρ satisfies the following two axioms:

1. ***Additivity:*** $\rho(X \oplus Y) = \rho(X) + \rho(Y)$ for X and Y in A-mod;
2. ***Triangle inequality:*** $\rho(Z) \leq \rho(Y) \leq \rho(X) + \rho(Z)$ for any right exact sequence of modules in A-mod

$$X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

A rank function taking values in $\mathbb{Z}$ is said to be ***integral***. A rank function ρ is called a ***Sylvester rank function*** if $\rho(A) = 1$.

Example 2.2. Let A be a left Artinian ring. In this context, the finitely presented A-modules are exactly those that have finite (Jordan–Hölder) length. Note that the length of an A-module yields an integral rank function. However, this is not a Sylvester rank function in general.

Example 2.3. Let A be an integral domain and let k be its field of fractions. Denote by h the embedding of A into k. One can check that the corresponding integral Sylvester rank function rk_h coincides with the classic notion of rank of a module over an integral domain mentioned previously.

Example 2.4. Let $h : A \to S$ be a ring morphism, where S is a simple left Artinian ring. By the Wedderburn–Artin theorem, S is isomorphic to a matrix ring over a division algebra, i.e. $S \cong M_n(k)$, for some $n \in \mathbb{N}$ and a division ring k. Define

$$\mathrm{rk}_h(X) := \frac{\mathrm{length}(S \otimes_A X)}{n}.$$

Note that this assignment generalises the previous definition of rank of X with respect to h and yields a Sylvester rank function taking values in $\mathbb{Q}$.

Much of what we have seen so far hints at a close relationship between rank functions on A-mod and ring morphisms from A to ‘uncomplicated’ rings. This relationship was made precise in the work of Cohn, Malcolmson and Scholfield.

Theorem 2.5 ([Sch85]). *There is a bijection between:*

1. *integral Sylvester rank functions on A-mod;*
2. *equivalence classes of ring epimorphisms from A to a division ring.*

The inverse correspondence maps a ring epimorphism $h : A \to k$ to rk_h.

Theorem 2.6 ([Sch85]). *Let A be an algebra over a field. There is a bijection between:*

1. *Sylvester rank functions on A-mod taking values on $\frac{1}{n}\mathbb{Z}$ with $n \in \mathbb{Z}_{>0}$;*
2. *equivalence classes of ring morphisms from A to a simple Artinian ring.*

The inverse correspondence maps a ring morphism $h : A \to S$ to rk_h.

3 Rank functions on triangulated categories

Triangulated categories play a prominent role in algebraic topology, algebraic geometry and representation theory. They provide a framework that generalises several key structures found in these areas, such as derived categories, homotopy categories of stable cofibration categories and stable categories of Frobenius categories. A triangulated category is a category $\mathcal{C}$ equipped with an additional structure subject to certain axioms: an equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$, called a ***translation functor***, together with a class of sequences of three composable morphisms, called ***distinguished triangles***. The precise definition of a triangulated category is lengthy and detailing it here would detract from the main focus of this text, which is to provide a general understanding of rank functions on triangulated categories and their connection to other types of 'measure' functions on different categories. For more details on triangulated categories, we refer, for instance, to [KS06, Nee01].

Suppose henceforth that $\mathcal{C}$ is a skeletally small triangulated category with translation functor $\Sigma : \mathcal{C} \to \mathcal{C}$. The concept of a rank function on a triangulated category was recently introduced by Chuang and Lazarev in [CL21]. Their motivation stemmed from the work by Cohn, Malcolmson and Schofield on Sylvester rank functions ([Coh08, Sch85]).

Definition 3.1 ([CL21]). A ***rank function on objects*** ρ_{ob} of $\mathcal{C}$ assigns to each object X in $\mathcal{C}$ an element $\rho_{ob}(X) \in \mathbb{R}_{\geq 0}$ such that ρ_{ob} satisfies the axioms:

1. ***Additivity:*** $\rho_{ob}(X \oplus Y) = \rho_{ob}(X) + \rho_{ob}(Y)$ for X and Y in $\mathcal{C}$;
2. ***Triangle inequality:*** $\rho_{ob}(Y) \leq \rho_{ob}(X) + \rho_{ob}(Z)$ for any triangle
$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X;$$
3. Σ***-invariance:*** $\rho_{ob}(\Sigma X) = \rho_{ob}(X)$ for every X in $\mathcal{C}$.

Alternatively, rank functions can be defined on morphisms of $\mathcal{C}$.

Definition 3.2 ([CL21])**.** A ***rank function*** ρ on $\mathcal{C}$ assigns to each morphism f in $\mathcal{C}$ an element $\rho(f) \in \mathbb{R}_{\geq 0}$ such that ρ satisfies the following axioms:

1. ***Additivity:*** $\rho(f \oplus g) = \rho(f) + \rho(g)$ for any two morphisms f and g;
2. ***Rank-nullity:*** $\rho(f) + \rho(g) = \rho(\mathbf{1}_Y)$ for any distinguished triangle
$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X;$$
3. Σ***-invariance:*** $\rho(\Sigma f) = \rho(f)$ for any morphism f in $\mathcal{C}$.

By [CL21], both definitions above are equivalent by setting $\rho_{ob}(X) := \rho(\mathbf{1}_X)$ and
$$\rho(f : X \to Y) := \frac{\rho_{ob}(X) + \rho_{ob}(Y) - \rho_{ob}(\operatorname{Cone} f)}{2}.$$
Here, $\operatorname{Cone} f$ denotes the unique object (up to isomorphism) fitting into a distinguished triangle
$$X \xrightarrow{f} Y \longrightarrow \operatorname{Cone} f \longrightarrow \Sigma X.$$

A standard example of a rank function on a triangulated category is the dimension of the total cohomology of an object in the bounded derived category of a finite-dimensional algebra ([CGMZ24]). Mass functions associated with Bridgeland stability conditions provide further interesting examples ([CL21, Ike21]). As the next example shows, it is possible pull back a rank function along a triangulated functor. Note that a ***triangulated functor*** is simply a functor between triangulated categories that preserves the triangulated structure, i.e. it commutes with the respective translation functors and maps distinguished triangles to distinguished triangles.

Example 3.3. Let $h : \mathcal{C} \to \mathcal{D}$ be a triangulated functor an let ρ be a rank function on $\mathcal{D}$. The assignment $\rho_h(f) \coloneqq \rho(h(f))$ defines a rank function on $\mathcal{C}$.

Developing the analogy between rank functions on triangulated categories and rank functions for modules, and recalling Theorems 2.5 and 2.6, it is natural to wonder which rank functions on $\mathcal{C}$ arise as a pull back of a rank function on an 'uncomplicated' triangulated category $\mathcal{D}$ along a triangulated functor $h : \mathcal{C} \to \mathcal{D}$, as explained in the previous example. To partially answer this question, some further definitions are needed.

Definition 3.4. A rank function ρ on $\mathcal{C}$ is:

1. ***integral*** if $\rho(f) \in \mathbb{Z}$ for every morphism f in $\mathcal{C}$;
2. ***localising*** if it is integral and moreover, if $\rho(f) = 0$ implies that the morphism f factors through some object Z such that $\rho_{ob}(Z) = 0$;
3. ***prime*** if it is integral and $\mathcal{C}$ has a thick generator X such that $\rho(X) = 0$.

In [CL21], Chuang and Lazarev established a connection between prime localising rank functions and certain functors to ***simple triangulated categories***, that is, triangulated categories with an indecomposable generator such that any triangle is a direct sum of split triangles. Compare the next result with Theorems 2.5 and 2.6.

Theorem 3.5 ([CL21])**.** *Suppose that the triangulated category $\mathcal{C}$ has a thick generator. There is a bijection between:*

1. *prime localising rank functions on $\mathcal{C}$;*
2. *thick subcategories $\mathcal{K}$ of $\mathcal{C}$ such that the Verdier quotient $\mathcal{C}/\mathcal{K}$ is a simple triangulated category.*

The inverse correspondence maps a thick subcategory $\mathcal{K}$ of $\mathcal{C}$ to the rank function ρ_h (see Example 3.3), where $h : \mathcal{C} \to \mathcal{C}/\mathcal{K}$ is the Verdier quotient functor and ρ is the unique prime rank function on $\mathcal{C}/\mathcal{K}$.

If $\mathcal{C}$ is the perfect category Per A *of a differential graded algebra A, then 1. and 2. above are also in bijection with:*

3. *equivalence classes of finite homological epimorphisms $A \to S$ with A a simple Artinian differential graded algebra.*

In the subsequent sections we demonstrate how functorial methods can further the analogy between rank functions for modules and rank functions on triangulated categories. This will enhance and generalise the theorem by Chuang and Lazarev discussed above.

4 From rank functions to additive functions

Functorial methods have a history of success in algebra and representation theory. In the following paragraphs, we will explain how these methods can be used to relate rank functions on triangulated categories to similar 'measure' functions on more familiar categories. For this, we will carry on with an analogy between two mathematical realms: additive categories with cokernels and triangulated categories.

An instance of an additive category with cokernels is the category A-mod of finitely presented modules over a ring A. Definition 2.1 remains applicable for an additive category $\mathcal{C}$ with cokernels and there is a well-developed theory of integral rank functions in this broader context ([Cra94a]). The results on rank functions in [Cra94a] are characterised by the following distinctive approach. An abelian category $\mathcal{A}_\mathcal{C}$ is constructed from the initial additive category $\mathcal{C}$ and the rank functions on $\mathcal{C}$ are lifted to certain 'measure' functions on $\mathcal{A}_\mathcal{C}$, called additive. Working with additive functions on abelian categories is more straightforward, and the results obtained in this context ([Cra94b, Her97]) can be translated back into theorems about rank functions on additive categories with cokernels ([Cra94a]).

In order to apply this strategy to triangulated categories, the first step is to associate an abelian category with a triangulated category $\mathcal{C}$, so that a rank function on $\mathcal{C}$ lifts to an additive function on the associated abelian category. Before explaining how to do this, let us first provide the definition of additive function.

Definition 4.1. An ***additive function*** $\widetilde{\rho}$ on a skeletally small abelian category $\mathcal{A}$ is an assignment $\widetilde{\rho}(X) \in \mathbb{R}_{\geq 0}$ to each object X of $\mathcal{A}$ which satisfies $\widetilde{\rho}(Y) = \widetilde{\rho}(X) + \widetilde{\rho}(Z)$ for any short exact sequence in $\mathcal{A}$

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

An additive function is said to be ***integral*** if it takes values in $\mathbb{Z}$.

The category Mod-$\mathcal{C}$ of all additive contravariant functors from a skeletally small triangulated category $\mathcal{C}$ to the category of abelian groups is an abelian category with remarkable properties. Let mod-$\mathcal{C}$ be the full subcategory of Mod-$\mathcal{C}$ whose objects are the functors which occur as cokernels of morphisms between ***representable functors***, i.e. between functors naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \mathrm{Ab}$ for some X in $\mathcal{C}$. The category mod-$\mathcal{C}$ turns out to be a skeletally small abelian category with an equivalence $\Sigma^* : \text{mod-}\mathcal{C} \to \text{mod-}\mathcal{C}$ induced by Σ via $\Sigma^*(F) := F \circ \Sigma^{-1}$. Furthermore, any rank function ρ on the triangulated category $\mathcal{C}$ lifts to an additive function $\widetilde{\rho}$ on mod-$\mathcal{C}$ by setting $\widetilde{\rho}(F) := \rho(f)$, where f is a morphism in $\mathcal{C}$ satisfying $F \cong \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(-, f))$. This assignment yields a bijection between rank functions on $\mathcal{C}$ and the additive functions $\widetilde{\rho}$ on mod-$\mathcal{C}$ satisfying $\widetilde{\rho}(\Sigma^* F) = \widetilde{\rho}(F)$.

Theorem 4.2 ([CGMZ24])**.** *Let $\mathcal{C}$ be a skeletally small triangulated category. There is a bijection between:*

1. *Σ-invariant additive functions $\widetilde{\rho}$ on* mod-$\mathcal{C}$*;*
2. *rank functions ρ on $\mathcal{C}$.*

5 Integral rank functions on triangulated categories

By Theorem 4.2, a rank function ρ on a triangulated category $\mathcal{C}$ can be reinterpreted as an additive function $\widetilde{\rho}$ on the category mod-C of finitely presented additive contravariant functors from $\mathcal{C}$ to abelian groups. Through this reinterpretation, the deeply developed theory of additive functions on abelian categories ([Cra94a, Her97]) can be employed to obtain various results about rank functions on triangulated categories. We present some of these results in this section.

The first result is a unique decomposition theorem.

Theorem 5.1 ([CGMZ24])**.** *Every integral rank function on a skeletally small triangulated category $\mathcal{C}$ can be uniquely decomposed as a locally finite sum of irreducible rank functions.*

Here, an ***irreducible*** rank function is a non-zero integral rank function that cannot be further decomposed into a sum of two non-zero integral rank functions. Every prime rank function, for instance, can be shown to be irreducible.

The functorial approach also allows to extend the bijection between localising prime rank functions on $\mathcal{C}$ and thick subcategories of $\mathcal{C}$ with simple

Verdier quotient, established in Theorem 3.5. For this purpose, one needs to introduce a new class of rank functions, called ***exact***, which contains all localising rank functions, and to use the notion of a ***CE-quotient functor***, introduced by Krause in [Kra05]. Instead of functors to simple triangulated categories, one should now consider functors to ***locally finite triangulated categories***, that is, triangulated categories $\mathcal{D}$ such that mod-$\mathcal{D}$ is an abelian category where every object has finite (Jordan–Hölder) length. Instead of prime rank functions, ***basic*** ones need to be used (these are integral rank functions with no repetitions in the irreducible summands appearing in their decomposition).

Theorem 5.2 ([CGMZ24])**.** *Let $\mathcal{C}$ be a skeletally small triangulated category. There is a bijection between:*

1. *exact basic rank functions on $\mathcal{C}$;*
2. *equivalence classes of CE-quotient functors from $\mathcal{C}$ to locally finite triangulated categories.*

Moreover, the underlying rank function is localising if and only if the CE-quotient is equivalent to a Verdier quotient.

In case $\mathcal{C}$ is the perfect derived category of a differential graded algebra A, the following partial generalisation of Theorem 3.5 can be obtained.

Theorem 5.3 ([CGMZ24])**.** *Let A be a cofibrant differential graded algebra. There is a bijection between:*

1. *idempotent basic rank functions on* $\operatorname{Per} A$*;*
2. *equivalence classes of homological epimorphisms $A \to B$ with* $\operatorname{Per} B$ *locally finite.*

Referências

[CGMZ24] Teresa Conde, Mikhail Gorsky, Frederik Marks, and Alexandra Zvonareva. A functorial approach to rank functions on triangulated categories. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2024(811):135–181, 2024.

[CL21] Joseph Chuang and Andrey Lazarev. Rank functions on triangulated categories. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2021(781):127–164, 2021.

[Coh08] Paul Cohn. *Skew fields. Theory of general division rings.*, volume 57. Cambridge: Cambridge University Press, 2008.

[Cra94a] William Crawley-Boevey. Additive functions on locally finitely presented Grothendieck categories. *Communications in Algebra*, 22(5):1629–1639, 1994.

[Cra94b] William Crawley-Boevey. Locally finitely presented additive categories. *Communications in Algebra*, 22(5):1641–1674, 1994.

[DF04] David Dummit and Richard Foote. *Abstract algebra.* Chichester: Wiley, 3rd ed. edition, 2004.

[Her93] Ivo Herzog. Elementary duality of modules. *Transactions of the American Mathematical Society*, 340(1):37–69, 1993.

[Her97] Ivo Herzog. The Ziegler spectrum of a locally coherent Grothendieck category. *Proceedings of the London Mathematical Society. Third Series*, 74(3):503–558, 1997.

[Ike21] Akishi Ikeda. Mass growth of objects and categorical entropy. *Nagoya Mathematical Journal*, 244:136–157, 2021.

[Kra05] Henning Krause. Cohomological quotients and smashing localizations. *American Journal of Mathematics*, 127(6):1191–1246, 2005.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren Math. Wiss.* Berlin: Springer, 2006.

[Nee01] Amnon Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies.* Princeton, NJ: Princeton University Press, 2001.

[Sch85] Aidan Schofield. *Representation of rings over skew fields*, volume 92. Cambridge University Press, Cambridge. London Mathematical Society, London, 1985.

From the Teukolsky equation to a system of wave equations on Schwarzschild

Rita Teixeira da Costa
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
e-mail: `rita.t.costa@dpmms.cam.ac.uk`

Abstract: We review the proof of energy boundeness and decay for solutions of the Teukolsky equation on the Schwarzschild geometry. This result was first shown by Dafermos, Holzegel and Rodniasnki (Acta. Math. 22(1):1–214, 2019). The proof here is based on an analysis of a transformed system of wave equations obtained by appropriately differentiating the Teukolsky equation. This approach was developed in collaboration with Shlapentokh-Rothman in a series of joint works (arXiv: 2007.07211, 2302.08916) concerning the Teukolsky equation on the more general Kerr family of rotating black holes.

Contents

1 Introduction

The stability of the Schwarzschild black hole,

$$g_M = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma, \quad M>0,$$

as a solution to the Einstein vacuum equations, has been a subject of great interest to the Mathematics and Physics communities since the pioneering work of Regge and Wheeler [RW57]. In this survey article, we review the proof of boundedness and decay estimates for the Teukolsky equation [BP73, Teu73]

$$\left(\Box^{[s]}_{g_M} + \frac{2s}{r}\left(1-\frac{M}{r}\right)\partial_r - \frac{2s}{r}\left(1-\frac{3M}{r}\right)\partial_t\right)\alpha^{[s]} = 0 \tag{1.1}$$

with $s=\pm 2$, which is one of the fundamental equations governing linearized perturbations of the Schwarzschild geometry. Here, $\Box^{[s]}_{g_M}$ is the spin-weighted d'Alembertian associated to the Schwarzschild metric g_M, see already Section 2.1.2 for a definition. Though we defer to Theorems 3.1 and 4.1 for the precise statements, a rough version of our main result is:

Theorem 1.1. *Fix $s\in\{\pm1,\pm2\}$ and $M>0$. On the Schwarzschild manifold, general solutions to the Teukolsky equation (1.1) arising from sufficiently regular initial data on a Cauchy surface ($\{\tau=0\}$)*

(i) satisfy a suitable version of "energy boundedness" without derivative loss: schematically,

$$\mathbb{E}(\tau_2) \leq B(M,s)\mathbb{E}(\tau_1)\,, \qquad \forall \tau_2\geq\tau_1\geq 0\,;$$

(ii) satisfy a suitable version of "integrated local energy decay" with loss of one derivative at trapping ($r=3M$): schematically,

$$\mathbb{I}^{\mathrm{deg}}(\tau_1,\tau_2) \leq B(M,s)\mathbb{E}(\tau_1)\,, \qquad \forall \tau_2\geq\tau_1\geq 0\,;$$

(iii) satisfy suitable, inverse-polynomial, energy and pointwise decay estimates with derivative loss.

The approach presented in this article is largely based on the approach in recent joint work with Shlapentokh-Rothman [SRTdC20, SRTdC23], where Theorem 1.1 is established in the more general class of Kerr black holes rotating at any speed below the maximal threshold[1]. In the Schwarzschild class considered in this work, boundedness and decay for (1.1) was first shown by Dafermos, Holzegel and Rodnianski [DHR19b] for $s = \pm 2$. Their analysis was later extended to the $s = \pm 1$ case, where (1.1) is an important component of the Maxwell system on Schwarzschild, by Pasqualotto [Pas19]. The case $s = 0$, the scalar wave equation on Schwarzschild, is by now classical, and the reader may find a proof and references to the original works in the lecture notes [DR13]. For sharp decay results, we direct the reader to [AAG23, MZ22] and the references therein.

The proof of boundedness and decay for (1.1) in [DHR19b] was a cornerstone result in the understanding of the stability properties of Schwarzschild spacetimes. Not long after its original proof [DHR19b] appeared, it was used by Klainerman and Szeftel [KS20] to establish nonlinear stability of Schwarzschild under polarized axisymmetric perturbations. The definitive result on stability of Schwarzschild, which makes no symmetry assumptions[2], was very recently obtained by Dafermos, Holzegel, Rodnianski and Taylor [DHRT21] again building on the (proof of) [DHR19b].

Having motivated the study of (1.1) and Theorem 1.1, let us turn to a discussion of its proof. As in [DHR19b, Pas19], our approach is to consider specific differential transformations of $\alpha^{[s]}$, introduced in [DHR19b] and inspired by Chandrasekhar's transformations [Cha75]. This allows one to, in particular, replace the Teukolsky equation (1.1) with a spin-weighted wave equation

$$\Box^{[s]}_{g_M}\Phi^{[s]} = 0 \tag{1.2}$$

for a new variable $\Phi^{[s]}$; for $s = \pm 2$, this equation is sometimes called Regge–Wheeler equation after [RW57]. Equation (1.2) can be treated using scalar wave methods which, as we mentioned, are by now classical (see e.g. the aforementioned lecture notes [DR13]). Thus, one can obtain energy boundedness, integrated local energy decay, and then decay for its solutions. To conclude the proof of Theorem 1.1, one must then upgrade these estimates for $\Phi^{[s]}$ to suitable estimates for $\alpha^{[s]}$.

[1]We direct the reader to these works for further details and references concerning the stability problem for rotating Kerr black holes, and for a more exaustive account of the literature on perturbations of Schwarzschild.

[2]As Schwarzschild is a member of the larger Kerr family of black holes, stability can only hold up to a co-dimension 3 "submanifold" of the moduli space of initial data for the Einstein vacuum equations, corresponding to perturbations which eventually radiate away all angular momentum. In this work, the authors teleologically identify this set of data and thus show the nonlinear stability of the Schwarzschild subfamily in full codimension.

It is in this recovery procedure that the proof of Theorem 1.1 differs from the boundedness and decay results of [DHR19b] for (1.1). In the former, estimates for $\alpha^{[s]}$ are obtained by integrating the transport equations which define $\Phi^{[s]}$; indeed,

$$\Phi^{[s]} \doteq \phi^{[s]}_{(|s|)}, \qquad \phi^{[s]}_{(k)} \doteq j_k (j_0 \mathcal{L})^k (j_{-1}\alpha^{[s]}) \text{ with } k \in \{0, \dots, |s|\},$$

for some appropriate choices of weights $j_i = j_i(r)$, $i = -1, \dots, |s|$, and with $\mathcal{L}$ a particular null vector field. In the present article, as in [SRTdC20, SRTdC23], we instead rely primarily on the system of wave equations for the new variables $\phi^{[s]}_{(k)}$ that can be derived from (1.1). For instance, in the case $s = \pm 2$, this system takes the form

$$\begin{aligned}
&\Box^{[2]}_{g_M} \phi^{[\pm 2]}_{(2)} - 4\left(1 - \frac{2M}{r}\right)\phi^{[\pm 2]}_{(2)} = 0,\\
&\Box^{[2]}_{g_M} \phi^{[\pm 2]}_{(1)} - 4\left(1 - \frac{2M}{r}\right)\phi^{[\pm 2]}_{(1)} = \pm\frac{2(r-3M)}{r^2}\phi^{[\pm 2]}_{(2)} \mp 6M\phi^{[\pm 2]}_{(0)},\\
&\Box^{[2]}_{g_M} \phi^{[\pm 2]}_{(0)} - 2\left(1 + \frac{2M}{r}\right)\phi^{[\pm 2]}_{(0)} = \pm\frac{4(r-3M)}{r^2}\phi^{[\pm 2]}_{(1)},
\end{aligned}$$

see already Remark 2.6 for more examples. With this approach, our Theorem 1.1 obtains slight improvements over [DHR19b, Theorem 2, Propositions 12.3.1-12.3.2]:

- In Theorem 4.1, we close energy boundedness estimates without derivative loss, and integrated estimates with 1 derivative loss only at trapping, for *all* derivatives of $\alpha^{[s]}$ at or below a certain level. In [DHR19b], derivatives in certain directions are not explicitly controlled.
- In Theorem 4.1, the estimates are at the level of $|s| + 1$ derivatives and below, while in [DHR19b] the authors require $|s| + 3$ derivatives in their energy boundedness statement to avoid derivative loss.
- There is an ϵ loss in the r-weights of the energy norms used in [DHR19b] compared to those in Theorem 1.1.

Finally, let us note that both the approach of this article, and that of the aforementioned [DHR19b, Pas19] to the Teukolsky equation (1.1) can, in principle, be extended to other $s \in \mathbb{Z}$ not contained in the statement of Theorem 1.1, see Remark 3.3 below. However, since only $s \in \{0, \pm 1, \pm 2\}$ are expected to be physically meaningful, we have chosen to exclude other values of s from our main result.

We conclude the introduction with an outline of the rest of the paper:

- Section 2 is a preliminary section introducing the Schwarzschild manifold, the relevant PDE, the spin-weighted function space on which they live, and the norms we will employ in our analysis.
- Section 3 considers the decoupled equation (1.2), and establishes energy boundedness, integrated local energy decay, energy decay and pointwise decay for its solutions.
- Section 4 then establishes similar results for all transformed variables $\upphi^{[s]}_{(k)}$ with $k < |s|$, thus concluding the proof of Theorem 1.1.

Acknowledgments. The main ideas in this survey are the result of collaboration with Y. Shlapentokh-Rothman, whom the author gratefully acknowledges.

2 Preliminaries

2.1 Geometric preliminaries

2.1.1 The Schwarzschild family

As is usual, we introduce the manifold structure of Schwarzschild independently of the black hole parameter M. To this end, consider a manifold-with-boundary $\mathcal{M} = \mathbb{R} \times \mathbb{R}^+_0 \times \mathbb{S}^2$. Coordinates $t^* \in \mathbb{R}$, $y^* \in \mathbb{R}^+_0$ and (θ, ϕ) the usual polar coordinates on $\mathbb{S}^2$ induce a global (modulo the usual degeneration of polar coordinates) differentiable structure on $\mathcal{M}$. The boundary, $\partial\mathcal{M} = \{y^* = 0\}$, is called the event horizon, $\mathcal{H}^+ \doteq \partial\mathcal{M}$. The interior, $\mathrm{int}(\mathcal{M})$, is called the domain of outer communications. We can also introduce the smooth vector fields $T \doteq \partial_{t^*}$ and $Z \doteq \partial_\phi$.

Let $r = r(y)$ be a new coordinate, smoothly depending on y, whose range is $r \in [2M, \infty)$. For $M > 0$, the Schwarzschild family is the (1-parameter) family of Lorenztian manifolds $(\mathcal{M}, g_M)$ with

$$g_M = -\left[1 - \mu_M(r)\right] (\mathrm{d}t^*)^2 + 2\mu_M(r)\mathrm{d}t^*\mathrm{d}r + (1 + \mu_M(r))\, dr^2 + r^2 \mathring{\not{g}}_{\mathbb{S}^2},$$
$$\mu_M(r) \doteq \frac{2M}{r},$$

where $\mathring{\not{g}}_{\mathbb{S}^2} = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ is the usual metric on the round sphere with volume form $\mathrm{d}\sigma \doteq \sin\theta \mathrm{d}\theta \mathrm{d}\phi$. The coordinates (t^*, r, θ, ϕ) are called ingoing Eddington–Finkelstein coordinates. We can use these to define a hyperboloidal folliation of interest for $\mathcal{M}$: following [SRTdC23], we take

$$\Sigma_\tau \doteq \{\tilde{t}^* = \tau\},$$

$$\tilde{t}^* \doteq t^* + \tilde{\zeta}_1(r)\left(r - \frac{M}{2}\log r\right) - \tilde{\zeta}_2(r)\left(r + 2M\log r - \frac{3M^2}{r}\right),$$

with $\tilde{\zeta}_1$ and $\tilde{\zeta}_2$ smooth cutoff functions satisfying: $\tilde{\zeta}_1 = 1$ for $r_+ \leq r \leq 3M/2$ and $\tilde{\zeta}_1 = 0$ for $r \geq 3M$; $\tilde{\zeta}_2 = 0$ for $r \leq 3M$ and $\tilde{\zeta}_2 = 1$ for $r \geq 4M$.

In the domain of outer communications alone, instead of Eddinton–Finkelstein coordinates, we can consider other coordinates t and r, as well as t and r^*, which are induced by the transformations

$$\frac{\mathrm{d}r^*}{dr} = \frac{1}{1-\mu_M(r)}, \quad r^*(3M) = 0; \qquad t = t^* - r^* + r - 3M + 2M\log M.$$

In coordinates (t, r, θ, ϕ) or (t, r^*, θ, ϕ), called Schwarzschild coordinates, the induced metric on $\mathrm{int}(\mathcal{M})$ is given by

$$\begin{aligned} g_M &= -\left[1-\mu_M(r)\right]\left[\mathrm{d}t^2 - (\mathrm{d}r^*)^2\right] + r^2 \mathring{\not{g}}_{\mathbb{S}^2} \\ &= -\left[1-\mu_M(r)\right]\mathrm{d}t^2 + \frac{1}{1-\mu_M(r)}\mathrm{d}r^2 + r^2 \mathring{\not{g}}_{\mathbb{S}^2}. \end{aligned}$$

Schwarzschild coordinates are used predominantly in this work. We also note the often-used function

$$w \doteq \frac{1-\mu_M(r)}{r^2}.$$

We often drop the subscript on the function μ_M for readability.

Finally, of interest to us are the M-dependent vector fields

$$L \doteq \partial_{r^*} + T\,, \quad \underline{L} \doteq -\partial_{r^*} + T\,, \qquad g(L, \underline{L}) = -2\,,$$

defining two principal null directions on $\mathrm{int}(\mathcal{M})$. Note that L and $(1-\mu_M)^{-1}\underline{L}$ can be extended to $\mathcal{M}$ using the coordinate transformations above.

2.1.2 The spin-weighted structure

In this section, we introduce a spin-weighted structure on $\mathcal{M}$.

To start with, let us define spin-weighted functions and operators on the round sphere $\mathbb{S}^2$. Letting (θ, ϕ) denote standard spherical coordinates in $\mathbb{S}^2$, consider the operators[3]

$$\begin{aligned} \tilde{Z}_1 &= -\sin\phi\partial_\theta + \cos\phi\left(-is\csc\theta - \cot\theta\partial_\phi\right)\,, \\ \tilde{Z}_2 &= -\cos\phi\partial_\theta - \sin\phi\left(-is\csc\theta - \cot\theta\partial_\phi\right)\,, \quad \tilde{Z}_3 = \partial_\phi\,. \end{aligned} \tag{2.1}$$

[3]The operators $\tilde{Z}_1$, $\tilde{Z}_2$ and $\tilde{Z}_3$ arise from the action of the canonical orthonormal frame on $\mathbb{S}^3$, viewed as the Hopf bundle over $\mathbb{S}^2$, on complex-valued functions with a particular s-dependence along the $\mathbb{S}^1$ fibers, see for instance [?] for more details.

Definition 2.1 (Smooth spin-weighted functions on $\mathbb{S}^2$)**.** Fix some $s \in \frac{1}{2}\mathbb{Z}$. Let f be a complex-valued function of $(\theta, \phi) \in \mathbb{S}^2$. We say f is a smooth s-spin-weighted function on $\mathbb{S}^2$, and write $f \in \mathscr{S}_\infty^{[s]}$, if for any $k_1, k_2, k_3 \in \mathbb{N}_0$,

$$(\tilde{Z}_1)^{k_1}(\tilde{Z}_2)^{k_2}(\tilde{Z}_3)^{k_3} f$$

is a function of (θ, ϕ) which is smooth for $\theta \neq 0, \pi$ and such that

$$e^{is\phi}(\tilde{Z}_1)^{k_1}(\tilde{Z}_2)^{k_2}(\tilde{Z}_3)^{k_3} f \text{ and } e^{-is\phi}(\tilde{Z}_1)^{k_1}(\tilde{Z}_2)^{k_2}(\tilde{Z}_3)^{k_3} f$$

extend continuously to, respectively, $\theta = 0$ and $\theta = \pi$.

For any $s \in \frac{1}{2}\mathbb{Z}$, the *spin-weighted laplacian*

$$\begin{aligned} \mathring{\triangle}^{[s]} &= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\partial_\phi^2 - 2is\frac{\cos\theta}{\sin^2\theta}\partial_\phi + s^2\cot^2\theta - s \\ &= -\tilde{Z}_1^2 - \tilde{Z}_2^2 - \tilde{Z}_3^2 - s - s^2 = \mathring{\triangle} - 2is\frac{\cos\theta}{\sin^2\theta}\partial_\phi + s^2\cot^2\theta - s \end{aligned} \quad (2.2)$$

where $\tilde{Z}_1$, $\tilde{Z}_2$ and $\tilde{Z}_3$ are defined in (2.1) and $\mathring{\triangle}$ is the usual laplacian on the round sphere is a example of a smooth differential operator on $\mathscr{S}_\infty^{[s]}$. Another example is provided by the *spinorial gradient*

$$\mathring{\nabla}^{[s]} = (\partial_\theta\,,\ \partial_\phi + is\cos\theta)\,, \quad (2.3)$$

which arises naturally in connection with the spin-weighted laplacian: for $\Xi\,, \Pi \in \mathscr{S}_\infty^{[s]}$,

$$\int_{\mathbb{S}^2}\left(\mathring{\triangle}^{[s]}\Xi\right)\overline{\Pi}\mathrm{d}\sigma = \int_{\mathbb{S}^2}\left[\mathring{\nabla}^{[s]}\Xi\cdot\overline{\mathring{\nabla}^{[s]}\Pi}\right]_{\mathbb{S}^2}\mathrm{d}\sigma. \quad (2.4)$$

Let us state two useful results concerning these important operators:

Lemma 2.2 (Spin-weighted spherical harmonics)**.** *Fix $s \in \frac{1}{2}\mathbb{Z}$. On $\mathscr{S}_\infty^{[s]}$, the operator $\mathring{\triangle}^{[s]}$ has a countable set of eigenfunctions referred to as spin-weighted spherical harmonics, forming a complete orthogonal basis of $\mathscr{S}_\infty^{[s]}$.*

As is standard, we index the spin-weighted spherical harmonics and their eigenvalues by parameters m and ℓ chosen to satisfy $m - s \in \mathbb{Z}$ and $\ell \geq \max\{|s|, |m|\}$ such that:

- *the (m, ℓ) spin-weighted spherical harmonic is denoted by $S_{m\ell}^{[s]}(\theta, \phi)$;*
- *the corresponding eigenvalue is $\lambda_{m\ell}^{[s]} = \ell(\ell+1) - s^2$.*

Lemma 2.3 (Spinorial gradient)**.** *Let $s \in \mathbb{Z}$ and $\mathring{\nabla}^{[s]}$ be as defined in (2.3). Then, for any $\Xi \in \mathscr{S}_\infty^{[s]}$, one has the Poincaré inequality*

$$\int_{\mathbb{S}^2} \left|\mathring{\nabla}^{[s]}\Xi\right|^2 \mathrm{d}\sigma \geq |s| \int_{\mathbb{S}^2} |\Xi|^2 \,\mathrm{d}\sigma. \tag{2.5}$$

As the Schwarzschild manifold $\mathcal{M}$ is spherically symmetric, it inherits the spin-weighted structure over $\mathbb{S}^2$:

Definition 2.4 (Smooth spin-weighted functions on $\mathcal{M}$)**.** Fix some $s \in \frac{1}{2}\mathbb{Z}$. We say f is a smooth s-spin-weighted function on $\mathcal{M}$, and write $f \in \mathscr{S}_\infty^{[s]}(\mathcal{M})$, if f is smooth in the Eddington–Finkelstein t^* and r, and its restriction to constant (t^*, r) yields a smooth s-spin-weighted function on $\mathbb{S}^2$.

For some $s \in \frac{1}{2}\mathbb{Z}$, the spin-weighted d'Alembertian

$$\Box_{g_M}^{[s]} = \Box_{g_M} + \frac{2is}{r^2}\frac{\cos\theta}{\sin^2\theta}\partial_\phi + \frac{1}{r^2}(s - s^2\cot^2\theta)$$

is an example of a smooth differential operator on $\mathscr{S}_\infty^{[s]}(\mathcal{M})$. In Schwarzschild coordinates it is given by

$$\Box_{g_M}^{[s]} = \frac{1}{r^2}\partial_r\left(r^2(1-\mu_M)\partial_r\right) - \frac{1}{1-\mu_M}\partial_t^2 - \frac{1}{r^2}\mathring{\triangle}^{[s]}.$$

2.2 Analytical preliminaries

2.2.1 The Teukolsky equation

We say that a function $\alpha^{[s]}$ such that $(1-\mu)^s\alpha^{[s]} \in \mathscr{S}_\infty^{[s]}(\mathcal{M})$ satisfies the Teukolsky equation if

$$\Box_{g_M}^{[s]}\alpha^{[s]} - \frac{2s}{r}\left(1 - \frac{3M}{r}\right)T\alpha^{[s]} + \frac{2s}{r}\left(1 - \frac{M}{r}\right)\partial_r\alpha^{[s]} = 0, \tag{2.6}$$

or, in Schwarzschild coordinates,

$$\frac{1}{r^{2(1+s)}(1-\mu_M)^s}\partial_r\left(r^{2(s+1)}(1-\mu_M)^{s+1}\partial_r\right)\alpha^{[s]} - \frac{1}{r^2(1-\mu_M)}\left(T + \frac{2s}{r}\left(1 - \frac{3M}{r}\right)\right)T\alpha^{[s]} - \mathring{\triangle}^{[s]}\alpha^{[s]} = 0.$$

2.2.2 The DHR transformed system

In this section, inspired by the transformations of Dafermos, Holzegel and Rodnianski in [DHR19b, DHR19a] and of Pasqualotto [Pas19], we introduce a system of equations derived from the Teukolsky equation (2.6) which was obtained with Shlapentokh-Rothman [SRTdC20, SRTdC23].

Fix $s \in \mathbb{Z}$, and consider the following rescaling of the Teukolsky variable

$$\psi_{(0)}^{[s]} = w^{|s|(1+\operatorname{sgn} s)/2} r^{1-|s|(1-\operatorname{sgn} s)} \alpha^{[s]}. \tag{2.7}$$

For $k = 1, \ldots, |s|$, we then define $\psi_{(k)}^{[s]}$ by the system of transport equations

$$\psi_{(k)}^{[s]} = w^{-1} \mathcal{L} \psi_{(k-1)}^{[s]}, \qquad k = 1, ..., |s|, \tag{2.8}$$

with $\mathcal{L} = L$ if $s < 0$, $\mathcal{L} = \underline{L}$ if $s > 0$ and $\mathcal{L}$ being the identity if $s = 0$. We will sometimes use the notation $\Psi^{[s]} \doteq \psi_{(|s|)}^{[s]}$. It will also be useful to introduce a further rescaling

$$\phi_{(k)}^{[s]} \doteq r^{-1} \psi_{(k)}^{[s]}, \quad \Phi^{[s]} \doteq \phi_{(|s|)}^{[s]} = r^{-1} \Psi^{[s]}.$$

Proposition 2.5 (DHR transformed system)**.** *Fix $s \in \mathbb{Z}$ and let $\alpha^{[s]}$ solve the Teukolsky equation (2.6). Then, for $k = 0, ..., |s|$, $\psi_{(k)}^{[s]}$ given in (2.7)–(2.8) satisfy the following system of wave equations*

$$\Box_g^{[s]} \phi_{(k)}^{[s]} - U_{(k)}^{[s]}(r) \phi_{(k)}^{[s]} = -\operatorname{sgn} s(|s| - k) \frac{w'}{w} \phi_{(k+1)}^{[s]} - \sum_{j=0}^{k-1} \mathfrak{I}_{(k),(j)}^{[s]}[\psi_{(j)}^{[s]}], \tag{2.9}$$

where

$$U_{(k)}^{[s]} \doteq |s| + k(2|s| - k - 1) - \frac{2M}{r}\Big(3|s| - 2s^2 + 3k(2|s| - k - 1)\Big), \tag{2.10}$$

$$\mathfrak{I}_{(k),(j)}^{[s]}[\psi_{(j)}^{[s]}] \doteq c_{s,k,j}^{\mathrm{id}}(r) \psi_{(j)}^{[s]}, \tag{2.11}$$

and $c_{s,k,j}^{\mathrm{id}}$, $j = 0, ..., k-1$, denote functions of r which can be explicitly computed for through a recursive relation (2.16) initialized by (2.17), and which have the following properties:

- *if $0 < k < |s|$ and $j = k - 1$, $c_{s,k,j}^{\mathrm{id}}$ is a nonzero constant;*
- *otherwise, $c_{s,k,j}^{\mathrm{id}}$ vanishes;*

Equivalently, $\psi^{[s]}_{(k)}$ *satisfy the equations*

$$\mathfrak{R}^{[s]}_{(k)}\psi^{[s]}_{(k)} = \operatorname{sgn} s(|s|-k)w'\psi^{[s]}_{(k+1)} + w\sum_{j=0}^{k-1}\mathfrak{J}^{[s]}_{(k),(j)}[\psi^{[s]}_{(j)}]\,, \tag{2.12}$$

where

$$\mathfrak{R}^{[s]}_{(k)} \doteq \frac{1}{2}\left(L\underline{L}+\underline{L}L\right) + \frac{2Mw}{r} + w\mathring{\triangle}^{[s]} + wU^{[s]}_{(k)}. \tag{2.13}$$

In particular, for $k=|s|$, $\Phi^{[s]}$ *(equivalently* $\Psi^{[s]}$*) solves a decoupled wave equation*

$$\Box^{[s]}_g\Phi^{[s]} = U^{[s]}(r)\Phi^{[s]}, \qquad \mathfrak{R}^{[s]}\Psi^{[s]}=0, \tag{2.14}$$

with $\mathfrak{R}^{[s]} \doteq \mathfrak{R}^{[s]}_{(|s|)}$ *and* $U^{[s]} \doteq U^{[s]}_{(|s|)} = s^2\mu(r)$*. For* $k<|s|$*, using* (2.8)*, we can recast* (2.12) *as the constraint equation*

$$\begin{aligned}&\underline{\mathcal{L}}\psi^{[s]}_{(k+1)} - \operatorname{sgn} s(|s|-k-1)\frac{w'}{w}\psi^{[s]}_{(k+1)}\\ &\qquad = -\mathring{\triangle}^{[s]}\psi^{[s]}_{(k)} - \left[U^{[s]}_{(k)}(r)+\frac{2M}{r}\right]\psi^{[s]}_{(k)} + \sum_{j=0}^{k-1}\mathfrak{J}^{[s]}_{(k),(j)}[\psi^{[s]}_{(j)}].\end{aligned} \tag{2.15}$$

Proof. This proof was given in [SRTdC20] for general rotating Kerr black holes; here we repeat the main points for Schwarzschild black holes.

The result follows by recursion in k. To obtain the wave equation (2.12) at kth level, we start from the $(k-1)$th wave equation (2.12). Noting $\underline{L}L+L\underline{L}=2\underline{\mathcal{L}}\mathcal{L}$, we use the definition of $\psi^{[s]}_{(k)}$ in (2.8); we thus obtain the $(k-1)$th transport relation (2.15). Then, we divide by w and apply $\mathcal{L}$. We once again simplify terms using the definitions of $\psi^{[s]}_{(j)}$, $j=0,\dots,k$, except for the term $\mathcal{L}\underline{\mathcal{L}}\psi^{[s]}_{(k)}$.

With our choice of weights, the PDEs for the transformed quantities take the form

$$\mathcal{L}\underline{\mathcal{L}}\psi^{[s]}_{(k)} - \operatorname{sgn} s(|s|-k)\frac{w'}{w}\mathcal{L}\psi^{[s]}_{(k)} + \sum_{X\in\mathfrak{X}}\sum_{j=0}^{k}c^X_{s,k,j}\psi^{[s]}_{(j)} = 0\,,$$

where $\mathfrak{X}=\{\mathring{\triangle}^{[s]},\mathrm{id}\}$ and coefficients $c^X_{s,k,j}$ satisfy the recursive relation

$$\begin{cases} c^X_{s,k,0} &= -\operatorname{sgn} s\left(\dfrac{c^X_{s,k-1,0}}{w}\right)' \\ c^X_{s,k,j} &= -\operatorname{sgn} s\left(\dfrac{c^X_{s,k-1,j}}{w}\right)' + c^X_{s,k-1,j-1} \quad \text{for } \; j=1,\dots,k-1 \end{cases}, \tag{2.16}$$

initialized by the relations, for $j = k = 0, ..., |s|$,

$$
\begin{aligned}
c^{\mathring{\triangle}}_{s,k,k} &= w, \\
c^{\mathrm{id}}_{s,k,k} &= w\left(|s| + \frac{2M(1-3|s|+2s^2)}{r}\right) + \frac{k(2|s|-k-1)}{2}\left(\frac{w'}{w}\right)' \\
&= w\left[|s| + k(2|s|-k-1)\right] + \frac{2Mw}{r}\left[1-3|s|+2s^2-3k(2|s|-k-1)\right].
\end{aligned}
\tag{2.17}
$$

Since $(1/r)' = -w$, we can deduce the properties of $c^{\mathrm{id}}_{s,k,j}$, for $j = 0, \ldots, k-1$, claimed in the statement. □

Remark 2.6 (Full computation of the wave systems in particular cases)**.** We carry out the computations of (the proof of) Proposition 2.5 for particular cases $s \in \mathbb{Z}$. For $|s| = 1$, (2.9) become

$$
\begin{aligned}
\Box_g^{[1]}\phi^{[\pm1]}_{(1)} - \left(1 - \frac{2M}{r}\right)\phi^{[\pm1]}_{(1)} &= 0 \\
\Box_g^{[1]}\phi^{[\pm1]}_{(0)} - \left(1 - \frac{2M}{r}\right)\phi^{[\pm1]}_{(0)} &= \pm\frac{2(r-3M)}{r^2}\phi^{[\pm1]}_{(1)}.
\end{aligned}
$$

For $|s| = 2$, (2.9) become

$$
\begin{aligned}
\Box_g^{[2]}\phi^{[\pm2]}_{(2)} - 4\left(1 - \frac{2M}{r}\right)\phi^{[\pm2]}_{(2)} &= 0, \\
\Box_g^{[2]}\phi^{[\pm2]}_{(1)} - 4\left(1 - \frac{2M}{r}\right)\phi^{[\pm2]}_{(1)} &= \pm\frac{2(r-3M)}{r^2}\phi^{[\pm2]}_{(2)} \mp 6M\phi^{[\pm2]}_{(0)}, \\
\Box_g^{[2]}\phi^{[\pm2]}_{(0)} - 2\left(1 + \frac{2M}{r}\right)\phi^{[\pm2]}_{(0)} &= \pm\frac{4(r-3M)}{r^2}\phi^{[\pm2]}_{(1)}.
\end{aligned}
$$

For $|s| = 3$, (2.9) become

$$
\begin{aligned}
\Box_g^{[3]}\phi^{[\pm3]}_{(3)} - 9\left(1 - \frac{2M}{r}\right)\phi^{[\pm3]}_{(3)} &= 0, \\
\Box_g^{[3]}\phi^{[\pm3]}_{(2)} - 9\left(1 - \frac{2M}{r}\right)\phi^{[\pm3]}_{(2)} &= \pm\frac{2(r-3M)}{r^2}\phi^{[\pm2]}_{(3)} \mp 16M\phi^{[\pm3]}_{(1)}, \\
\Box_g^{[3]}\phi^{[\pm3]}_{(1)} - \left(7 - \frac{6M}{r}\right)\phi^{[\pm3]}_{(1)} &= \pm\frac{4(r-3M)}{r^2}\phi^{[\pm3]}_{(2)} \mp 20M\phi^{[\pm3]}_{(0)}, \\
\Box_g^{[3]}\phi^{[\pm3]}_{(0)} - 3\left(1 + \frac{6M}{r}\right)\phi^{[\pm3]}_{(0)} &= \pm\frac{4(r-3M)}{r^2}\phi^{[\pm3]}_{(2)},
\end{aligned}
$$

Notice that there is no smallness parameter in the coupling of the kth equation to any of the $j = 0, \ldots, k-1$ equations nor to the $(k+1)$th equation.

Remark 2.7 (Rescaling the transformed variables)**.** In what follows, it will be useful to consider rescalings of the unknowns in (2.8). If we let $\psi^{[s]}_{(k)} = c_k(r)\tilde{\psi}^{[s]}_{(k)}$, then (2.8) becomes

$$\mathcal{L}\tilde{\psi}^{[s]}_{(k)} = \frac{wc_{k+1}}{c_k}\tilde{\psi}^{[s]}_{(k+1)} + \operatorname{sgn} s\frac{c_k'}{c_k}\tilde{\psi}^{[s]}_{(k)}, \tag{2.18}$$

and the PDEs (2.12) become

$$\begin{aligned}&\left[\mathfrak{R}^{[s]}_{(k)} - \left(\frac{c_k'}{c_k}\right)' - \operatorname{sgn} s\frac{c_k'}{c_k}\underline{\mathcal{L}}\right]\tilde{\psi}^{[s]}_{(k)} \\ &\quad= -\operatorname{sgn} s\left(\frac{c_k'}{c_k} - (|s|-k)\frac{w'}{w}\right)\frac{wc_{k+1}}{c_k}\tilde{\psi}^{[s]}_{(k+1)} + w\frac{c_j}{c_k}\mathfrak{J}^{[s]}_{(k),(j)}[\tilde{\psi}^{[s]}_{(j)}].\end{aligned} \tag{2.19}$$

2.2.3 Energy norms

In this section, we introduce the definitions of the energy norms we will use throughout the rest of the paper. Fix $s \in \mathbb{Z}$, $-\infty \leq \tau_1 < \tau < \tau_2 \leq \infty$ and recall the definitions (2.7) and (2.8). Then, set

$$\tilde{\psi}^{[s]}_{(k)} \doteq c_k^{-1}(r)\psi^{[s]}_{(k)}, \quad c_k(r) = \begin{cases} r^{-(|s|-k)}, & s>0 \\ (1-\mu)^{|s|-k}, & s<0 \end{cases}.$$

First order energy norms. For $k \in \{0, \dots, |s|\}$, $p \in (-1, 2]$, and $q \in [0, 1]$, we define energy fluxes

$$\begin{aligned}\mathbb{E}^{[s]}_{(k),p,q}(\tau) &\doteq \int_{\Sigma_\tau}\left(r^p|L\tilde{\psi}^{[s]}_{(k)}|^2 + \frac{r^{-2}}{(1-\mu)^{1+q}}|\underline{L}\tilde{\psi}^{[s]}_{(k)}|^2\right)\mathrm{d}r\mathrm{d}\sigma \\ &\quad + \int_{\Sigma_\tau} r^{-2}|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2\mathrm{d}r\mathrm{d}\sigma, \\ \mathbb{E}^{[s]}_{(k),\mathcal{H}^+,q}(\tau_1,\tau_2) &\doteq \int_{\mathcal{H}^+_{(\tau_1,\tau_2)}}\left(|\underline{L}\tilde{\psi}^{[s]}_{(k)}|^2 + \mathbb{1}_{\{q=1\}}|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2\right)\mathrm{d}\sigma\mathrm{d}\tau', \\ \mathbb{E}^{[s]}_{(k),\mathcal{I}^+,p}(\tau_1,\tau_2) &\doteq \lim_{v\to\infty}\int_{S_{(\tau_1,\tau_2)}(v)}\left(r^p|L\tilde{\psi}^{[s]}_{(k)}|^2 + \mathbb{1}_{\{p=2\}}|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2\right)\mathrm{d}\sigma\mathrm{d}\tau',\end{aligned}$$

where if $p = q = 0$ we drop these superscripts, and $\mathbb{1}_{\{x=y\}}$ is 1 if $x = y$ and 0 otherwise. Here, $S_{(\tau_1,\tau_2)}(v)$ denote null hypersurfaces which approach $\mathcal{I}^+_{(\tau_1,\tau_2)}$ as $v \to \infty$, and $\mathcal{H}^+_{(\tau_1,\tau_2)} = \mathcal{H}^+ \cap \left(\cup_{\tau'\in[\tau_1,\tau_2]}\Sigma_\tau\right)$.

We also define bulk energies as follows. For $p \in [0,2]$ and $q \in [0,1]$,

$$\begin{aligned}
\mathbb{I}^{[s]}_{\delta,p,q}(\tau_1,\tau_2) \doteq& \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{-3+p}(2-p+r^{-1})|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+ \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{p-1}(p+r^{-1})|L\tilde{\psi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+ \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{r^{-1-\delta}}{(1-\mu)^{1+q}}|\underline{L}\tilde{\psi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau', \\
\mathbb{I}^{[s],\deg}_{\delta,p,q}(\tau_1,\tau_2) \doteq& \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{-3+p}(2-p+r^{-1})\left(1-\frac{3M}{r}\right)^2|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+ \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{p-1}(p+r^{-1})\left(1-\frac{3M}{r}\right)^2|L\tilde{\psi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+ \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{r^{-1-\delta}}{(1-\mu)^{2+q}}(1+q(1-\mu))\left(1-\frac{3M}{r}\right)^2|\underline{L}\tilde{\psi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+ \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} r^{-1-\delta}|(L-\underline{L})\tilde{\psi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau',
\end{aligned}$$

where if $\delta = 1$, $p = q = 0$ we may drop the subscripts. Note that these two definitions differ by the fact that in the second one the r-weights on some of the derivative degenerates (hence the notation "deg") at $r = 3M$.

Now take $k \in \{0,\dots,|s|-1\}$. For $s > 0$, $p \in [0,2)$ and $q \in [0,1]$, define

$$\begin{aligned}
\mathbb{I}^{[s]}_{(k),p,q}(0,\tau) \doteq& \int_0^\tau\int_{\Sigma_{\tau'}} r^{-1}\left(r^p|L\tilde{\psi}^{[s]}_{(k)}|^2 + \frac{1}{(1-\mu)^2}|\underline{L}\tilde{\psi}^{[s]}_{(k)}|^2\right)\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+ \int_0^\tau\int_{\Sigma_{\tau'}} r^{p-2}|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau'.
\end{aligned}$$

We can may also denote this same norm by $\mathbb{I}^{[s]}_{(k),\delta,p,q}(0,\tau)$ for convenience, even though it does not depend on the parameter δ. For $s < 0$, $q \in [0,1]$, $\delta \in (0,1]$ and $p \in [0,2]$, define

$$\begin{aligned}
\mathbb{I}^{[s]}_{(k),\delta,p,q}(0,\tau) \doteq& \int_0^\tau\int_{\Sigma_{\tau'}} \left(r^{\max\{p,1\}}|L\tilde{\psi}^{[s]}_{(k)}|^2 + \frac{r^{-1-\delta}}{(1-\mu)^{2+q}}(1+q(1-\mu))|\underline{L}\tilde{\psi}^{[s]}_{(k)}|^2\right)\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau' \\
&+ \int_0^\tau\int_{\Sigma_{\tau'}} r^{\max\{p,1\}-2}(2-p+r^{-1})|\mathring{\nabla}\tilde{\psi}^{[s]}_{(k)}|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau'.
\end{aligned}$$

Higher order energy norms. If X is a vector field on $\mathcal{M}$ or $\mathrm{int}(\mathcal{M})$, adding a superscript X to any of the previous norms denotes the same quantity with $\tilde{\psi}^{[s]}_{(k)}$ replaced by $X\tilde{\psi}^{[s]}_{(k)}$. We also set

$$\mathbb{E}^{[s],\,J}_{(k),p,q}(\tau) \doteq \sum_{X\in\{\mathrm{id},T,r^{-1}\mathring{\nabla}^{[s]}\}} \mathbb{E}^{[s],\,J-1,\,X}_{(k),p,q}(\tau),$$

with $\mathbb{E}^{[s],\,0}_{(k),p,q} = \mathbb{E}^{[s]}_{(k),p,q}$ defined above. We take definitions for the fluxes $\mathbb{E}^{[s],\,J}_{(k),\mathcal{H}^+,q}(0,\tau)$ and $\mathbb{E}^{[s],\,J}_{(k),\mathcal{I}^+,p}(0,\tau)$, as well as for the non-degenerate bulk term $\mathbb{I}^{[s],\,J}_{(k),\delta,p,q}(0,\tau)$. Finally, for $k<|s|$ and $J=|s|$, we define

$$\mathbb{I}^{[s],\,\deg,\,|s|-k}_{(k),\delta,p,q}(0,\tau) \doteq \mathbb{I}^{[s],\,|s|-k-1}_{(k),p,q}(0,\tau) + \sum_{X\in\{T,r^{-1}\mathring{\nabla}^{[s]}\}} \mathbb{I}^{[s],\,\deg,\,|s|-k-1,\,X}_{(k),p,q}(0,\tau).$$

2.3 Notational conventions

In our estimates throughout this article, we use B to denote possibly large positive constants and b to denote possibly small positive constants depending only on $M>0$ and $s\in\mathbb{Z}$. Whenever the constant depends, additionally, on another parameter that has not (yet) been fixed, say x, we write $B(x)$ or $b(x)$. We rarely keep track of changes in such constants, and thus by convention we have

$$B+B=BB=B, \qquad b+b=bb=b, \qquad B+b=B,$$
$$Bb=B, \qquad b^{-1}=B, \qquad \text{etc.}$$

3 Estimates for the top level wave equation

In this section, we prove Theorem 1.1 for the transformed wave equation (2.14). To be precise, we will show:

Theorem 3.1 (EB and ILED for $k=|s|$)**.** *Fix $M>0$, and $s\in\mathbb{Z}$ with $|s|\le 4$. For any $p\in[0,2]$, $q\in\{0,1\}$, and $\delta\in(0,1]$, and all $\tau>0$, we have the following uniform-in-τ estimates:*

- *energy boundedness without derivative loss:*

$$\mathbb{E}^{[s]}_{(|s|),p,q}(\tau) \le B\mathbb{E}^{[s]}_{(|s|),p,q}(0); \tag{3.1}$$

- *integrated local energy decay with loss of one derivative at trapping:*

$$\mathbb{I}^{[s],\,\deg}_{(|s|),\delta,p,q}(0,\tau) \le B(\delta)\mathbb{E}^{[s]}_{(|s|),p,q}(0). \tag{3.2}$$

Corollary 3.2 (Decay for $k=|s|$)**.** *Fix $M>0$, and $s\in\mathbb{Z}$ with $|s|\leq 4$. We have the following uniform-in-τ estimates:*

- *energy decay with derivative loss:*
$$\mathbb{E}^{[s]}_{(|s|),0,1}(\tau)\leq (1+\tau)^{-2}B\mathbb{E}^{[s],\,2}_{(|s|),2,1}(0); \tag{3.3}$$
- *pointwise decay with derivative loss: for any $\delta>0$,*
$$\sup_{\Sigma_\tau}\left|\Psi^{[s]}\right|^2\leq (1+\tau)^{-2}B\mathbb{E}^{[s],\,4}_{(|s|),2,1}(0). \tag{3.4}$$

Remark 3.3 (Restrictions on s)**.** While the restriction to $s\in\mathbb{Z}$ is important to obtain the DHR transformed equation (2.14) in Proposition 2.5, we expect the restriction to $|s|\leq 4$ to be merely technical. This restriction is introduced so that a particular choice of s-independent multiplier estimate goes through, see the proof of Proposition 3.4 below. It seems plausible that a more refined, possibly s-dependent, choice of multiplier would allows us to obtain the same statement for all $s\in\mathbb{Z}$. However, in view of the fact that, among integer s, only the cases $s\in\{0,\pm1,\pm2\}$ are of physical significance, we have chosen not to attempt this optimization.

3.1 Energy boundedness and integrated local energy decay

In this section, we prove Theorem 3.1. We note that the proof given here is by no means novel: Theorem 3.1 is a classical result that can be obtained by following the methods of the lecture notes [DR13] or the more recent [DHR19b, Section 11]. We start with a slightly less sharp version:

Proposition 3.4 (Basic EB and ILED)**.** *Fix $s\in\mathbb{Z}$ with $|s|\leq 4$. The following estimates hold:*
$$\mathbb{E}^{[s]}_{(|s|)}(\tau)+\mathbb{E}^{[s]}_{|s|,\mathcal{H}^+}(0,\tau)+\mathbb{E}^{[s]}_{(|s|),\mathcal{I}^+}(0,\tau)\leq B\mathbb{E}^{[s]}_{(|s|)}(0), \tag{3.5}$$
$$\mathbb{I}^{[s],\,\mathrm{deg}}_{(|s|)}(0,\tau)\leq B\mathbb{E}^{[s]}_{(|s|)}(0). \tag{3.6}$$

Proof. Estimate (3.5) is obtained by using T as a multiplier for (2.14): i.e. we multiply (2.14) by $\overline{T\Phi}$, take the real part, and integrate by parts over $\mathrm{int}(\mathcal{M})\cap\{0\leq t^*\leq\tau\}$. Estimate (3.6) requires more work, and we will split its proof into several steps.

Step 0: decomposition in angular modes. To simplify the proof of (3.6), we start by expanding Ψ in spin-weighted spherical harmonics using Lemma 2.2:
$$\Psi^{[s]}_\ell(t,r)\doteq\int_{\mathbb{S}^2}\Psi^{[s]}(t,r,\theta,\phi)\sum_{m\leq\ell}S^{[s]}_{m\ell}(\theta,\phi)\mathrm{d}\sigma,\quad \ell\geq|s|,$$

satisfy the PDE

$$(\Psi_\ell^{[s]})'' - T^2\Psi_\ell^{[s]} - \mathcal{V}_\ell^{[s]}(r)\Psi_\ell^{[s]} = 0, \tag{3.7}$$
$$\mathcal{V}_\ell^{[s]}(r) = \frac{1-\mu}{r^2}\left(\ell(\ell+1) + \frac{2M}{r}(1-s^2)\right).$$

In what follows, we drop the superscript $[s]$ for readability, both from the transformed variable and its norms. We will add a superscript ℓ to the energy norms of Section 2.2.3 to indicate norms where $\Psi^{[s]}$ has been replaced by $\Psi_\ell^{[s]}$, and we will drop the subscript $(|s|)$.

Step 1: Morawetz estimate. Now we apply multiplier estimates to the ℓ-dependent PDE (3.7). We set

$$f(r) \doteq \begin{cases} \left(1-\frac{3M}{r}\right)\left(1+\frac{M}{r}\right) & \text{if } \ell \geq \max\{1,|s|\} \\ 1 & \text{if } s=\ell=0 \end{cases},$$
$$y(r) \doteq \left(1-\frac{3M}{r}\right)^3.$$

For the first multiplier estimate, we multiply (3.7) by $\overline{f'\Psi_\ell + 2f\Psi_\ell'}$, take the real part, and integrate by parts in $\{0 \leq t^* \leq \tau\}$ to obtain

$$\begin{aligned} b\int_0^\tau\int_{\Sigma_{\tau'}} &\left[\frac{1-\mu}{r^2}|\Psi_\ell'|^2 + \frac{(1-\mu)}{r^3}\left[\ell(\ell+1)\left(1-\frac{3M}{r}\right)+r^{-1}\right]|\Psi_\ell|^2\right]\mathrm{d}r^*\mathrm{d}\tau' \\ &\leq \int_0^\tau\int_{\Sigma_{\tau'}}\left[2f'|\Psi_\ell'|^2 - f\mathcal{V}_\ell'|\Psi_\ell|^2 - \frac{1}{2}f'''|\Psi_\ell|^2\right]\mathrm{d}r^*\mathrm{d}\tau' \\ &= \left(\int_{\Sigma_\tau} - \int_{\Sigma_0}\right)\left(\mathrm{Re}[(f'\Psi_\ell + 2f\Psi_\ell')\overline{T\Psi_\ell}]\right)\mathrm{d}r^*\mathrm{d}\sigma \\ &\quad + \int_0^\tau\int_{\Sigma_{\tau'}}\left[f|\Psi_\ell'|^2 - f|T\Psi_\ell|^2\right]'\mathrm{d}r^*\mathrm{d}\tau' \\ &\leq B\mathbb{E}^\ell(\tau) + B\mathbb{E}^\ell(0) + B\mathbb{E}^\ell_{\mathcal{H}^+}(0,\tau) + B\mathbb{E}^\ell_{\mathcal{I}^+}(0,\tau), \end{aligned}$$

as long as[4] $|s| \leq 4$. Then, we multiply (3.7) by $\overline{2y\Psi_\ell'}$, take the real part, and integrate by parts in $\{0 \leq t^* \leq \tau\}$ to obtain, in an entirely analogous manner,

$$b\int_0^\tau\int_{\Sigma_{\tau'}}\frac{(1-\mu)}{r^2}\left(1-\frac{3M}{r}\right)|T\Psi_\ell|^2\mathrm{d}r^*\mathrm{d}\tau'$$

[4] The restriction here arises from our choice of factor $(1+M/r)$ in the definition of f. It is conceivable that other choices, perhaps depending on $|s|$, would allow us to drop this requirement. See Remark 3.3.

$$\begin{aligned}
&\leq \int_0^\tau \int_{\Sigma_{\tau'}} \left[y'|\Psi_\ell'|^2 + y'|T\Psi_\ell|^2\right] \mathrm{d}r^*\mathrm{d}\tau' \\
&= \int_0^\tau \int_{\Sigma_{\tau'}} (y\mathcal{V}_\ell)'|\Psi_\ell|^2 \mathrm{d}r^*\mathrm{d}\tau' + \left(\int_{\Sigma_\tau} - \int_{\Sigma_0}\right)\left(2y\,\mathrm{Re}[\Psi_\ell'\overline{T\Psi_\ell}]\right)\mathrm{d}r^*\mathrm{d}\sigma \\
&\quad + \int_0^\tau \int_{\Sigma_{\tau'}} \left[y|\Psi_\ell'|^2 - y|T\Psi_\ell|^2\right]' \mathrm{d}r^*\mathrm{d}\tau' \\
&\leq B\int_0^\tau \int_{\Sigma_{\tau'}} \frac{(1-\mu)}{r^3}\left(\ell(\ell+1)+r^{-1}\right)|\Psi_\ell|^2\mathrm{d}r^*\mathrm{d}\tau' \\
&\quad + B\mathbb{E}^\ell(\tau) + B\mathbb{E}^\ell(0) + B\mathbb{E}^\ell_{\mathcal{H}^+}(0,\tau) + B\mathbb{E}^\ell_{\mathcal{I}^+}(0,\tau).
\end{aligned}$$

By combining the two previous estimates, we have

$$\mathbb{I}^{\mathrm{deg},\ell}(0,\tau) \leq B\mathbb{E}^\ell(\tau) + B\mathbb{E}^\ell(0) + B\mathbb{E}^\ell_{\mathcal{H}^+}(0,\tau) + B\mathbb{E}^\ell_{\mathcal{I}^+}(0,\tau).$$

Step 2: sum over angular modes. Making use of Lemma 2.2, we can use the L^2 orthogonality of the angular modes to conclude to sum the previous estimate for all $\ell \geq |s|$: we have

$$\mathbb{I}^{\mathrm{deg}}(0,\tau) \leq B\mathbb{E}(\tau) + B\mathbb{E}(0) + B\mathbb{E}_{\mathcal{H}^+}(0,\tau) + B\mathbb{E}_{\mathcal{I}^+}(0,\tau),$$

and thus (3.6) follows after applying (3.5). □

Having obtained an energy boundedness statement and an integrated energy decay statement, we can now improve their r weights in the two limits $r \to \infty$ and $r \to 2M$:

Proof of Theorem 3.1. In Proposition 3.4, we have already shown a version of the statement with $p = q = 0$ and $\delta = 1$. To conclude the proof, we just need to show that the large r weights can be improved using parameters $p \in (0, 2]$ and $\delta \in (0, 1]$ and that the r weights close to $r = 2M$ can be improved using a parameter $q \in (0, 1]$.

Weights at $r \to \infty$. For the improvement in the bulk term only, related to the new δ constant, we employ a large r Morawetz multiplier: in the language of the proof of Proposition 3.4, we choose $y(r) = (1 - r^{-\delta})\chi$ for χ supported at sufficiently large r, so that the error is contained in a bounded $|r|$ region where Proposition 3.4 can be used. For the improvement in the bulk term and energy fluxes related to the p constant, we rely on the r^p-weighted estimates of [DR10]: we multiply (2.14) by $r^p(1 + 4Mr^{-1})\chi\overline{L\Psi}$ for χ supported at sufficiently large r, take the real part and integrate by parts. It is clear from, for instance, the identities in [DHR19a, Page 46] or [SRTdC23, Section 4.1.4] that the errors produced can be controlled by Proposition 3.4.

Weights at $r \approx 2M$. This improvement can be done through the redshift multiplier introduced in [DR09]: we multiply (2.14) by $(1-\mu)^{-q} r^4 \chi \overline{\underline{L}\Psi}$ for χ supported at sufficiently small $r-2M$, take the real part and integrate by parts. It is clear from, for instance, the identities in [DHR19a, Page 92] or [SRTdC23, Section 4.1.4] that the cost to obtain the improvement can be controlled by Proposition 3.4. □

We conclude the section with a trivial corollary of the previous results that will be useful in the next section:

Lemma 3.5 (Nondegenerate ILED)**.** *Fix $s \in \mathbb{Z}$ with $|s| \leq 4$. For any $p \in [0,2]$ and $q \in [0,1]$, the following estimate holds:*

$$\mathbb{I}^{[s]}_{(|s|),1,p,q}(0,\tau) \leq B\mathbb{E}^{[s]}_{|s|,p,q}(0) + B\mathbb{E}^{[s],T}_{|s|,0,0}(0).$$

3.2 Decay of the energy and the solution

In this section, we prove Corollary 3.2. The proof given here is, again, not novel: it is based on Dafermos and Rodnianski's r^p method [DR10].

Proof of Corollary 3.2. In this proof, we will lighten the notation as follows: we drop the superscript $[s]$ and the subscript $(|s|)$ from our definition of norms; we take $\delta = 1$ and drop this subscript.

Take $p \in [1,2]$, $0 < \tau_A < \tau_B < \infty$. By direct inspection of the definitions of the norms,

$$\int_{\tau_A}^{\tau_B} \mathbb{E}_{p-1,1}(\tau)\mathrm{d}\tau \leq B\mathbb{I}_{p,1}(\tau_A,\tau_B)$$

and thus, by Lemma 3.5, we have

$$\int_{\tau_A}^{\tau_B} \mathbb{E}_{p-1,1}(\tau)\mathrm{d}\tau \leq B\mathbb{E}_{p,1}(\tau_A) + B\mathbb{E}^T_{0,0}(\tau_A).$$

Step 1: decay along dyadic sequences. Fix some $0 < \tau_0 < \infty$. Taking $p = 1$ in the above, we have

$$\int_{\tau_0}^{\infty} \left[\mathbb{E}^T_1(\tau) + \mathbb{E}^0_{1,1}(\tau)\right] \mathrm{d}\tau \leq B\mathbb{E}_{2,1}(\tau_0) + B\mathbb{E}^T_{2,1}(\tau_0) + B\mathbb{E}^{TT}_{0,0}(\tau_0)$$

$$\implies \begin{cases} \mathbb{E}_1^T(\tau_n) \leq \frac{B}{\tau_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}_{2,1}^T(\tau_0)+\mathbb{E}_{0,0}^{TT}(\tau_0)\right), \\ \mathbb{E}_1^0(\tau_n) \leq \frac{B}{\tau_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}_{2,1}^T(\tau_0)+\mathbb{E}_{0,0}^{TT}(\tau_0)\right), \end{cases}$$

along a dyadic sequence such as $\tau_n = 2^n\tau_0$.

We now state two estimates for $p = 0$. Firstly, we note

$$\begin{aligned} \int_{\tau_n}^{\tau_{n+1}} \mathbb{E}_{0,1}^T(\tau)\mathrm{d}\tau &\leq B\mathbb{E}_{1,1}^T(\tau_n) + B\mathbb{E}_{0,0}^{TT}(\tau_n) \\ &\leq \frac{B}{\tau_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}_{2,1}^T(\tau_0)\right)+\mathbb{E}_{0,0}^{TT}(\tau_0) \\ \implies \mathbb{E}_{0,1}^T(\overline{\tau}_n) &\leq \frac{B}{\overline{\tau}_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}_{2,1}^T(\tau_0)+\mathbb{E}_{0,0}^{TT}(\tau_0)\right), \end{aligned}$$

along another dyadic sequence $\overline{\tau}_n \in [\frac{3}{4}\tau_n, \frac{5}{4}\tau_n]$. The last inequality in the first line follows by the first estimate in this step, which is used to control $\mathbb{E}_{1,1}^T(\tau_n)$, and energy boundedness (Proposition 3.4), which is used to control $\mathbb{E}_{0,1}^{TT}(\tau_n)$. Secondly, we have

$$\begin{aligned} \int_{\overline{\tau}_n}^{\overline{\tau}_{n+1}} \mathbb{E}_{0,1}(\tau)\mathrm{d}\tau &\leq B\mathbb{E}_{1,1}(\overline{\tau}_n) + B\mathbb{E}_{0,0}^T(\overline{\tau}_n) \leq B\mathbb{E}_{1,1}(\tau_n) + B\mathbb{E}_{0,0}^T(\overline{\tau}_n) \\ &\leq \frac{B}{\overline{\tau}_n}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}_{2,1}^T(\tau_0)+\mathbb{E}_{0,0}^{TT}(\tau_0)\right) \\ \implies E_{0,1}(\overline{\overline{\tau}}_n) &\leq \frac{B}{\overline{\overline{\tau}}_n^2}\left(\mathbb{E}_{2,1}(\tau_0)+\mathbb{E}_{2,1}^T(\tau_0)+\mathbb{E}_{0,0}^{TT}(\tau_0)\right), \end{aligned}$$

along another dyadic sequence $\overline{\overline{\tau}}_n \in [\frac{3}{4}\overline{\tau}_n, \frac{5}{4}\overline{\tau}_n]$. In the first line, we have used energy boundedness (Theorem 3.1) to control $\mathbb{E}_{1,1}(\overline{\tau}_n)$ and then the previous estimates.

Step 2: decay along foliation. By combining the decay of $\mathbb{E}_{0,1}$ along a dyadic sequence established in Step 1 with the energy boundedness of Proposition 3.4, we deduce that $\mathbb{E}_{0,1}(\tau)$ decays in τ. Sobolev estimates using commutation with angular symmetries then imply pointwise decay for the field Ψ away from $r = 2M$. □

4 Estimates for the lower level wave equations

In this section, we prove Theorem 1.1 for the transformed wave equation (2.14). To be precise, we will show:

Theorem 4.1 (EB and ILED for $k \leq |s|$)**.** *Fix $M > 0$, $s \in \mathbb{Z}$ with $|s| \leq 4$, and $k \in \{0, \ldots, |s|-1\}$. For any $p \in [0,2)$, $q \in [0,1]$, and $\delta \in (0,1]$, and all $\tau > 0$, we have the following uniform-in-τ estimates:*

- *energy boundedness without derivative loss:*

$$\sum_{k=0}^{|s|} \mathbb{E}^{[s],|s|-k}_{(k),p,q}(\tau) \leq B \sum_{k=0}^{|s|} \mathbb{E}^{[s],\,|s|-k}_{(k),p,q}(0); \tag{4.1}$$

- *integrated local energy decay with loss of one derivative at trapping:*

$$\sum_{k=0}^{|s|} \mathbb{I}^{[s],\,\deg,\,|s|-k}_{(k),\delta,p,q}(0,\tau) \leq B(\delta) \sum_{k=0}^{|s|} \mathbb{E}^{[s],\,|s|-k}_{(k),p,q}(0). \tag{4.2}$$

Corollary 4.2 (Decay for $k \leq |s|$)**.** *Fix $M > 0$, and $s \in \mathbb{Z}$ with $|s| \leq 4$. We have the following uniform-in-τ estimates:*

- *energy decay with derivative loss: for any $\eta \in (0,1)$,*

$$\sum_{k=0}^{|s|} \mathbb{E}^{[s],|s|-k}_{(j),0,1}(\tau) \leq (1+\tau)^{-2+\eta} B(\eta) \sum_{k=0}^{|s|} \mathbb{E}^{[s],\,|s|-k+2}_{(|s|),2-\eta,1}(0); \tag{4.3}$$

- *pointwise decay with derivative loss: for any $\delta > 0$ and $\eta \in (0,1)$,*

$$\sup_{k \leq |s|} \sup_{\Sigma_\tau} |\tilde{\psi}^{[s]}_{(k)}|^2 \leq (1+\tau)^{-2+\eta} B(\eta) \sum_{k=0}^{|s|} \mathbb{E}^{[s],\,|s|-k+4}_{(|s|),2-\eta,1}(0). \tag{4.4}$$

If $s < 0$, the above holds with $\eta = 0$ as well.

In these statements, as in those of Section 3, we expect the condition that $|s| \leq 4$ to be suboptimal, see Remark 3.3 for details.

4.1 First order energy boundedness and integrated local energy decay

In this section, we obtain a weak form of Theorem 4.1. Namely, we will show that all first order energy norms of $\psi^{[s]}_{(k)}$ are controlled up to zeroth order energy norms involving $\psi^{[s]}_{(k+1)}$:

Proposition 4.3 (First order EB and ILED)**.** *Take $s \in \pm\mathbb{Z}_{\leq 4}$. If $s < 0$, for all $\delta \in (0,1]$, $p \in [0,2]$ and $q \in [0,1]$, we have*

$$\mathbb{E}^{[s]}_{(k),p,q}(\tau) + \mathbb{E}^{[s]}_{(k),\mathcal{H}^+,q}(0,\tau) + \mathbb{E}^{[s]}_{(k),\mathcal{I}^+,p}(0,\tau) + b(\delta)\mathbb{I}^{[s]}_{(k),\delta,p,q}(0,\tau)$$

$$\begin{aligned}&\leq B\sum_{j=0}^{k}\mathbb{E}^{[s]}_{(j),p,q}(0)+B\left(\int_{\Sigma_\tau}+\int_{\Sigma_0}\right)w|\tilde{\psi}^{[s]}_{(k+1)}|\mathrm{d}r^*\mathrm{d}\sigma\\&\quad+B\int_0^\tau\int_{\Sigma_{\tau'}}\frac{w}{r}(1+r\mathbb{1}_{\{p=2\}})|\tilde{\psi}^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'.\end{aligned}$$

If $s>0$, for all $p\in[0,2)$ and $q\in[0,1]$, we have

$$\begin{aligned}&\mathbb{E}^{[s]}_{(k),p,q}(\tau)+\mathbb{E}^{[s]}_{(k),\mathcal{H}^+,q}(0,\tau)+\mathbb{E}^{[s]}_{(k),\mathcal{I}^+,p}(0,\tau)+\mathbb{I}^{[s]}_{(k),p,q}(0,\tau)\\&\quad\leq B\sum_{j=0}^{k}\mathbb{E}^{[s]}_{(j),p,q}(0)+B\left(\int_{\Sigma_\tau}+\int_{\Sigma_0}\right)w|\tilde{\psi}^{[s]}_{(k+1)}|\mathrm{d}r^*\mathrm{d}\sigma\\&\quad\quad+B\int_0^\tau\int_{\Sigma_{\tau'}}wr^{p-1}|\tilde{\psi}^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'.\end{aligned}$$

The proof of Proposition 4.3 follows by combining the results of the subsections below: Lemma 4.4 with either Lemmas 4.7 and 4.8 (if $s<0$) or Lemmas 4.6 and 4.9 (if $s>0$). The last four lemmas, though nontrivial, can be viewed as mostly technical. It is Lemma 4.4 which is the heart of the proof of Proposition 4.3 and, indeed, Theorem 4.1.

4.1.1 Estimates with suboptimal weights

We begin by closing estimates for the transformed variables with weaker weights as $r\to\infty$ and $r\to 2M$ than those in Proposition 4.3. This will be loosely based on the insights and, to large extent, approach of [SRTdC20], though see Remark 4.5 for a comparison. Our result of the section is:

Lemma 4.4 (EB and ILED with suboptimal weights)**.** *Fix $s\in\mathbb{Z}$ with $1\leq|s|\leq 4$. Then, for all $k\in\{0,\dots,|s|-1\}$, we have the estimate*

$$\begin{aligned}&\int_0^\tau\int_{\Sigma_{\tau'}}w\left[|\partial_{r^*}\psi^{[s]}_{(k)}|^2+|T\psi^{[s]}_{(k)}|^2+r^{-1}|\mathring{\nabla}^{[s]}\psi^{[s]}_{(k)}|^2+r^{-1}|\psi^{[s]}_{(k)}|^2\right]\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\&\quad+\int_{\Sigma_\tau}\left[r^{-2}|\underline{L}\psi^{[s]}_{(k)}|^2+(1-\mu)|L\psi^{[s]}_{(k)}|^2+w|\mathring{\nabla}^{[s]}\psi^{[s]}_{(k)}|^2\right]dr^*d\sigma\\&\leq B\int_0^\tau\int_{\Sigma_{\tau'}}wr^{-1}|\tilde{\psi}^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'+B\int_{\Sigma_\tau}w|\psi^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma\\&\quad+B\int_{\Sigma_0}w|\psi^{[s]}_{(k+1)}|^2\mathrm{d}r^*\mathrm{d}\sigma+B\sum_{j=0}^{k}\mathbb{E}_j(0).\end{aligned}$$

Let us start with the easier case $k=0$. Note that, for $|s|=1$, this is the only case we need to study.

Proof of Lemma 4.4 if $k=0$. Consider the wave equation (2.12) with $k=0$; in particular notice that, in this case, $\mathfrak{J}^{[s]}_{(k),(j)}\equiv 0$. The proof follows in two steps. First, in Step 1, we repeat the proof of Proposition 3.4, now applied to (2.12) with $k=0$, to obtain a (conditional) energy boundedness statement and a degenerate integrated local energy decay statement. Then, in Step 2, we will remove the degeneracy, thus recovering integrated (conditional) control over all derivatives of the $k=0$ transformed variable.

In this proof, and all other proofs of Section 4 from this point onwards, we will drop the superscripts $[s]$ and the parenthesis in the subscripts for readibility.

Step 1: energy estimate and degenerate bulk estimate. One can check easily that the same choices of multiplier currents from the proof of Proposition 3.4 lead, for (2.12) with $k=0$, to the estimate

$$\begin{aligned}
&\int_0^\tau\int_{\Sigma_{\tau'}} w\left[|\psi_0'|^2+\left(1-\frac{3M}{r}\right)\left(|T\psi_0|^2+\frac{1}{r}|\mathring{\slashed{\nabla}}\psi_0|^2\right)+\frac{1}{r}|\psi_0|^2\right]\mathrm{d}r^*\mathrm{d}\tau'\\
&\quad\leq B\mathbb{E}_0(0)+B\int_0^\tau\int_{\Sigma_{\tau'}} w\left(r^{-1}|\psi_1\psi_0'|+rw|\psi_1\psi_0|+rw|\psi_1|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau\\
&\quad\leq B\mathbb{E}_0(0)+B\int_0^\tau\int_{\Sigma_{\tau'}} wr^{-2}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau,
\end{aligned}$$

at least for $|s|\leq 6$, and

$$\begin{aligned}
&\int_{\Sigma_\tau}\left[r^{-2}|\underline{L}\psi_0|^2+(1-\mu)|L\psi_0|^2+w|\mathring{\slashed{\nabla}}\psi_0|^2\right]dr^*d\sigma\\
&\quad\leq B\mathbb{E}_0(0)+B\int_0^\tau\int_{\Sigma_{\tau'}} wr^{-2}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau.
\end{aligned}$$

We have used here that (2.8) $\Leftrightarrow T\psi_0=w\psi_1+\operatorname{sgn}s\,\psi_0'$ to simplify the error terms arising from application of the T multiplier.

Step 2: removing the degeneration. Now take the constraint equation (2.15) with $k=0$, multiply by $\overline{\psi_0}$ and integrate by parts to obtain

$$\begin{aligned}
&\int_{\mathbb{S}^2}\frac{w}{r}\left(|\mathring{\slashed{\nabla}}\psi_0|^2+\left(|s|+\frac{2M}{r}(1-3|s|+2s^2)\right)|\psi_0|^2\right)\mathrm{d}\sigma\\
&\quad=\int_{\mathbb{S}^2}\frac{w}{r}\left[w|\psi_1|^2+2\operatorname{sgn}s\operatorname{Re}[\psi_1\overline{\psi_0}']+\operatorname{sgn}s\left(|s|\frac{w'}{w}-\frac{1}{r}\right)\operatorname{Re}[\psi_1\overline{\psi_0}]\right]\mathrm{d}\sigma\\
&\qquad-\int_{\mathbb{S}^2}\underline{\mathcal{L}}\left(\frac{w}{r}\operatorname{Re}[\overline{\psi_0}\psi_1]\right)\mathrm{d}\sigma.
\end{aligned}$$

Thus, we have the estimate

$$\begin{aligned}
&\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-1}\left[|\mathring{\nabla}\psi_0|^2+|\psi_0|^2\right]\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau \\
&\quad\leq B\int_0^\tau \int_{\Sigma_{\tau'}} |\psi_0'|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau + B\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-2}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau \\
&\qquad + B\int_{\Sigma_\tau} wr^{-1}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma + B\int_{\Sigma_0} wr^{-1}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma + B\mathbb{E}_0(0).
\end{aligned}$$

Using Step 1, and the fact that (2.8) $\Leftrightarrow T\psi_0 = w\psi_1 + \operatorname{sgn} s\psi_0'$ (which allows us to directly estimate the bulk term in $T\psi_0$), we obtain

$$\begin{aligned}
&\int_0^\tau \int_{\Sigma_{\tau'}} w\left[|\psi_0'|^2+|T\psi_0|^2+r^{-1}|\mathring{\nabla}\psi_0|^2+r^{-1}|\psi_0|^2\right]\mathrm{d}r^*\mathrm{d}\tau' \\
&\qquad + \int_{\Sigma_\tau}\left[r^{-2}|\underline{L}\psi_0|^2+(1-\mu)|L\psi_0|^2+w|\mathring{\nabla}\psi_0|^2\right]dr^*d\sigma \\
&\quad\leq B\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-2}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau + B\int_{\Sigma_\tau}\frac{w}{r}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma \\
&\qquad + B\int_{\Sigma_0}\frac{w}{r}|\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma + B\mathbb{E}_0(0).
\end{aligned}$$

This concludes the proof of Lemma 4.4 in the case $k=0$. □

Studying the $k=0$ case has us provided a strategy to try to address the $k<|s|$ wave equations (2.12). However, in the more involved $k\geq 1$ case, as we will see shortly, this must be supplemented with new insights:

Proof of Lemma 4.4 if $k \geq 1$. Consider the wave equation (2.12), now with $k \neq 0$. We start by trying to follow the strategy of the proof of Lemma 4.4 with $k=0$. Applying the same steps as in that case, is not hard to see that such multiplier currents and use of the constraint equation (2.15) will yield the estimate

$$\begin{aligned}
&\int_0^\tau \int_{\Sigma_{\tau'}} w\left[|\psi_k'|^2+|T\psi_k|^2+r^{-1}|\mathring{\nabla}\psi_k|^2+r^{-1}|\psi_k|^2\right]\mathrm{d}r^*\mathrm{d}\tau' \\
&\quad\leq B\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-2}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\int_{\Sigma_\tau}\frac{w}{r}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma \\
&\qquad + B\int_{\Sigma_0}\frac{w}{r}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma + B\mathbb{E}_k(0) \qquad (4.5)\\
&\qquad + B\sum_{j=0}^{k-1}\int_0^\tau \int_{\Sigma_{\tau'}} w|\psi_j|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',
\end{aligned}$$

for $|s| \leq 4$ and all $1 \leq k < |s|$, and for $|s| = 5, 6$ only if $1 \leq k < |3|$, as well as the estimate

$$\int_{\Sigma_\tau} \left[r^{-2}|\underline{L}\psi_k|^2 + (1-\mu)|L\psi_k|^2 + w|\mathring{\nabla}\psi_k|^2 \right] dr^* d\sigma$$
$$\leq B \int_0^\tau \int_{\Sigma_{\tau'}} w\Big(r^{-2}|\psi_{k+1}|^2 + \sum_{j=0}^{k-1} |\psi_j|^2 \Big) \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' + B\mathbb{E}_k(0),$$

Notice that, since $\mathfrak{J}_{k,j} \not\equiv 0$ for $k \neq 0$, we now have coupling errors involving ψ_j with $j < k$. In order to conclude the proof we must control these errors. By separately examining the $s < 0$ and $s > 0$ cases below, we will see that on Schwarzschild this can be achieved by supplementing the wave estimate (4.5) with appropriate transport estimates for (2.8).

The case $s < 0$. Let us consider the transport equation (2.8) with $s < 0$. By multiplying it by $c(r)\overline{\psi_k}$, and taking the real part, we can derive the identity

$$-c'(r)|\psi_k|^2 = -L\left(c(r)|\psi_k|^2\right) + 2c(r)w\,\mathrm{Re}[\psi_{k+1}\overline{\psi_k}].$$

Now choose $c(r) = r^{-1}$ to obtain

$$\begin{aligned}\int_0^\tau \int_{\Sigma_{\tau'}} w|\psi_k|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \leq{}& B\int_{\Sigma_0} r^{-3}|\psi_k|^2 \mathrm{d}r^* \mathrm{d}\sigma \\ &+ B\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-2}|\psi_{k+1}|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \\ \leq{}& B\mathbb{E}_k(0) + B\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-2}|\psi_{k+1}|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau'.\end{aligned} \tag{4.6}$$

Notice that we have used the additional decay of ψ_k compared to $\tilde{\psi}_k$ to estimate the integrals over Σ_0 by our data norms.

Using the transport estimate (4.6), we deduce that the coupling errors in the last line of (4.5) satisfy

$$\sum_{j=0}^{k-1} \int_0^\tau \int_{\Sigma_{\tau'}} w|\psi_j|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \leq B\int_0^\tau \int_{\Sigma_{\tau'}} \frac{w}{r}|\psi_k|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' + B\sum_{j=0}^{k-1} \mathbb{E}_j(0).$$

Therefore, if the derivative terms dominate the left hand side of (4.5), i.e. if

$$\int_0^\tau \int_{\Sigma_{\tau'}} w\left(|T\psi_k|^2 + r^{-1}|\mathring{\nabla}\psi_k|^2\right) \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \geq \frac{1}{\epsilon}\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-1}|\psi_k|^2 \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau',$$

for sufficiently small ϵ, the coupling errors in (4.5) can be absorbed by the left hand side; we can fix ϵ so that this is the case. Then, (4.5) holds without the last line as long as we add $B\sum_{j=0}^{k-1}\mathbb{E}_j(0)$ to the right hand side.

If, on the contrary, we have

$$\int_0^\tau\int_{\Sigma_{\tau'}} w\left(|T\psi_k|^2 + r^{-1}|\mathring{\nabla}\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' < \frac{1}{\epsilon}\int_0^\tau\int_{\Sigma_{\tau'}} wr^{-1}|\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',$$

then, since we have already fixed ϵ,

$$\begin{aligned}&\int_0^\tau\int_{\Sigma_{\tau'}} w\left[|\psi_k'|^2 + |T\psi_k|^2 + r^{-1}|\mathring{\nabla}\psi_k|^2 + r^{-1}|\psi_k|^2\right]\mathrm{d}r^*\mathrm{d}\tau'\\ &\le B\int_0^\tau\int_{\Sigma_{\tau'}} w|\psi_k|^2\mathrm{d}r^*\mathrm{d}\tau' \le B\int_0^\tau\int_{\Sigma_{\tau'}} wr^{-1}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\tau',\end{aligned}$$

by noting that $\psi_k' = w\psi_{k+1} - T\psi_k$ and then using the transport estimate (4.6) directly.

The case $s > 0$. If we repeat the procedure from the previous case now for transport equation (2.8) with $s > 0$, we can derive the identity

$$c'(r)|\psi_k|^2 = -\underline{L}\left(c(r)|\psi_k|^2\right) + 2c(r)w\,\mathrm{Re}[\psi_{k+1}\overline{\psi_k}].$$

A natural choice (analogous to that in the previous case) is $c(r) = 1-\mu$, which leads to

$$\begin{aligned}\int_0^\tau\int_{\Sigma_{\tau'}} w|\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' &\le B\int_{\Sigma_0}(1-\mu)^2|\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\\ &\quad + B\int_0^\tau\int_{\Sigma_{\tau'}} w(1-\mu)|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\ &\le B\mathbb{E}_k(0) + B\int_0^\tau\int_{\Sigma_{\tau'}} w(1-\mu)|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',\end{aligned}$$

where we have again used the additional decay of ψ_k compared to $\tilde{\psi}_k$ to estimate the integrals over Σ_0 by our data norms. In this estimate, the bulk term in ψ_{k+1} on the right hand side has too weak decay as $r\to\infty$ for us to carry on emulating the argument of the previous case case. However, it is enough to show that the coupling errors in the last line of (4.5) satisfy

$$\sum_{j=0}^{k-1}\int_0^\tau\int_{\Sigma_{\tau'}} w|\psi_j|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' \le B\int_0^\tau\int_{\Sigma_{\tau'}} w|\psi_{k-1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\sum_{j=0}^{k-2}\mathbb{E}_j(0).$$

If we seek to change the r-weights in our transport estimate, a first obvious attempt is to modify the choice of $c(r)$. For instance, consider

$$c(r) = -(r^*)^{-1}\mathbb{1}_{\{r^*\geq R^*\}} - (r^* - R^* + 1)/R^*\mathbb{1}_{\{R^*-1\leq r^*\leq R^*\}}$$

for some arbitrary, but fixed, $R^* \geq 1$. Then, we have

$$\int_0^\tau \int_{\Sigma_{\tau'}} w|\psi_k|^2 \mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' \leq B\mathbb{E}_k(0) + B\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-1}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\int_0^\tau \int_{\Sigma_{\tau'}\cap\{R^*-1\leq r^*\leq R^*\}} |\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',$$

where the r-weight in the ψ_{k+1} bulk term has enough decay as $r\to\infty$; this comes at the cost of having to control a bounded r term bulk term in ψ_k. The coupling terms in (4.5) therefore are controlled by

$$\begin{aligned} &B\int_0^\tau \int_{\Sigma_{\tau'}} w|\psi_{k-1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\sum_{j=0}^{k-2}\mathbb{E}_j(0) \\ &\quad\leq B\int_0^\tau \int_{\Sigma_{\tau'}} \frac{w}{r}|\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\int_0^\tau \int_{\Sigma_{\tau'}\cap\{0\leq|r^*-R^*|\leq 1\}} |\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' \\ &\qquad + B\sum_{j=0}^{k-1}\mathbb{E}_j(0). \end{aligned}$$

As in the case $s<0$, if the terms involving T and $\mathring{\nabla}$ derivatives dominate the left hand side of (4.5), we can now conclude using their comparative largeness.

In contrast, if the terms involving T and $\mathring{\nabla}$ derivatives on the left hand side of (4.5) are not dominant, the above approach will not help us. Let us instead consider the modified transport equation (2.18). Multiplying (2.18) by $c(r)\overline{\tilde{\psi}_k}$ yields for $s>0$

$$\begin{aligned} &\left(c'(r) + 2rw(|s|-k)c(r)\right)|r^{|s|-k}\psi_k|^2 \\ &\quad = -\underline{L}\left(c(r)|\tilde{\psi}_k|^2\right) + 2c(r)r^{2(|s|-k)}w\,\mathrm{Re}[\psi_{k+1}\overline{\psi_k}]. \end{aligned}$$

Now notice that choosing $c(r)=r^{-2}$, we find that

$$\begin{aligned} &(|s|-k-1)\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-1}|r^{|s|-k}\psi_k|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' \\ &\quad\leq B\mathbb{E}_k(0) + B\int_0^\tau \int_{\Sigma_{\tau'}} wr^{-1}|\tilde{\psi}_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'. \end{aligned}$$

Unless $(s,k)=(+1,0)$, this transport estimate represents the desired gain in r-weights as $r\to\infty$, and it is enough to conclude, arguing similarly to the $s<0$ case, when neither T nor $\mathring{\nabla}$ derivatives are dominant. $\square$

As required, we have obtained energy boundedness and integrated local energy decay estimates for $\psi^{[s]}_{(k)}$ conditional on lower order statements holding for $\psi^{[s]}_{(k+1)}$. However, as we have mentioned, these estimates have suboptimal r-weights, and this is easy to see: for instance, we expect $\psi^{[s]}_{(k)}\to 0$ as $r^*\to(\operatorname{sgn} s)\infty$, that is, we expect that $\psi^{[s]}_{(k)}$ are not radiation fields. In the next two sections, accordingly, we will improve the r-weights of Lemma 4.4 as $r^*\to(\operatorname{sgn} s)\infty$ and beyond.

Before doing so, and because Lemma 4.4 is the heart of the proof of Theorem 4.1, let us wrap up the section with a reflection on its proof and outlook to the rotating Kerr case in [SRTdC20]:

Remark 4.5 (From $k=0$ to $k\geq 1$ to rotating Kerr)**.** In the above proofs of Lemma 4.4, we saw that in turning from the case of $k=0$ to $k\geq 1$ the significant difficulty we encounter is the fact that, in the latter case, the wave equation (2.12) is coupled not only to the $(k+1)$th equation but also to the jth wave equations, with $0\leq j<k$. Since the coupling constant is not small, for us to close any wave-type estimates, we must find smallness elsewhere.

For the Schwarzschild case considered here, we easily find smallness of this backward coupling (coupling to $j<k$ equations) in the time/angular derivative-dominated regime by making use of the constraint equation (2.15), obtained by combining the wave equation (2.12) and the transport equation (2.8). In the complementary regime, we avoid wave-type estimates altogether, using transport estimates for (2.8) instead. Crucially, we use the fact that we never need to gain any smallness in the forward coupling, i.e. the coupling to the $(k+1)$th equation; because we control the top, $k=|s|$, transformed variable *unconditionally* by the results of Section 3, our task for $k<|s|$ is simply to convert the $k<|s|$ errors into (zeroth order) bulk terms involving $k=|s|$.

Turning to the rotating Kerr case analyzed in [SRTdC20] presents a further layer of difficulties. First of all, the coupling to the $j<k$ wave equations occurs through angular derivatives as well, making the gain of smallness in the time/angular derivative-dominated regime significantly harder to achieve. Secondly, in the complementary regime, we cannot hope to rely

on transport estimates alone, as the top, $k = |s|$, wave equation is coupled (with an $O(M)$ coupling constant) to all the $k < |s|$ equations.

4.1.2 Improving weights near $\mathcal{I}^+$

To improve the r-weights in the estimates of Lemma 4.4, we start with the asymptotically flat region, $r \to \infty$. We emphasize that the statements and proofs contained in this section are not new: they have appeared, in a more condensed form, in [SRTdC23]. For simplicity, we will state them (and prove them) with a simplified notation, where we drop the superscript $[s]$ and the parenthesis in the subscript in the transformed variables and their norms.

Lemma 4.6 (Improving weights for $s > 0$)**.** *Fix $s > 0$. Let $p \in [0, 2)$ and $R^* > 0$ be sufficiently large. Then, for all $k \in \{0, \dots, |s| - 1\}$, we have the estimate*

$$
\begin{aligned}
&\mathbb{E}_{k,p,0}(\tau) + \mathbb{E}_{k,\mathcal{I}^+,p}(0,\tau) + \mathbb{I}_{k,p,0}(0,\tau) \\
&\quad \leq B \sum_{j=0}^{k} \mathbb{E}_{j,p,0}(0) + B \int_{\Sigma_0} w|\tilde{\psi}_{k+1}|^2 \mathrm{d}r^* \mathrm{d}\sigma \\
&\qquad + B \int_{\Sigma_\tau} w|\tilde{\psi}_{k+1}|^2 \mathrm{d}r^* \mathrm{d}\sigma + B \int_0^\tau \int_{\Sigma_{\tau'}} w r^{p-1} |\tilde{\psi}_{k+1}| \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \\
&\qquad + B(R^*) \int_0^\tau \int_{\Sigma_{\tau'} \cap \{|r^*| \leq R^*\}} \Big(|L\tilde{\psi}_k|^2 + |\mathring{\nabla}\tilde{\psi}_k|^2 + \sum_{j=0}^{k} |\tilde{\psi}_j|^2 \Big) \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \\
&\qquad + B \int_0^\tau \int_{\Sigma_{\tau'} \cap \{r^* \leq -R^*\}} (1-\mu) \Big(\frac{1}{1-\mu} |\underline{L}\tilde{\psi}_k|^2 + |L\tilde{\psi}_k|^2 + |\mathring{\nabla}\tilde{\psi}_k|^2 \Big) \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \\
&\qquad + B \int_{\Sigma_\tau \cap \{r^* \leq -R^*\}} (1-\mu) \Big(\frac{1}{1-\mu} |\underline{L}\tilde{\psi}_k|^2 + |L\tilde{\psi}_k|^2 + |\mathring{\nabla}\tilde{\psi}_k|^2 \Big) \mathrm{d}r^* \mathrm{d}\sigma.
\end{aligned}
$$

Lemma 4.7 (Improving weights for $s < 0$)**.** *Fix $s < 0$. Let $p \in [0, 2]$ and $R^* > 0$ be sufficiently large. Then, for all $k \in \{0, \dots, |s| - 1\}$, we have the estimate*

$$
\begin{aligned}
&\int_{\Sigma_\tau \cap \{r^* \geq R^*\}} r^{-2} \left(r^{p+2} |L\psi_k|^2 + |\underline{L}\psi_k|^2 + |\mathring{\nabla}\psi_k|^2 + |\psi_k|^2 \right) \mathrm{d}r^* \mathrm{d}\sigma \\
&\quad + \mathbb{E}_{\mathcal{I}^+,p}(0,\tau) + \int_0^\tau \int_{\Sigma_{\tau'} \cap \{r^* \geq R^*\}} \left(r^{p-1} |L\psi_k|^2 + \delta r^{-1-\delta} |\underline{L}\psi_k|^2 \right) \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau' \\
&\quad + \int_0^\tau \int_{\Sigma_{\tau'} \cap \{r^* \geq R^*\}} r^{p-3} \left(2 - p + r^{-1} \right) \left(|\mathring{\nabla}\psi_k|^2 + |\psi_k|^2 \right) \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau'
\end{aligned}
$$

$$\leq B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{R^*-1\leq r^*\leq R^*\}}\Big(|\underline{L}\psi_k|^2+|\mathring{\nabla}\psi_k|^2+\sum_{j=0}^k|\psi_j|^2\Big)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*-1\}}r^{p-4}(1+r^{-1})|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+B\sum_{j=0}^k E_{j,p,0}(0)+B\int_{\Sigma_0}w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma.$$

Proof of Lemma 4.6. We consider the rescaled equation (2.19) for $s>0$, focusing especially on the case $r\gg 1$ where the rescaling weight $c_k(r)$ becomes prominent.

In the proof, we will consider the cases $k=0$ (Step 1) and $k>0$ (Step 2) separately in the basic setting $p=0$, $\delta=1$. In both instances, and for all the lemmas in Sections 4.1.2 and 4.1.3, the proof follows by considering multipliers $z(r)L$ and $z(r)\underline{L}$ for some function z: that is, by considering the identities generated by multiplying (2.19) by $z(r)\overline{L\tilde{\psi}_k}$ and $z(r)\overline{\underline{L}\tilde{\psi}_k}$, respectively, taking the real part, and integrating by parts. The precise choices in integration by parts depend on the situation; thus, it will be useful to keep in mind the notation

$$\begin{aligned}&\left(\int_{\Sigma_\tau}-\int_{\Sigma_0}\right)\left(v_1(r)r^{-2}F^X_L+v_2(r)(1-\mu)F^X_{\underline{L}}\right)\mathrm{d}r^*\mathrm{d}\sigma\\&\qquad+\int_{\mathcal{H}^+_{(0,\tau)}}v_3(r)F^X_{\underline{L}}\mathrm{d}\sigma\mathrm{d}\tau'+\int_{\mathcal{I}^+_{(0,\tau)}}v_4(r)F^X_L\mathrm{d}\sigma\mathrm{d}\tau'\\&+\int_0^\tau\int_{\Sigma_{\tau'}}v_5(r)I^X\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'=0,\end{aligned}\tag{4.7}$$

for the identity induced by a given multiplier X, not just in this proof but in the following three proofs; here $b\leq v_i(r)\leq B$ only depending on M. In Step 3, we sketch the proof for general $p\in[0,2)$, $\delta\in(0,1]$. For readability, we have kept the discussion in these steps at the level of the $r\gg 1$ region only; in Step 4 we explain how to make these estimates become the stated almost global estimates.

Step 1: $k=0$. Since (2.19) with $k=0$ couples only to $\tilde{\psi}_1$, our estimates in this case are slightly easier. First, we apply the multiplier $z(r)L$ to obtain an identity of the form of (4.7) with

$$F^{zL}_{\underline{L}}=z|L\tilde{\psi}_0|^2,$$
$$F^{zL}_L=\frac{1}{2}zw|\mathring{\nabla}\tilde{\psi}_0|^2+\frac{Mzw}{r}(1-|s|+2s^2)|\tilde{\psi}_0|^2,$$

$$I^{zL} = (|s|rwz + z')\,|L\tilde{\psi}_0|^2 - \frac{(zw)'}{2}|\mathring{\nabla}\tilde{\psi}_0|^2 - \left(\frac{Mzw}{r}\right)'(1-3|s|+2s^2)|\tilde{\psi}_0|^2$$
$$+ z|s|w\left(1-\frac{4M}{r}\right)\operatorname{Re}\left[\overline{L\tilde{\psi}_0}\tilde{\psi}_1\right].$$

Here, we have treated the terms $(U_0 + \frac{2M}{r})w - \left(\frac{c_0'}{c_0}\right)'$ in (2.19) together. Choosing z to be a cutoff function which equals 1 for $r^* \geq R^*+1$ and equals 0 for $r^* \leq R^*$, we have

$$\int_{\Sigma_\tau\cap\{r^*\geq R^*+1\}}\left(|L\tilde{\psi}_0|^2 + r^{-4}|\mathring{\nabla}\tilde{\psi}_0|^2 + r^{-4}|\tilde{\psi}_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma$$
$$+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*+1\}}\left(r^{-1}|L\tilde{\psi}_0|^2 + r^{-3}|\mathring{\nabla}\tilde{\psi}_0|^2 + r^{-3}|\tilde{\psi}_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$\leq B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\frac{w}{r}|\tilde{\psi}_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau' + B\mathbb{E}_0(0)$$
$$+ B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{R^*\leq r^*\leq R^*+1\}}\left(|L\tilde{\psi}_0|^2 + |\mathring{\nabla}\tilde{\psi}_0|^2 + |\tilde{\psi}_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',$$

as long as R^* is chosen to be sufficiently large.

To improve the weights on Σ_τ in this estimate, we then consider an identity generated by using the $z(r)\underline{L}$ multiplier, specifically taking the form

$$F_L^{z\underline{L}} = z|\underline{L}\tilde{\psi}_0|^2 - \frac{1}{2}s^2w^2r^2z|\tilde{\psi}_0|^2,$$
$$F_{\underline{L}}^{z\underline{L}} = \frac{1}{2}zw|\mathring{\nabla}\tilde{\psi}_0|^2 + \frac{Mzw}{r}(1-|s|+2s^2)\tilde{\psi}_0|^2,$$
$$I^{z\underline{L}} = -z'|\underline{L}\tilde{\psi}_0|^2 + z|s|r^2w^2\operatorname{Re}[\tilde{\psi}_1\overline{\underline{L}\tilde{\psi}_0}]$$
$$+\frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_0|^2 + \frac{1}{2}\left[w\left(s^2+\frac{2M}{r}(1-|s|+s^2)\right)z\right]'|\tilde{\psi}_0|^2$$
$$+|s|z(r-4M)r^2w^3|\tilde{\psi}_1| - s^2(r-4M)r^2w^3z\operatorname{Re}[\tilde{\psi}_1\overline{\tilde{\psi}_0}],$$

in the notation of (4.7). To obtain these expressions, we have treated the terms $(U_0+\frac{2M}{r})w - \left(\frac{c_0'}{c_0}\right)'$ in (2.19) together again. We also used the fact that $\underline{L}\tilde{\psi}_0 = wr\left(\tilde{\psi}_1 - |s|\tilde{\psi}_0\right)$ to treat the other new terms in (2.19) which are absent from (2.12), and integrated by parts any terms involving $\operatorname{Re}[\overline{\tilde{\psi}_0}L\tilde{\psi}_0]$. With $z(r)$ the cutoff function from before, we deduce

$$\int_{\mathcal{I}^+_{(0,\tau)}}|\underline{L}\tilde{\psi}_0|\mathrm{d}\sigma\mathrm{d}\tau' + \int_{\Sigma_\tau\cap\{r^*\geq R^*+1\}}\left(r^{-2}|\underline{L}\tilde{\psi}_0|^2 + r^{-2}|\mathring{\nabla}\tilde{\psi}_0|^2 + r^{-2}|\tilde{\psi}_0|^2\right)$$

$$\leq B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}} r^{-3}\left(r^2|\underline{L}\psi_0|^2+|\mathring{\nabla}\tilde\psi_0|^2+|\tilde\psi_0|^2+|\tilde\psi_1|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+B\mathbb{E}_0(0),$$

for sufficiently large R^*. Adding a small multiple of this second estimate with the previous one, and then using the identity $\underline{L}\tilde\psi_0=rw\left(\tilde\psi_1-|s|\tilde\psi_0\right)$ to directly estimate the $\underline{L}\tilde\psi_0$ bulk terms, we obtain suitable flux and bulk estimates for large r:

$$\int_{\Sigma_\tau\cap\{r^*\geq R^*+1\}}\left(|L\tilde\psi_0|^2+r^{-2}|\mathring{\nabla}\tilde\psi_0|^2+r^{-2}|\tilde\psi_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma$$
$$+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*+1\}}\frac{1}{r}\left(|L\tilde\psi_0|^2+|\underline{L}\tilde\psi_0|^2+\frac{1}{r^2}|\mathring{\nabla}\tilde\psi_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$\leq B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\frac{w}{r}|\tilde\psi_1|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'+B\mathbb{E}_0(0)$$
$$+B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{R^*\leq r^*\leq R^*+1\}}\left(|L\tilde\psi_0|^2+|\mathring{\nabla}\tilde\psi_0|^2+|\tilde\psi_0|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'.$$

Step 2: $k>0$ and $|s|>1$. Our strategy will be similar to that in the previous step; however, because (2.19) with $0<k<|s|$ is coupled to the $(k+1)$th equation *and* all the jth equations with $j<k$, the implementation will be much more involved. The main difference is that we will need to work harder to obtain multiplier identities generated by the $z(r)L$ and $z(r)\underline{L}$ multipliers which are suitable for our intended estimates. We will use the notation

$$c(r)\doteq\frac{c_k'}{c_k}-(|s|-k)\frac{w'}{w}=\frac{(|s|-k)(r-4M)}{r^2}=(|s|-k)rw-\frac{2M}{r^2}(|s|-k).$$

It will be useful to note that for $0<k<|s|$ and $|s|\geq 2$,

$$w\left(U_k+\frac{2M}{r}\right)-\left(\frac{c_k'}{c_k}\right)'$$
$$=k(2|s|-k)w+\frac{2M}{r}w\left[1-|s|+2s^2+k(1+3k-6|s|)\right]\geq 2w,$$
$$w\left(U_k+\frac{2M}{r}\right)-\left(\frac{c_k'}{c_k}\right)'+2\left[\left(\frac{c_k'}{2c_k}\right)'-\frac{c_k'}{c_k}(|s|-k)rw\right]$$
$$=w[2s(|s|-k)+|s|-k+k^2]+\frac{2M}{r}w[1-2k(2|s|-k-3)-3s]\geq 3w.$$

Let us first state and explain the the relevant multiplier identities. Firstly, the $z\underline{L}$ multiplier applied to (2.12) yields an identity of the form

(4.7) with

$$
\begin{aligned}
F^{z\underline{L}}_{L} &= z|\underline{L}\tilde{\psi}_k|^2 - \frac{c_k'}{c_k}z\operatorname{Re}[\underline{L}\tilde{\psi}_k\overline{\tilde{\psi}_k}] + \frac{1}{2}\left(\frac{c_k'}{c_k}\right)^2|\tilde{\psi}_k|^2 \\
F^{z\underline{L}}_{\underline{L}} &= \frac{1}{2}zw|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2}z\left[w\left(U_k+\frac{2M}{r}\right)-\left(\frac{c_k'}{c_k}\right)'\right]|\tilde{\psi}_k|^2 \\
&\quad + \left[\left(\frac{zc_k'}{2c_k}\right)' - \frac{c_k'}{c_k}zrw(|s|-k)\right]|\tilde{\psi}_k|^2 \\
I^{z\underline{L}} &= (cz-z')|\underline{L}\tilde{\psi}_k|^2 + \frac{2M}{r^2}(|s|-k)\frac{c_k'}{c_k}\operatorname{Re}[\underline{L}\tilde{\psi}_k\overline{\tilde{\psi}_k}] \\
&\quad + \frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2}\left(\left[w\left(U_k+\frac{2M}{r}\right)-\left(\frac{c_k'}{c_k}\right)'\right]z\right)'|\tilde{\psi}_k|^2 \\
&\quad + \left[\left(\frac{zc_k'}{2c_k}\right)'' - (|s|-k)\left(\frac{zwrc_k'}{c_k}\right)' - \left(\frac{z}{2}\left(\frac{c_k'}{c_k}\right)^2\right)'\right]|\tilde{\psi}_k|^2 \\
&\quad - \frac{c_k'}{c_k}z\left(w\left(U_k+\frac{2M}{r}\right)-\left(\frac{c_k'}{c_k}\right)' - \frac{c_k'}{c_k}c\right)|\tilde{\psi}_k|^2 \\
&\quad + z\sum_{j=0}^{k-1}\frac{wc_j}{c_k}c^{\mathrm{id}}_{s,k,j}\left(\frac{c_k'}{c_k}\operatorname{Re}\left[\overline{\tilde{\psi}_k}\tilde{\psi}_j\right] - \operatorname{Re}\left[\overline{\underline{L}\tilde{\psi}_k}\tilde{\psi}_j\right]\right).
\end{aligned}
$$

In deriving this identity, in addition to the obvious integration by parts argument for the wave operator $\mathfrak{R}_k - \left(\frac{c_k'}{c_k}\right)'$, we have used the fact that

$$
\begin{aligned}
&\frac{wc_{k+1}}{c_k}cz\operatorname{Re}\left[\tilde{\psi}_{k+1}\overline{\underline{L}\tilde{\psi}_k}\right] - \frac{c_k'}{c_k}z\operatorname{Re}\left[\overline{L\tilde{\psi}_k}\underline{L}\tilde{\psi}_k\right] \\
&\quad = cz|\underline{L}\tilde{\psi}_k|^2 - cz\frac{c_k'}{c_k}\operatorname{Re}\left[\psi_k\overline{\underline{L}\tilde{\psi}_k}\right] - \frac{c_k'}{c_k}z\operatorname{Re}\left[\overline{L\tilde{\psi}_k}\underline{L}\tilde{\psi}_k\right],
\end{aligned}
$$

where the second term (or a subset thereof) can be integrated in $\underline{L}$ and the third term can be integrated in L; one then uses (2.12) to simplify $L\underline{L}\tilde{\psi}_k$, and integrates by parts the resulting expression. Let us remark that, as $r\to\infty$,

$$
\begin{aligned}
&\left(\frac{c_k'}{2c_k}\right)'' - (|s|-k)\left(rw\frac{c_k'}{c_k}\right)' + c\left(\frac{c_k'}{c_k}\right)^2 - \left(\frac{1}{2}\left(\frac{c_k'}{c_k}\right)^2\right)' \\
&\quad - \frac{c_k'}{c_k}\left(w\left(U_k+\frac{2M}{r}\right)-\left(\frac{c_k'}{c_k}\right)'\right)
\end{aligned}
$$

$$= -\frac{c_k'}{c_k}\left(s^2 - (|s|-k) - 1\right) + O(r^{-4})$$

which is either positive, since $0 < k < |s|$ and $|s| > 1$ by assumption, or $O(r^{-4})$. Secondly, the zL multiplier applied to (2.12) yields an identity of the form (4.7) where we have

$$\begin{aligned}
F_{\underline{L}}^{zL} &= z|L\tilde{\psi}_k|^2 - \frac{1}{2}(|s|-k)(zr^{-1})'|\tilde{\psi}_k|^2 + \frac{1}{2}c^2|\tilde{\psi}_k|^2, \\
F_L^{zL} &= \frac{1}{2}zw|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2}z\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)'\right)|\tilde{\psi}_k|^2 \\
&\quad + cz\frac{wc_{k+1}}{c_k}\operatorname{Re}[\overline{\tilde{\psi}_k}\tilde{\psi}_{k+1}] - \frac{2M}{r^2}(|s|-k)\frac{c_k'}{c_k}z|\tilde{\psi}_k|^2, \\
I^{zL} &= \left(z' - \frac{c_k'}{c_k}z\right)|L\tilde{\psi}_k|^2 - \frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_k|^2 \\
&\quad + 2M(|s|-k)\left[\left(\frac{z}{r^2}\right)'\operatorname{Re}[\underline{L}\tilde{\psi}_k\overline{\tilde{\psi}_k}] + \frac{zc_k'}{r^2c_k}\operatorname{Re}[L\tilde{\psi}_k\overline{\tilde{\psi}_k}]\right] \\
&\quad - \frac{1}{2}\left[\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)' - c^2\right)z\right]'|\tilde{\psi}_k|^2 \\
&\quad + czw|\mathring{\nabla}\tilde{\psi}_k|^2 + cz\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)' - \frac{c_k'}{c_k}c\right)|\tilde{\psi}_k|^2 \\
&\quad + (|s|-k)\left[\left(\frac{c_k'}{c_k}zrw\right)' - \frac{1}{2}(zr^{-1})''\right]|\tilde{\psi}_k|^2 \\
&\quad - z\sum_{j=0}^{k_1}\frac{wc_j}{c_k}c_{s,k,j}^{\mathrm{id}}\left(\operatorname{Re}\left[\overline{L\tilde{\psi}_k}\tilde{\psi}_j\right] - c\operatorname{Re}\left[\overline{\tilde{\psi}_k}\tilde{\psi}_j\right]\right).
\end{aligned}$$

Here, in addition to the obvious integration by parts argument for the wave operator $\mathfrak{R}_k - \left(\frac{c_k'}{c_k}\right)'$, we have used the fact that

$$\frac{wc_{k+1}}{c_k}cz\operatorname{Re}\left[\tilde{\psi}_{k+1}\overline{L\tilde{\psi}_k}\right] = zc\operatorname{Re}\left[\left(\underline{L}\tilde{\psi}_k - \frac{c_k'}{c_k}\tilde{\psi}_k\right)\overline{L\tilde{\psi}_k}\right]$$

can be integrated by parts in $\underline{L}$ (first term) and L (second term), that we can use (2.12) to simplify $\underline{L}L\tilde{\psi}_k$, and that the resulting expression can also be integrated by parts. Let us remark that

$$cw|\mathring{\nabla}\tilde{\psi}_k|^2 + \left[c\left(w\left(U_k + \frac{2M}{r}\right) - \left(\frac{c_k'}{c_k}\right)'\right)\right]|\tilde{\psi}_k|^2$$

$$+\left[-\frac{c_k'}{c_k}c^2z+(|s|-k)\left(\frac{c_k'}{c_k}zrw\right)'-\frac{1}{2}(|s|-k)(zr^{-1})''+\frac{1}{2}(c^2z)'\right]|\tilde{\psi}_k|^2$$
$$=cw|\mathring{\nabla}\tilde{\psi}_k|^2+cw(s^2-2(|s|-k)-1)|\tilde{\psi}_k|^2+O(r^{-4})|\tilde{\psi}_k|^2$$

as $r\to\infty$; thus, in view of the properties of the spin-weighted laplacian, this expression is either positive for $0<k<|s|-1$ and $s\geq 1$, or it is $O(r^{-4})|\tilde{\psi}_k|^2$.

We now apply the above identities in an identical fashion to Step 1. We choose $z(r^*)=\chi(r^*)$ to be a cutoff function equal to 1 for $r^*\geq R^*$ and equal to zero for $r^*\leq R^*-1$ in both the multiplier estimates above. We add a small multiple of the $z\underline{L}$ identity to the zL identity. We deduce that:

$$\begin{aligned}&\int_{\mathcal{I}^+_{(0,\tau)}}|\underline{L}\tilde{\psi}_k|^2\mathrm{d}\sigma\mathrm{d}\tau'\\&\quad+\int_{\Sigma_\tau\cap\{r^*\geq R^*\}}\left(|L\psi_k|^2+r^{-1}|\underline{L}\psi_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2+r^{-2}|\tilde{\psi}_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\\&\quad+\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\frac{1}{r}\left(|L\psi_k|^2+|\underline{L}\psi_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2+r^{-2}|\tilde{\psi}_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\&\leq B\mathbb{E}_k(0)+B\int_{\Sigma_0}w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma+B\int_{\Sigma_\tau\cap\{r^*\geq R^*-1\}}w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\\&\quad+\frac{B}{(R^*)^2}\sum_{j=0}^{k-1}\int_0^\tau\int_{\Sigma_\tau'\cap\{r^*\geq R^*-1\}}w'|\tilde{\psi}_j|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\&\quad+B(R^*)\int_0^\tau\int_{\Sigma_\tau'\cap\{0\leq R^*-r^*\leq 1\}}\Big(|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+\sum_{j=0}^{k+1}|\tilde{\psi}_j|^2\Big)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau',\end{aligned}$$

as long as R^* is chosen to be sufficiently large.

Notice that there is a small parameter multiplying the large r error involving $\tilde{\psi}_j$ for $j<k$. Thus, making R^* even larger and then fixing it, we can iterate the estimates for $j=0,\dots k$ (thus making use of Step 1) to obtain:

$$\begin{aligned}&\int_{\mathcal{I}^+_{(0,\tau)}}|\underline{L}\tilde{\psi}_k|^2\mathrm{d}\sigma\mathrm{d}\tau'\\&\quad+\int_{\Sigma_\tau\cap\{r^*\geq R^*\}}\left(|L\psi_k|^2+r^{-1}|\underline{L}\psi_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2+r^{-2}|\tilde{\psi}_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\\&\quad+\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}}\frac{1}{r}\left(|L\psi_k|^2+|\underline{L}\psi_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2+r^{-2}|\tilde{\psi}_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\&\leq B\sum_{j=0}^{k}\mathbb{E}_j(0)+B\int_{\Sigma_0}w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma+B\int_{\Sigma_\tau\cap\{r^*\geq R^*-1\}}w|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\end{aligned}$$

$$+ B \int_0^{\tau} \int_{\Sigma'_{\tau} \cap \{0 \le R^* - r^* \le 1\}} \Big(|L\tilde{\psi}_k|^2 + |\mathring{\nabla}\tilde{\psi}_k|^2 + \sum_{j=0}^{k+1} |\tilde{\psi}_j|^2 \Big) \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau'.$$

Step 3: p-weighted norms. To improve the weights further using the parameter $p \in (0,2)$, we repeat the two previous steps but now choosing $z = r^p \chi$, with χ the cutoff function from the previous step, for the $z(r)L$ multiplier only. Notice that, for $k = 0$, the term

$$z|s|w \left(1 - \frac{4M}{r}\right) \mathrm{Re}[L\tilde{\psi}_0 \overline{\tilde{\psi}_1}], \quad z = r^p \chi,$$

prevents us from closing estimates with $p = 2$.

Step 4: almost-global fluxes on Σ_τ. By combining the previous steps, we have closed estimates for $\tilde{\psi}_k$ in the large r region in terms of $\tilde{\psi}_{k+1}$ errors (and data). It is not hard to see that the energy flux estimates can be made almost global in r: indeed, if we choose $z(r)$ in the $z(r)L$ multiplier and $z(r)\underline{L}$ multipliers to be supported not just at large r but for all r away from $r = 2M$, then we can control the derivatives of $\tilde{\psi}_k$ on $\Sigma_\tau \cap \{r^* \ge -R^*\}$ as long as we control $\tilde{\psi}_{k+1}$ and $\tilde{\psi}_j$, $j = 0, \dots k-1$, spacetime errors away from $r \approx 2M$. This final version of our estimates is the one given in the statement. □

Proof of Lemma 4.7. Consider the wave equation (2.12) for $s < 0$. As announced above, this proof will also rely on using $z(r)L$ and $z(r)\underline{L}$ multipliers for appropriate choices of functions $z(r)$.

We start by stating the forms of the multiplier identities we will consider, modeled on the notation in (4.7). For the $z(r)L$ multiplier, we have

$$\begin{aligned}
F_{\underline{L}}^{zL} &= z|L\psi_k|^2, \\
F_{L}^{zL} &= \frac{1}{2} zw |\mathring{\nabla}\psi_k|^2 + \frac{1}{2} zw \left(U_k + \frac{2M}{r}\right) |\psi_k|^2 - zw \sum_{j=0}^{k-1} c^{\mathrm{id}}_{s,k,j} \,\mathrm{Re}\left[\overline{\psi_k} \psi_j\right], \\
I^{zL} &= z' |L\psi_k|^2 - \frac{1}{2}(zw)' |\mathring{\nabla}\psi_k|^2 - \frac{1}{2}\left[w\left(U_k + \frac{2M}{r}\right) z\right]' |\psi_k|^2 \\
&\quad + (|s| - k) w' zw |\psi_{k+1}|^2 \\
&\quad + \sum_{j=0}^{k-1} c^{\mathrm{id}}_{s,k,j} \left(z' w \,\mathrm{Re}\left[\overline{\psi_k} \psi_j\right] + zw^2 \,\mathrm{Re}\left[\overline{\psi_k} \psi_{j+1}\right]\right),
\end{aligned}$$

which is obtained by the usual integration by parts procedure for terms generated by $\mathfrak{R}_k$, together with integration by parts of the terms $\mathrm{Re}[L\psi_k \overline{\psi_j}]$.

For the $z(r)\underline{L}$ multiplier, we have

$$F_L^{z\underline{L}} = z|\underline{L}\psi_k|^2, \qquad F_{\underline{L}}^{z\underline{L}} = \frac{1}{2}zw|\mathring{\nabla}\psi_k|^2 + \frac{1}{2}zw\left(U_k + \frac{2M}{r}\right)|\psi_k|^2,$$
$$I^{z\underline{L}} = -z'|\underline{L}\psi_k|^2 + \frac{1}{2}(zw)'|\mathring{\nabla}\psi_k|^2 + \frac{1}{2}\left[w\left(U_k + \frac{2M}{r}\right)z\right]'|\psi_k|^2$$
$$+ (|s|-k)w'z\,\mathrm{Re}[\psi_{k+1}\overline{\underline{L}\psi_k}] - z\sum_{j=0}^{k-1} wc_{s,k,j}^{\mathrm{id}}\,\mathrm{Re}\left[\overline{\underline{L}\psi_k}\psi_j\right].$$

Let χ be a smooth cutoff function such that $\chi = 1$ for $r^* \geq R^*$ and $\chi = 0$ for$r^* \leq R^* - 1$. Apply the $z(r)L$ identity with $z = r^p(1+4M/r)\chi$, and add a small multiple of the $z(r)\underline{L}$ identity with $z = \chi$. Since $c_k = 1 + O(r^{-1})$ as $r \to \infty$, we have

$$\int_{\mathcal{I}^+_{(0,\tau)}} \left[|L\psi_k|^2 + \mathbb{1}_{\{p=2\}}\left(|\mathring{\nabla}\psi_k|^2 + r^{p-2}|\psi_k|^2\right)\right] \mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ \int_{\Sigma_\tau\cap\{r^*\geq R^*\}} r^{-2}\left(r^{p+2}|L\psi_k|^2 + |\underline{L}\psi_k|^2 + |\mathring{\nabla}\psi_k|^2 + |\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma$$
$$+ \int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}} \left(r^{p-1}|L\psi_k|^2 + r^{-2}|\underline{L}\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ \int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*\}} r^{p-3}\left(2-p+r^{-1}\right)\left(|\mathring{\nabla}\psi_k|^2 + |\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$\leq B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{R^*-1\leq r^*\leq R^*\}} \left(|\underline{L}\psi_k|^2 + |\mathring{\nabla}\psi_k|^2 + |\psi_k|^2\right)\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*-1\}} r^{p-4}(1+r^{-1})|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ B\int_{\Sigma_0\cap\{r^*\geq R^*-1\}} r^{p-4}|\psi_{k+1}|^2\mathrm{d}r^*\mathrm{d}\sigma + BE_k(0)$$
$$+ B\sum_{j=0}^{k-1}\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\geq R^*-1\}} r^{p-3}\left((2-p)+r^{-1}\right)|\psi_j|^2\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'$$
$$+ B\sum_{j=0}^{k-1}\left(\int_{\Sigma_\tau\cap\{r^*\geq R^*-1\}} + \int_{\Sigma_0\cap\{r^*\geq R^*-1\}}\right) r^{2(p-3)}|\psi_j|^2\mathrm{d}r^*\mathrm{d}\sigma.$$

Clearly, for any choice of $p \in [0,2]$, we can use the improvement in the r-weights associated to $k+1$ and iterate the above estimate for $j = 0, \ldots, k$: thus, for R^* sufficiently large, the above estimate holds without the last two lines and with $\mathbb{E}_k(0)$ replaced by $\sum_{j=0}^k \mathbb{E}_j(0)$.

Finally, we can improve our weights on $\underline{L}\psi_k$ bulk terms by using the $z(r)\underline{L}$ identity with $z = (1+r^{-\delta})\chi$ instead of $z = \chi$. □

4.1.3 Improving weights near $\mathcal{H}^+$

Finally, in this we turn to the task of improving Lemma 4.4 in the near horizon region. We again emphasize that the statements and proofs contained in this section are not new, as they have appeared already in [SRTdC23]. As in the previous section, we will state our results (and prove them) with a simplified notation, where we drop the superscript $[s]$ and the parenthesis in the subscript in the transformed variables and their norms.

Lemma 4.8 (Improving weights for $s<0$)**.** *Fix $s<0$. Let $q\in[0,1]$ and $R^*>0$ be sufficiently large. Then, for all $k\in\{0,\dots,|s|-1\}$, we have the estimate*

$$\begin{aligned}
&\mathbb{E}_{k,0,q}(\tau)+\mathbb{E}_{k,\mathcal{H}^+,q}(0,\tau)+\mathbb{I}_{k,1,0,q}(0,\tau)\\
&\quad\le B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{|r^*|\le R^*\}}\Big[|\underline{L}\tilde{\psi}_k|^2+|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+\sum_{j=0}^{k}|\tilde{\psi}_j|^2\Big]dr^*d\sigma d\tau'\\
&\qquad+B\sum_{j=0}^{k}\mathbb{E}_{j,0,q}(0)+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\le -R^*\}}\frac{w}{r}|\tilde{\psi}_{k+1}|^2dr^*d\sigma d\tau'\\
&\qquad+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\ge R^*\}}\left[r^{-2}|\underline{L}\tilde{\psi}_k|^2+|L\tilde{\psi}_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2\right]dr^*d\sigma d\tau'\\
&\qquad+B\int_{\Sigma_\tau\cap\{r^*\ge R^*\}}\left[r^{-2}|\underline{L}\tilde{\psi}_k|^2+r^{-1}|L\tilde{\psi}_k|^2+r^{-2}|\mathring{\nabla}\tilde{\psi}_k|^2\right]dr^*d\sigma d\tau'.
\end{aligned}$$

Lemma 4.9 (Improving weights for $s>0$)**.** *Fix $s>0$. Let $q\in[0,1]$ and $R^*<0$ be sufficiently negative. Then, for all $k\in\{0,\dots,|s|-1\}$, we have the estimate*

$$\begin{aligned}
&\mathbb{E}_{k,\mathcal{H}^+,q}(0,\tau)+\int_{\Sigma_\tau\cap\{r^*\le R^*\}}(1-\mu)^{-q}|\underline{L}\psi_k|^2dr^*d\sigma\\
&\quad+\int_{\Sigma_\tau\cap\{r^*\le R^*\}}(1-\mu)\left[|L\psi_k|^2+|\mathring{\nabla}\psi_k|^2+|\psi_k|^2\right]dr^*d\sigma\\
&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\le R^*\}}\left[q(1-\mu)^{-q}|\underline{L}\psi_k|^2+(1-\mu)|L\psi_k|^2\right]dr^*d\sigma d\tau'\\
&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\le R^*\}}(1-\mu)^{1-q}\left((1-q)+(1-\mu)\right)\left[|\mathring{\nabla}\psi_k|^2+|\psi_k|^2\right]dr^*d\sigma d\tau'\\
&\le B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*-R^*\in[0,1]\}}\Big[|L\psi_k|^2+|\mathring{\nabla}\psi_k|^2+\sum_{j=0}^{k+1}|\psi_j|^2\Big]dr^*d\sigma d\tau'\\
&\qquad+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\le R^*\}}(1-\mu)|\tilde{\psi}_{k+1}|^2dr^*d\sigma d\tau'
\end{aligned}$$

$$+ B\int_{\Sigma_\tau\cap\{r^*\leq R^*\}}(1-\mu)|\tilde{\psi}_{k+1}|^2dr^*d\sigma + B\sum_{j=0}^{k}\mathbb{E}_{j,0,q}(0).$$

In view of the positive surface gravity of Schwarzschild black holes, and associated redshift effect [DR09], one can expect this region to be significantly easier to handle than the asymptotically flat $r\to\infty$ region. Indeed, this section is much less technically involved than the previous. Nevertheless, the main conceptual ideas are the same.

Proof of Lemma 4.8. Fix some $s<0$, and consider the rescaled equation (2.19) with $k\leq |s|-1$. This proof will follow a similar strategy to that of Lemma 4.6. To obtain the estimates, we derive suitable identities generated by multipliers $z(r)L$ and $z(r)\underline{L}$. Using the notation of (4.7), these can be stated as

$$\begin{aligned}
F_L^{z\underline{L}} &= z|\underline{L}\tilde{\psi}_k|^2,\\
F_{\underline{L}}^{z\underline{L}} &= \frac{1}{2}zw|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2}zw\left(U_k+\frac{2M}{r}\right)|\tilde{\psi}_k|^2,\\
I^{z\underline{L}} &= \left(\frac{c_k'}{c_k}z - z'\right)|\underline{L}\tilde{\psi}_k|^2 + \frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2}\left[w\left(U_k+\frac{2M}{r}\right)z\right]'|\tilde{\psi}_k|^2\\
&\quad - z\,\mathrm{Re}\left\{\overline{\underline{L}\tilde{\psi}_k}\left[\left(\frac{c_k'}{c_k}-(|s|-k)\frac{w'}{w}\right)\frac{wc_{k+1}}{c_k}\tilde{\psi}_{k+1} + \left(\frac{c_k'}{c_k}\right)'\tilde{\psi}_k\right]\right\}\\
&\quad - z\sum_{j=0}^{k-1}\frac{wc_j}{c_k}c_{s,k,j}^{\mathrm{id}}\,\mathrm{Re}[\overline{\underline{L}\tilde{\psi}_k}\tilde{\psi}_j],
\end{aligned}$$

for the $z(r)\underline{L}\tilde{\psi}_k$ multiplier, and for the $z(r)L\tilde{\psi}_k$,

$$\begin{aligned}
F_{\underline{L}}^{zL} &= z|L\tilde{\psi}_k|^2 - \frac{1}{2}z\left(\frac{c_k'}{c_k}\right)^2|\tilde{\psi}_k|^2\\
F_L^{zL} &= \frac{1}{2}zw|\mathring{\nabla}\tilde{\psi}_k|^2 + \frac{1}{2}zw\left(U_k+\frac{2M}{r}\right)|\tilde{\psi}_k|^2\\
I^{zL} &= z'|L\tilde{\psi}_k|^2 - \frac{1}{2}(zw)'|\mathring{\nabla}\tilde{\psi}_k|^2 - \frac{1}{2}\left[w\left(U_k+\frac{2M}{r}\right)z\right]'|\tilde{\psi}_k|^2\\
&\quad - \frac{1}{2}z'\left(\frac{c_k'}{c_k}\right)^2|\tilde{\psi}_k|^2 - z\left(\frac{wc_{k+1}}{c_k}\right)^2c(r)|\tilde{\psi}_{k+1}|^2\\
&\quad - z\frac{wc_{k+1}}{c_k}\,\mathrm{Re}\left\{\overline{\tilde{\psi}_{k+1}}\left[\left(\frac{c_k'}{c_k}\right)'\tilde{\psi}_k - \frac{c_k'}{c_k}\underline{L}\tilde{\psi}_k - c(r)\frac{c_k'}{c_k}\tilde{\psi}_k\right]\right\}
\end{aligned}$$

$$- z\sum_{j=0}^{k-1}\frac{wc_j}{c_k}c^{\mathrm{id}}_{s,k,j}\operatorname{Re}[\overline{L\tilde{\psi}_k}\tilde{\psi}_j].$$

Here, we have used the notation

$$c(r)\doteq\frac{c_k'}{c_k}-(|s|-k)\frac{w'}{w}=\frac{2(|s|-k)w}{r},$$

and obtained the latter identity by, after the usual integration by parts procedure for the $\mathfrak{R}_k$ operator, noticing

$$\begin{aligned}
&-\operatorname{Re}\left\{\overline{L\tilde{\psi}_k}\left[c\frac{wc_{k+1}}{c_k}\tilde{\psi}_{k+1}+\left(\frac{c_k'}{c_k}\right)'\tilde{\psi}_k-\frac{c_k'}{c_k}\underline{L}\tilde{\psi}_k\right]\right\}+\left(\frac{wc_{k+1}}{c_k}\right)^2c|\tilde{\psi}_{k+1}|^2\\
&\quad=-\frac{wc_{k+1}}{c_k}\operatorname{Re}\left\{\overline{\tilde{\psi}_{k+1}}\left[\left(\frac{c_k'}{c_k}\right)'\tilde{\psi}_k-\frac{c_k'}{c_k}\underline{L}\tilde{\psi}_k-c(r)\frac{c_k'}{c_k}\tilde{\psi}_{k+1}\right]\right\}\\
&\qquad+\frac{1}{2}\left[\left(\frac{c_k'}{c_k}\right)^2\right]|\tilde{\psi}_k|^2-\frac{1}{2}\left(\frac{c_k'}{c_k}\right)^2\underline{L}|\tilde{\psi}_k|^2.
\end{aligned}$$

Similarly to Lemma 4.6, let $\chi(r^*)$ be a smooth cutoff function localized to $r\approx 2M$: $\chi=1$ for $r^*\leq R^*$ and $\chi=0$ for $r^*\geq R^*+1$, $R^*<-1$. Then, summing the $z(r)\underline{L}$ identity together with a small multiple of the $z(r)L$ identity, choosing $z=\chi$ for both, yields an estimate of the form:

$$\begin{aligned}
&\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}\left[|\underline{L}\tilde{\psi}_k|^2+(1-\mu)\left(|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+|\tilde{\psi}_k|^2\right)\right]dr^*d\sigma d\tau'\\
&\qquad+\int_{\Sigma_\tau\cap\{r^*\leq R^*\}}\left[|\underline{L}\tilde{\psi}_k|^2+(1-\mu)\left(|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+|\tilde{\psi}_k|^2\right)\right]dr^*d\sigma\\
&\qquad+\mathbb{E}_{k,\mathcal{H}^+}(0,\tau)\\
&\quad\leq B\mathbb{E}_k(0)+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)|\tilde{\psi}_{k+1}|^2dr^*d\sigma d\tau'\\
&\qquad\quad+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*-R^*\in[0,1]\}}\left[|\underline{L}\tilde{\psi}_k|^2+|L\tilde{\psi}_k|^2+|\mathring{\nabla}\tilde{\psi}_k|^2+|\tilde{\psi}_k|^2\right]\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau'\\
&\qquad\quad+B\sum_{j=0}^{k-1}\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)^2|\tilde{\psi}_j|^2dr^*d\sigma d\tau',
\end{aligned}$$

for sufficiently negative R^*. By making $|R^*|$ even larger, we can use the additional factor of $(1-\mu)$ in the coupling to $\tilde{\psi}_j$ as a smallness parameter; then, iterating the estimate above for $j=0,\dots,k$, we deduce that it holds without the last line as long as $\mathbb{E}_k(0)$ is replaced by $\sum_{j=0}^k\mathbb{E}_j(0)$.

Similarly to Lemma 4.6, we can now refine the estimate even further. Adding the $z(r)\underline{L}$ identity now with $z(r) = (1-\mu)^{-q}r^4\chi$, we obtain improve the left hand side of the estimate above to

$$\int_{\Sigma_\tau\cap\{r^*\leq R^*\}} \left[(1-\mu)^{-q}|\underline{L}\tilde{\psi}_k|^2 + (1-\mu)\left(|L\tilde{\psi}_k|^2 + |\mathring{\nabla}\tilde{\psi}_k|^2 + |\tilde{\psi}_k|^2\right)\right] dr^* d\sigma$$
$$+ \int_0^\tau \int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}} (1-\mu)\left[\frac{1}{(1-\mu)^{1+q}}|\underline{L}\tilde{\psi}_k|^2 + |L\tilde{\psi}_k|^2\right] dr^* d\sigma d\tau'$$
$$+ \int_0^\tau \int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}} (1-\mu)^{1-q}[(1-q)+(1-\mu)]\left(|\mathring{\nabla}\tilde{\psi}_k|^2 + |\tilde{\psi}_k|^2\right) dr^* d\sigma d\tau'$$
$$+ \mathbb{E}_{k,\mathcal{H}^+,q}(0,\tau).$$

(The above estimate corresponds to $q = 0$). Thus, this new estimate can again be iterated as before for $j = 0, \dots k$ if R^* is chosen sufficiently negative.

Finally, to obtain the estimates in the form of the statement, we simply note that we can extend the support of χ to large, but finite, r^*. □

Proof of Lemma 4.9. Fix some $s > 0$, and consider the wave equation (2.12) with $k \leq |s| - 1$. Our strategy will be similar to that of Lemma 4.6 and Lemma 4.8 above: we will make use of multiplier identities for the multipliers $z(r)L$ and $z(r)\underline{L}$ acting on (2.12). With the notation from (4.7), the identities we will use are:

$$F_L^{z\underline{L}} = z|\underline{L}\psi_k|^2, \qquad F_{\underline{L}}^{z\underline{L}} = \frac{1}{2}zw|\mathring{\nabla}\psi_k|^2 + \frac{1}{2}zw\left(U_k + \frac{2M}{r}\right)|\psi_k|^2,$$
$$I^{z\underline{L}} = -z'|\underline{L}\psi_k|^2 + \frac{1}{2}(zw)'|\mathring{\nabla}\psi_k|^2 + \frac{1}{2}\left[w\left(U_k + \frac{2M}{r}\right)z\right]'|\psi_k|^2$$
$$- (|s|-k)w'zw|\psi_{k+1}|^2 - z\sum_{j=0}^{k-1} wc^{\mathrm{id}}_{s,k,j}\operatorname{Re}\left[\overline{\underline{L}\psi_k}\psi_j\right],$$

for $z(r)\underline{L}$, and for $z(r)L$

$$F_{\underline{L}}^{zL} = z|L\psi_k|^2,$$
$$F_L^{zL} = \frac{1}{2}zw|\mathring{\nabla}\psi_k|^2 + \frac{1}{2}zw\left(U_k + \frac{2M}{r}\right)|\psi_k|^2,$$
$$I^{zL} = z'|L\psi_k|^2 + \frac{1}{2}(zw)'|\mathring{\nabla}\psi_k|^2 + \frac{1}{2}\left[w\left(U_k + \frac{2M}{r}\right)z\right]'|\psi_k|^2$$
$$- (|s|-k)w'z\operatorname{Re}[\psi_{k+1}\overline{L\psi_k}] - z\sum_{j=0}^{k-1} wc^{\mathrm{id}}_{s,k,j}\operatorname{Re}\left[\overline{L\psi_k}\psi_j\right].$$

Let χ be a smooth cutoff function with $\chi=1$ for $r^*\leq R^*$ and $\chi=0$ for $r^*\geq R^*+1$, for $R^*<-1$. Then, we apply the $z\underline{L}$ identity with $z=(1-\mu)^{-q}\chi$ and add a small multiple of the the zL identity with $z=\chi$. We will obtain, for $q\in[0,1]$,

$$\begin{aligned}
&\mathbb{E}_{k,\mathcal{H}^+,0}(0,\tau)+\mathbb{E}_{k,\mathcal{H}^+,q}(0,\tau)+\int_{\Sigma_\tau\cap\{r^*\leq R^*\}}(1-\mu)^{-q}|\underline{L}\psi_k|^2dr^*d\sigma\\
&\quad+\int_{\Sigma_\tau\cap\{r^*\leq R^*\}}(1-\mu)\left[|L\psi_k|^2+|\mathring{\nabla}\psi_k|^2+|\psi_k|^2\right]dr^*d\sigma\\
&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}\left[(1-\mu)^{-q}|\underline{L}\psi_k|^2+(1-\mu)|L\psi_k|^2\right]dr^*d\sigma d\tau'\\
&\quad+\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)^{1-q}\left((1-q)+(1-\mu)\right)\left[|\mathring{\nabla}\psi_k|^2+|\psi_k|^2\right]dr^*d\sigma d\tau'\\
&\leq B(R^*)\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*-R^*\in[0,1]\}}\left[|L\psi_k|^2+|\mathring{\nabla}\psi_k|^2+\sum_{j=0}^{k+1}|\psi_j|^2\right]dr^*d\sigma d\tau'\\
&\quad+B\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)|\tilde{\psi}_{k+1}|^2dr^*d\sigma d\tau'\\
&\quad+B\int_{\Sigma_\tau\cap\{r^*\leq R^*\}}(1-\mu)|\tilde{\psi}_{k+1}|^2dr^*d\sigma+B\mathbb{E}_k(0)\\
&\quad+\frac{B}{|R^*|^q}\sum_{j=0}^{k}\int_0^\tau\int_{\Sigma_{\tau'}\cap\{r^*\leq R^*\}}(1-\mu)^{1-q}|\tilde{\psi}_j|^2dr^*d\sigma d\tau',
\end{aligned}$$

for sufficiently negative R^*. In particular, choosing some $q\in(0,1)$ to start with, we can iterate the estimate in $j=0,\dots,k$. Making $|R^*|$ even larger so that the terms in the last line are small, we deduce that the estimate holds without the last line, for all $q\in[0,1]$, if $\mathbb{E}_k(0)$ is replaced by $\sum_{j=0}^k\mathbb{E}_j(0)$. □

4.2 Higher order energy boundedness and integrated local energy decay

In the previous section, we have shown that all first derivatives of the kth transformed variable can be controlled in terms of zeroth order quantities in the $(k+1)$th transformed variable, see Proposition 4.3. In view of Theorem 3.1, which is closed at the level of *first* order derivatives of $|s|$th transformed variable (possibly with degeneration at $r=3M$), we should be able to upgrade Proposition 4.3 to higher order control for the $k<|s|$ variables. This is precisely the goal of the present section.

When trying to control higher order derivatives, it is common to commute with Killing fields, such as T, as we have done already in Section 3.2

to obtain Corollary 3.2. However, for the goal of obtaining Theorem 4.1, that would not be adequate: due to trapping, we cannot expect to control $T\partial\psi^{[s]}_{(k)}$ or $\mathring{\nabla}\partial\psi^{[s]}_{(k)}$ bulk terms without a degeneration. Instead, we will commute with the vector field which does not "see" trapping (cf. Theorem 3.1), that is, ∂_{r^*}.

Lemma 4.10 (Commutation with ∂_{r^*}). *Fix $s\in\mathbb{Z}$ and $k\in\{0,\dots,|s|\}$. We have the identity*

$$\begin{aligned}\mathfrak{R}^{[s]}_{(k)}\left[\left(\psi^{[s]}_{(k)}\right)'\right] &= w'T\psi^{[s]}_{(k+1)} + w'\operatorname{sgn} s(|s|-k+1)\left(\psi^{[s]}_{(k+1)}\right)' \\ &\quad + \operatorname{sgn} s\left((|s|-k)\left(\frac{w'}{w}\right)' + \frac{(w')^2}{w}\right)\psi^{[s]}_{(k+1)} \\ &\quad + w\left(U^{[s]}_{(k)}+\frac{2M}{r}\right)'\psi^{[s]}_{(k)} + \sum_{j=0}^{k-1} w\left(c^{\mathrm{id}}_{s,k,j}\psi^{[s]}_{(j)}\right)',\end{aligned}$$

where $\mathfrak{R}^{[s]}_{(k)}$ is the differential operator defined in (2.13).

Proof. An easy computation shows that

$$[\mathfrak{R}_k,\partial_{r^*}] = -w'\mathring{\triangle}\phi_k - \left[\left(U_k+\frac{2M}{r}\right)w\right]'\phi_k.$$

To conclude, we make use of the constraint equation (2.15). □

With this lemma, we are now ready to conclude the proof of our main result for the lower level wave equations:

Proof of Theorem 4.1. Take $s\in\mathbb{Z}$ and $k\in\{0,\dots,|s|-1\}$ We have already shown that the estimates of Theorem 4.1 holds for first order energies in the k transformed variables; thus, the we only have to show that we can bootstrap from first to second order energies.

Let us start by using Lemma 4.10. The fact that ∂_{r^*} is not a symmetry of the equation is manifest from the commutator terms of Lemma 4.10. If one repeats the arguments of the previous section, these additional terms will generate additional bulk errors. For all but the first such error, we can treat them using Cauchy–Schwarz: e.g. for the choices $p=0=q$ and $\delta=1$ (and dropping those subscripts from the norms), the errors produced by all but the first term in Lemma 4.10 are controlled by

$$\epsilon B\sum_{j=0}^{k}\left(\mathbb{I}^{\partial_{r^*}}_j(0,\tau)+\mathbb{E}^{\partial_{r^*}}_j(\tau)\right)$$

$$+ \epsilon^{-1} B \sum_{j=0}^{k} \Big(\mathbb{I}_{j,1}(0,\tau) + \mathbb{E}_j(\tau) + \mathbb{E}_{j,\mathcal{H}^+}(0,\tau) + \mathbb{E}_{j,\mathcal{I}^+}(0,\tau) \Big)$$
$$+ \epsilon^{-1} B \Big(\mathbb{I}^{\deg}_{k+1}(0,\tau) + \mathbb{E}_{k+1}(\tau) + \mathbb{E}_{k+1,\mathcal{H}^+}(0,\tau) + \mathbb{E}_{k+1,\mathcal{I}^+}(0,\tau) \Big)$$
$$+ \sum_{j=0}^{k+1} \mathbb{E}_j(0) + \sum_{j=0}^{k} \mathbb{E}_j^{\partial_{r^*}}(0), \tag{4.8}$$

for small $\epsilon > 0$. Here, the ∂_r^* superscript represents ∂_r^*-commuted norms. For the first term in Lemma 4.10, we need extra work: repeating the steps of the previous section for the wave equation (2.12) will cause it to produce an error of the form

$$\begin{aligned} &w' \operatorname{Re}[\overline{T\psi_{k+1}} X \psi_k'] \\ &\quad = \Big(w' \operatorname{Re}[\overline{T\psi_{k+1}} X\psi_k]\Big)' - T \operatorname{Re}[(w'\psi_{k+1})' \overline{X\psi_k}] + \operatorname{Re}[(w'\psi_{k+1})' \overline{XT\psi_k}] \\ &\quad = \Big(w' \operatorname{Re}[T\psi_{k+1} \overline{X\psi_k}]\Big)' - T \operatorname{Re}[(w'\psi_{k+1})' \overline{X\psi_k}] \\ &\qquad + w \operatorname{Re}[(w'\phi_{k+1})' \overline{X\phi_{k+1}}] + \operatorname{sgn} s \operatorname{Re}[(w'\phi_{k+1})' \overline{X\phi_k'}], \end{aligned}$$

where X denotes either zL or $z\underline{L}$ for some choice of $z(r)$ function (specified in the the relevant lemmas). After the integration by parts procedure carried out in this identity, each term can be treated using Cauchy–Schwarz and controlled by (4.8). Notice that the same reasoning holds for (2.19). Thus, making ϵ sufficiently small yields control over all appropriately r-weighted second order derivatives of $\tilde{\psi}_k$ where one of the derivatives is ∂_{r^*}.

To conclude, we need only show that we can estimate second order terms involving appropriately r-weighted angular and time derivatives:

$$T^2\tilde{\psi}_k, \quad T\mathring{\nabla}\tilde{\psi}_k, \quad \mathring{\Delta}\tilde{\psi}_k.$$

For the first two, we use the fact that $T\psi_k = w\psi_{k+1} + \operatorname{sgn} s\psi_k'$ to reduce it to terms that we already understand. For the last one, a similar reduction can be attained from the constraint equation (2.15).

In the above sketch, we have overlooked details regarding the precise computation of r-weights involved. In view of the level of detail of the preceding sections, the reader should be able to fill in the missing details to obtain Theorem 4.1 at last. □

4.3 Decay of the energy and the solution

In this section, we prove Corollary 4.2. The proof strategy is very similar to the that of Corollary 3.2: it is based on the r^p method introduced in [DR10].

However, because we Theorem 4.1 does not yield estimates for $p = 2$, we require an additional interpolation lemma as in [SRTdC23, Theorem 9.3].

Proof of Corollary 4.2. To lighten the notation, let us denote simply $\mathbb{E}_{p,q}(\tau)$ the sum $\sum_{k=0}^{|s|} \mathbb{E}^{|s|-k}_{(k),p,q}$ for any $q \in [0,1]$. Repeating Step 1 from the proof of Corollary 3.2 with p replaced by $p-\eta$ for $\eta \in (0,1)$, we have

$$\mathbb{E}_{-\eta,1}(\tau_n) \leq \frac{B}{\tau_n}\left(\mathbb{E}_{2-\eta,1}(\tau_0) + \mathbb{E}^T_{2-\eta,1}(\tau_0) + \mathbb{E}^{TT}_{0,0}(\tau_0)\right),$$

on a dyadic sequence $\{\tau_n\}_{n=0}^{\infty}$ satisfying $\tau_n \uparrow \infty$.

We have not shown uniform boundedness of the $\mathbb{E}_{-\eta,1}$ energy. However, with the interpolation inequality

$$\begin{aligned}\mathbb{E}_{0,1}(\tau_n) &\leq (\mathbb{E}_{-\eta,1}(\tau_n))^{1-\eta/2} (\mathbb{E}_{2-\eta,1}(\tau_n))^{\eta/2} \\ &\leq \frac{B}{\tau_n^{2-\eta}}\left(\mathbb{E}_{2-\eta,1}(\tau_0) + \mathbb{E}^T_{2-\eta,1}(\tau_0) + \mathbb{E}^{TT}_{0,0}(\tau_0)\right)^{1-\eta/2} (\mathbb{E}_{2-\eta,1}(\tau_0))^{\eta/2},\end{aligned}$$

we can now easily conclude in the same fashion as Step 2 of Corollary 3.2. □

References

[AAG23] Y. Angelopoulos, S. Aretakis, and D. Gajic. Price's Law and Precise Late-Time Asymptotics for Subextremal Reissner–Nordström Black Holes. *Ann. Henri Poincaré*, 24(9):3215–3287, 2023.

[BP73] J. M. Bardeen and W. H. Press. Radiation fields in the Schwarzschild background. *J. Math. Phys.*, 14(1):7–19, 1973.

[Cha75] S. Chandrasekhar. On the Equations Governing the Perturbations of the Schwarzschild Black Hole. *Proc. R. Soc. A Math. Phys. Eng. Sci.*, 343(1634):289–298, 1975.

[DHR19a] M. Dafermos, G. Holzegel, and I. Rodnianski. Boundedness and Decay for the Teukolsky Equation on Kerr Spacetimes I: The Case $|a| \ll M$. *Ann. PDE*, 5(2):1–118, 2019.

[DHR19b] M. Dafermos, G. Holzegel, and I. Rodnianski. The linear stability of the Schwarzschild solution to gravitational perturbations. *Acta Math.*, 222(1):1–214, 2019.

[DHRT21] M. Dafermos, G. Holzegel, I. Rodnianski, and M. Taylor. The non-linear stability of the Schwarzschild family of black holes. *Preprint*, 2021, arXiv:2104.08222.

[DR09] M. Dafermos and I. Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Commun. Pure Appl. Math.*, 62(7):859–919, 2009.

[DR10] M. Dafermos and I. Rodnianski. A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. In *XVIth International Congress on Mathematical Physics*, pages 421–432. World Scientific, 2010.

[DR13] M. Dafermos and I. Rodnianski. Lectures on black holes and linear waves. In *Evolution equations. Clay Mathematics Proceedings, vol. 17*, pages 97–205, Providence, Rhode Island, 2013. American Mathematical Society.

[KS20] S. Klainerman and J. Szeftel. *Global Nonlinear Stability of Schwarzschild Spacetime under Polarized Perturbations.* Annals of Mathematics Studies, 2020.

[MZ22] S. Ma and L. Zhang. Price's Law for Spin Fields on a Schwarzschild Background. *Ann. PDE*, 8(2):25, 2022.

[Pas19] F. Pasqualotto. The spin ±1 Teukolsky equations and the Maxwell system on Schwarzschild. *Ann. Henri Poincaré*, 20(4):1263–1323, 2019.

[RW57] T. Regge and J. A. Wheeler. Stability of a Schwarzschild Singularity. *Phys. Rev.*, 108(4):1063–1069, 1957.

[SRTdC20] Y. Shlapentokh-Rothman and R. Teixeira da Costa. Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range $|a| < M$: frequency space analysis. *Preprint*, 2020, arXiv:2007.07211.

[SRTdC23] Y. Shlapentokh-Rothman and R. Teixeira da Costa. Boundedness and decay for the Teukolsky equation on Kerr in the full subextremal range $|a| \leq M$: physical space analysis. *Preprint*, 2023, arXiv:2302.08916.

[Teu73] S. A. Teukolsky. Perturbations of a Rotating Black Hole. I. Fundamental Equations for Gravitational, Electromagnetic, and Neutrino-Field Perturbations. *Astrophys. J.*, 185:635–647, 1973.

Malliavin differentiability of McKean-Vlasov SDEs with locally Lipschitz coefficients

Gonçalo dos Reis
University of Edinburgh, School of Mathematics, Edinburgh, UK
and
Center for Mathematics and Applications (NOVA Math),
NOVA School of Science and Technology (NOVA FCT), PT
e-mail: `G.dosReis@ed.ac.uk`

Zac Wilde
University of Edinburgh, School of Mathematics, Edinburgh, UK
e-mail: `Z.J.Wilde@sms.ed.ac.uk`

Resumo: Nesta pequena nota estabelecemos a diferenciabilidade no sentido do cálculo de Malliavin das soluções das Equações Diferenciais Estocásticas de McKean-Vlasov (MV-SDEs), onde se assume que o coeficiente de tendência ('drift') é localmente Lipschitz e satisfaz uma condição Lipschitz 'lateral' enquanto o coeficiente de difusão satisfaz a condição Lipschitz usual.

Como contribuição secundária, é investigado como é que a diferenciabilidade no sentido do cálculo de Malliavin se transfere através do limite de sistemas de partículas que converge para a solução da MV-SDE original. Estabelecer este resultado requer que os coeficientes da equação sejam diferenciáveis na componente espaço e na componente da medida de probabilidade, e logo o resultado apresentado não é tão geral quanto o resultado da primeira parte do artigo (onde não é necessária a diferenciabilidade em medida, apenas a propriedade Lipschitz na métrica de Wasserstein). No entanto, a metodologia é de interesse independente por ser esta a primeira vez a aparecer na literatura (ao melhor do nosso conhecimento). A apresentação da secção é de carácter didático e a secção conclui com uma discussão alargada sobre técnicas de molificação para a derivada de Lions (i.e., derivadas na variável da medida de probabilidade).

Abstract: In this short note, we establish Malliavin differentiability of McKean-Vlasov Stochastic Differential Equations (MV-SDEs) with drifts satisfying both a locally Lipschitz and a one-sided Lipschitz assumption, and where the diffusion coefficient is assumed to be uniformly Lipschitz in its variables.

As a secondary contribution, we investigate how Malliavin differentiability transfers across the interacting particle system associated with the McKean-Vlasov equation to its limiting equation. This final result requires both spatial and measure differentiability of the coefficients and doubles as a standalone result of independent interest since the study of Malliavin derivatives of weakly interacting particle systems seems novel to the literature. The presentation is didactic and finishes with a discussion on mollification techniques for the Lions derivative.

Keywords: McKean-Vlasov SDEs, Malliavin differentiability, superlinear growth, interacting particle systems

Contents

1 Introduction

The main object of our study are McKean-Vlasov Stochastic Differential Equations (MV-SDE), also known as mean-field equations or distribution-dependent SDEs. They differ from standard SDEs by means of the presence of the law of the solution process in the coefficients. Namely

$$dZ_t = b(t, Z_t, \mu_t)dt + \sigma(t, Z_t, \mu_t)dW_t, \quad X_0 = \xi,$$

for some measurable coefficients, where $\mu_t = \mathrm{Law}(Z_t)$ denotes the law of process Z at time t (with W a Brownian motion; ξ a random initial condition). Similar to standard SDEs, MV-SDEs are shown to be well-posed under a variety of frameworks, for instance, under locally Lipschitz and super-linear growth conditions alongside random coefficients, see e.g. [AdRR$^+$22] or [dRST19]. In this setting there are also many studies on their numerical approximation e.g. [dRES22, CdR22, CdR24], ergodicity [CdS25] and large deviations [dRST19, AdRR$^+$22].

Many mean-field models exhibit drift dynamics that include superlinear growth and non-global Lipschitz growth, for example, mean-field models for neuronal activity (e.g. stochastic mean-field FitzHugh-Nagumo models or the network of Hodgkin-Huxley neurons) [BFFT12], [BCC11], [BFT15] appearing in biology or the physics of modelling batteries [DGG$^+$11],[DFG$^+$16]. Quoting [GPV20], systems of weakly-interacting particles and their limiting processes, so-called McKean-Vlasov or mean-field equations appear in a wide variety of applications, ranging from plasma physics and galactic dynamics to mathematical biology, the social sciences, active media, dynamical density functional theory (DDFT) and machine learning. They can also be used in models for co-operative behavior, opinion formation, risk management, as well as in algorithms for global optimization.

Our 1st contribution: Malliavin differentiability of MV-SDEs under locally Lipschitz conditions. We extend Malliavin variational results to McKean-Vlasov SDEs with locally Lipschitz drifts satisfying a so-called one-sided Lipschitz condition. The result is new to the best of our knowledge. Malliavin differentiability of MV-SDEs has been addressed in [CM18, Proposition 3.1] and [RW19], and in both cases, their assumptions revolve around the differentiable Lipschitz case. Our proof methodology is inspired by that of [CM18] – both there and here, the result is established by appealing to the celebrated [Nua06, Lemma 1.2.3].

Our 2nd contribution: transfer of Malliavin differentiability across the particle system limit. Another large aspect of McKean-Vlasov SDE theory, is the study of the large weakly-interacting particle systems and their particle limit that recovers the MV-SDE in the limit. This latter limit result is called Propagation of Chaos [Szn91] (also [BCC11],[GPV20], [CCD22], [AdRR$^+$22]). In the second part of this note, we study the Malliavin differentiability of the interacting particle system and how the Malliavin regularity transfers across the particle limit to the limiting equation. To the best of our knowledge, this particular proof methodology is new to the literature.

From a methodological viewpoint, our point of attack is the *projections over empirical measures* approach [CCD22, CD18a, dRP23]. This approach allows us to use the best available Malliavin differentiability results for standard (multidimensional) SDEs [Nua06, IdRS19], and then carry them to the MV-SDE setting via the particle limit using Propagation of Chaos and [Nua06, Lemma 1.2.3]. Our variational results are limited only by the SDE results we cite. If better results are found, one only needs to replace the reference in the appropriate place. Lastly, in relation to our 1st contribution, this 2nd contribution is established under a full global Lipschitz and differentiability (space and measure) assumption on the coefficients.

Organization of the paper. In section 2, we set notation and review a few concepts necessary for the main constructions. In section 3, we prove Malliavin differentiability of MV-SDEs under superlinear drift growth assumptions and in section 4 we prove Malliavin differentiability of MV-SDEs under the weaker global Lipschitz assumptions via the convergence of interacting particle systems, providing a lengthy remark about mollification in Wasserstein spaces.

2 Notation and preliminary results

2.1 Notation and Spaces

For collections of vectors, let the upper indices denote the distinct vectors, whereas the lower index is a vector component, i.e. x^l_j denote the j-th component of l-th vector. Let $\mathbf{x} = (x^1, \cdots, x^N)$ denote a vector in $\mathbb{R}^{dN}$ where $x^i := (x^i_1, \cdots, x^i_d)$ for $i = 1, \cdots, N$. For matrices M and N of agreeing dimensions, define the inner product $M : N = \text{Trace}(M^T N)$ and the norm induced by this inner product (the Hilbert-Schmidt norm) as $|M| = \sqrt{\text{Trace}(M^T M)}$.

For $g : \mathbb{R}^m \to \mathbb{R}^n$, writing $g(x) = (g^1(x), \cdots, g^n(x))$ and define $\nabla_x g$ as the Jacobian matrix $(g_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$ where $g_{ij} = \partial_{x^i} g^j$.

Take $T \in [0, \infty)$ and let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered probability space carrying a m-dimensional Brownian Motion on the interval $[0, T]$ and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The filtration is the one generated by the Brownian motion and augmented by the $\mathbb{P}$-null sets, and with an additionally sufficiently rich sub σ-algebra $\mathcal{F}_0$ independent of W. We denote by $\mathbb{E}[\cdot] = \mathbb{E}^{\mathbb{P}}[\cdot]$ the usual expectation operator with respect to $\mathbb{P}$.

The space of probability measures on $\mathbb{R}^d$ with finite second moment, $\mathcal{P}_2(\mathbb{R}^d)$ is Polish under the 2-Wasserstein distance

$$W_2(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi(dx,dy) \Big)^{\frac{1}{2}}, \quad \mu,\nu \in \mathcal{P}_2(\mathbb{R}^d),$$

where $\Pi(\mu,\nu)$ is the set of couplings for μ and ν such that $\pi \in \Pi(\mu,\nu)$ is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\pi(\cdot \times \mathbb{R}^d) = \mu$ and $\pi(\mathbb{R}^d \times \cdot) = \nu$. Let $\mathrm{Supp}(\mu)$ denote the support of $\mu \in \mathcal{P}(\mathbb{R}^d)$.

Let $p \in [2,\infty)$. We introduce the following spaces.

- Let $\mathcal{X}$ be a metric space. We denote by $C(\mathcal{X})$ as the space of continuous functions $f : \mathcal{X} \to \mathbb{R}$ endowed with the uniform norm and $C_b(\mathcal{X})$ its subspace of bounded functions endowed with the sup norm $\|f\|_\infty = \sup_{x\in\mathcal{X}} |f(x)| < \infty$; For $k \in \mathbb{N}$ denote $\mathcal{C}^k(\mathbb{R}^d)$ the space of k-times continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}^d$, equipped with a collection of seminorms $\{\|g\|_{\mathcal{C}^p(K)} := \sup_{x\in K}(|g(x)| + \sum_{j=1}^k |\partial_x^j g(x)|), g \in \mathcal{C}^k(\mathbb{R}^d)\}$, indexed by the compact subsets $K \subset \mathbb{R}^d$.

- $L^p(\Omega) := L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, $t \in [0,T]$ is the space of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random variables $X : \Omega \to \mathbb{R}^d$ with norm $\|X\|_{L^p(\Omega)} = \mathbb{E}[\,|X|^p]^{1/p} < \infty$.

- $\mathcal{S}^p([0,T]) := \mathcal{S}^p([0,T], \mathbb{P}; \mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued measurable $\mathbb{F}$-adapted processes $(Y_t)_{t\in[0,T]}$ satisfying $\|Y\|_{\mathcal{S}^p([0,T])} = \mathbb{E}[\sup_{t\in[0,T]} |Y(t)|^p]^{1/p} < \infty$.

2.2 Malliavin Calculus

Let $\mathcal{H}$ be a Hilbert space and $W : \mathcal{H} \to L^2(\Omega)$ a Gaussian random variable. The space $W(\mathcal{H})$ endowed with an inner product $\langle W(h_1), W(h_2)\rangle = \mathbb{E}[W(h_1)W(h_2)]$ is a Gaussian Hilbert space. Let $C_p^\infty(\mathbb{R}^n;\mathbb{R})$ be the space of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ which have partial derivatives of all orders, each with polynomial growth. Let $\mathbb{S}$ be the collection of random variables $F : \Omega \to \mathbb{R}$ such that for $n \in \mathbb{N}$, $f \in C_p^\infty(\mathbb{R}^n;\mathbb{R})$ and $h_i \in \mathcal{H}$ can be written as $F = f(W(h_1),\dots,W(h_n))$. Then we define the derivative of F to be the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^n \partial_{x_i} f(W(h_1),\dots,W(h_n))h_i.$$

The Malliavin derivative from $L^p(\Omega)$ into $L^p(\Omega, \mathcal{H})$ is closable and the domain of the operator is defined to be $\mathbb{D}^{1,p}$, defined to be the closure of the of the set $\mathbb{S}$ with respect to the norm

$$\|F\|_{1,p} = \Big[\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathcal{H}}^p]\Big]^{\frac{1}{p}}.$$

We also define the directional Malliavin derivative $D^h F = \langle DF, h\rangle_{\mathcal{H}}$ for any choice of $h \in \mathcal{H}$. For more details, see [Nua06].

3 Malliavin differentiability under local Lipschitz assumptions

3.1 McKean-Vlasov Equations with locally Lipschitz coefficients

In this manuscript, we work with so-called McKean-Vlasov SDEs described by the following dynamics for $0 \le t \le T < \infty$,

$$dZ_t = b(t, Z_t, \mu_t)dt + \sum_{l=1}^{m} \sigma^l(t, Z_t, \mu_t)dW_t^l, \quad Z_0 = \xi, \tag{1}$$

where μ_t denotes the law of the process Z at time t, i.e. $\mu_t = \mathbb{P} \circ Z_t^{-1}$ and W^l, $l = 1, \ldots, m$ are 1-dimensional independent Brownian motions. We write $W = (W^1, \ldots, W^m)$ as the corresponding $m-$dimensional Brownian motion. In this chapter, we work under the locally Lipschitz case as the below assumption describes.

3.1.1 Assumptions

Assumption 3.1. *Let $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ and for $l = 1, \ldots, m$, $\sigma^l : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$. Then there $\exists L > 0$ such that:*

1. *For some $p \ge 2$, ξ is $\mathcal{F}_0$-measurable and $\xi \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$.*
2. *σ^l is continuous in time and Lipschitz in space-measure $\forall t \in [0,T]$, $\forall x, x' \in \mathbb{R}^d$ and $\forall \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ we have*

$$|\sigma^l(t,x,\mu) - \sigma^l(t,x',\mu')| \le L\Big(|x - x'| + W_2(\mu, \mu')\Big).$$

3. *b is continuous in time and satisfies the* one-sided Lipschitz condition *in space and is Lipschitz in measure:* $\forall t \in [0,T]$, $\forall x, x' \in \mathbb{R}^d$ *and* $\forall \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ *we have that*

$$\langle x - x', b(t,x,\mu) - b(t,\omega,x',\mu)\rangle_{\mathbb{R}^d} \leq L|x-x'|^2,$$
$$|b(t,x,\mu) - b(t,x,\mu')| \leq LW_2(\mu,\mu').$$

4. *b is Locally Lipschitz:* $\forall t \in [0,T]$, $\forall \mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\forall x, x' \in \mathbb{R}^d$ *such that* $|x|, |x'| < N$ *we have that* $\exists L_N > 0$ *such that*

$$|b(t,x,\mu) - b(t,x',\mu)| \leq L_N|x-x'|.$$

Throughout, denote by σ, the $d \times m$ matrix with columns $(\sigma^1, \cdots, \sigma^m)$.

Observe that time continuity is a sufficient condition for integrability of b and σ^l since we are working on a compact time interval. The above assumption ensures existence, uniqueness and related stability while the following will be used to ensure the differentiability results.

Assumption 3.2. *Let Assumption 3.1 hold. For any* $t \geq 0$, $\mu \in \mathcal{P}(\mathbb{R}^d)$ *the maps* $x \mapsto b(t,x,\mu)$ *and* $x \mapsto \sigma^l(t,x,\mu)$ *are* $C^1(\mathbb{R}^d)$. *The derivative maps are jointly continuous in their variables.*

Remark 3.3. *One recognises that the above assumptions can be weakened in a few ways. For instance, the Lipschitz constant can be a non-negative function of time* L_t, *under an* L^1*-integrability condition:* $\int_0^T L_s ds < \infty$. *Further, the time continuity* $t \mapsto b(t,0,\delta_0)$ *can be exchanged for an integrability condition:* $\int_0^T |b(s,0,\delta_0)|ds < \infty$, *while the time-continuity of* $t \mapsto \sigma(t,0,\delta_0)$ *can be exchanged for a square-integrability condition:* $\int_0^T |\sigma(s,0,\delta_0)|^2 ds < \infty$. *We leave these points open for the interested reader.*

3.1.2 Well-posedness and moment estimates

The first result establishes well-posedness, moment estimates and continuity in time for the solution of (1).

Theorem 3.4. *Let Assumption 3.1 hold with some* $p \geq 2$. *Then, MV-SDE* (1) *is well-posed and has a unique solution* $Z \in \mathcal{S}^p([0,T])$. *Moreover, it* *satisfies*

$$\mathbb{E}\left[\sup_{t\in[0,T]}|Z_t|^p\right]\leq Ce^{CT}\bigg(\mathbb{E}\big[\,|\xi|^p\big]+\mathbb{E}\Big[\Big(\int_0^T|b(s,0,\delta_0)|ds\Big)^p\Big]$$
$$+\mathbb{E}\Big[\Big(\int_0^T|\sigma(s,0,\delta_0)|^2ds\Big)^{\frac{p}{2}}\Big]\bigg),$$

for some positive constant C*. Lastly,* Z *has* $\mathbb{P}$*-almost surely continuous paths and its law* $[0,T]\ni t\mapsto\mu_t$ *is continuous under the* W_2*-distance.*

This result also yields estimates for standard SDEs (which have no measure dependency).

Proof. Well-posedness and the moment estimate follow from [AdRR$^+$22, Theorem 3.2] as their assumption (with $\mathcal{D}=\mathbb{R}^d$, x_0 and deterministic continuous maps b,σ) subsumes our Assumption 3.1. The continuity of the sample paths of Z and its law in W_2 is trivial. □

3.2 Malliavin differentiability with locally Lipschitz coefficients

We state the first main result of this work, the Malliavin differentiability of the solution of (1).

Theorem 3.5. *Let* $p\geq 2$*. Let Assumption 3.2 hold. Denote by* Z *the unique solution* (1) *in* $\mathcal{S}^p([0,T])$*. Then* Z *is Malliavin differentiable, i.e.* $Z\in\mathbb{D}^{1,2}(\mathcal{S}^2)\cap\mathbb{D}^{1,p}(\mathcal{S}^p)$*, and the Malliavin derivative against* W *satisfies for* $0\leq s\leq t\leq T$*,*

$$D_sZ_t=\sigma(s,Z_s,\mu_s)+\int_s^t(\nabla_x b)(r,Z_r,\mu_r)D_sZ_rdr$$
$$+\sum_{l=1}^m\int_s^t(\nabla_x\sigma^l)(r,Z_r,\mu_r)D_sZ_rdW_r^l. \tag{2}$$

If $s>t$ *then* $D_sZ_t=0$ $\mathbb{P}$*-a.s.*

Moreover, we have

$$\sup_{0\leq s\leq T}\|D_sZ\|^p_{\mathcal{S}^p([0,T])}\leq\sup_{0\leq s\leq T}\mathbb{E}\Big[\sup_{0\leq t\leq T}|D_sZ_t|^p\Big]\leq C\big(1+\|Z\|^p_{\mathcal{S}^p([0,T])}\big)$$
$$\leq C\big(1+\|\xi\|^p_{L^p(\Omega)}\big)<\infty,$$

and, reflecting that DZ_t is a Hilbert space valued random variable we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big(\int_0^T |D_sZ_t|^2 ds\Big)^{\frac{p}{2}}\Big] \leq C(1+\|\xi\|^p_{L^p(\Omega)}) < \infty. \tag{3}$$

This proof is inspired by that appearing in [CM18] but with critical differences to allow for the superlinear growth of the drift and the general Malliavin differentiability of [IdRS19].

We comment that using the proof methodology we present below, the result above can be extended in several ways – we leave these as open questions. The first is to allow for random drift and diffusion coefficients: [AdRR+22, Theorem 3.2] provides well-posedness and moment estimates and concluding Malliavin differentiability via [IdRS19, Theorem 3.2 or Theorem 3.7].

Another open setting, with continuous deterministic coefficients, is to establish our Theorem 3.5 with a drift map b having super-linear growth in the measure component: for instance, by allowing convolution type measure dependencies. Lastly, the differentiability requirements for b and σ can be weakened via mollification; see [IdRS19, Remark 3.4].

Example 3.6 (Linear interaction kernels)**.** *Take MV-SDE* (1) *with solution* $(Z_t,\mu_t)_{t\geq 0}$ *under the assumptions of Theorem 3.5 and let the drift function* b *take a specific convolutional form. Concretely, let* $b(t,x,\mu) = (\tilde{b}*\mu)(x) = \int_{\mathbb{R}^d}\tilde{b}(x-y)\mu(dy)$ *for some* $\tilde{b}\in C^1(\mathbb{R}^d)$ *and let* $\sigma=\sigma_0 I_d$ *for* $\sigma_0\neq 0$ *a constant.*

Then, we have for any $0\leq s\leq t\leq T$

$$D_sZ_t = \sigma_0 \exp\Big(\int_s^t (\nabla_x\tilde{b}*\mu_r)(Z_r)dr\Big). \tag{4}$$

We can in fact reduce the C^1 *in space differentiability assumption to a Lipschitz one, exploiting* [Nua06, Proposition 1.2.4]*. That is, we can take a sequence of mollifiers* $\tilde{b}_n := \tilde{b}*\rho_n$ *for a smoothing kernel* $(\rho_n)_n$ *such that* $C^\infty(\mathbb{R}^d)\ni\rho_n(x)\to x$ *uniformly, and the expression* (4) *still holds.*

We can generalise the above form of the drift to linear interaction kernels of the form $b(t,x,\mu)=\int_{\mathbb{R}^d}\widehat{b}(x,y)\mu(dy)$*, where* $\widehat{b}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ *and* $x\mapsto\widehat{b}(x,\cdot)$ *is Lipschitz uniformly to obtain for any* $0\leq s\leq t\leq T$

$$D_sZ_t = \sigma_0 \exp\Big(\int_s^t \Big\{\int_{\mathbb{R}^d}(\nabla_x\widehat{b})(Z_r,y)\mu_r(dy)\Big\}dr\Big).$$

Proof of Theorem 3.5. The Malliavin differentiability of (1) is shown by appealing to [Nua06, Lemma 1.2.3]. One builds a convenient sequence of Picard iterations which converge to the McKean-Vlasov Equation and use [Nua06, Lemma 1.2.3] to ensure that the limit is also Malliavin differentiable in $\mathbb{D}^{1,2}(\mathcal{S}^2)$. Lastly, by showing that the Malliavin derivative of Z is $\mathcal{S}^p$-integrable, then [Nua06, Proposition 1.5.5] yields that $Z \in \mathbb{D}^{1,p}(\mathcal{S}^p)$.

Step 1.0. The Picard sequence. We start by defining a Picard sequence approximation for (1), namely set $Z^0_\cdot = \xi$ and $\mu^0_t = \mathbb{P} \circ \xi^{-1} = \mathrm{Law}(\xi)$ for any $t \geq 0$; we have $t \mapsto \mu^0_t$ is a W_2-continuous map. For any $n \geq 1$ define

$$dZ^{n+1}_t = b(t, Z^{n+1}_t, \mu^n_t)dt + \sum_{l=1}^{m} \sigma^l(t, Z^{n+1}_t, \mu^n_t)dW^l_t, \quad Z^{n+1}_0 = \xi. \tag{5}$$

(5) is a standard SDE with added time dependence induced by $t \mapsto \mu^n_t$ with drift b satisfying a one-sided Lipschitz condition (in space) and σ uniformly Lipschitz (in space).

Step 1.1. Existence and uniqueness of Z^n. Take μ^{n-1} such that $t \mapsto b(t, x, \mu^{n-1}_t)$ and $t \mapsto \sigma^l(t, x, \mu^{n-1}_t)$ are continuous, then (a slight variation of) Theorem 3.4, given Assumption 3.1, yields the existence of a unique solution $Z^{n+1} \in \mathcal{S}^2([0,T])$. Moreover, an easy variation of [dRST19, Proposition 3.4] yields that for $n \geq 0$, $t \mapsto \mu^{n+1}_t$ is continuous (in 2-Wasserstein distance) if $t \mapsto \mu^n_t$ is. We can conclude that $\{Z^n\}_{n\geq 0}$ exists and is well defined.

Using that $W_2(\delta_0, \mu^n_\cdot)^2 \leq \mathbb{E}[|Z^n_\cdot|^2]$ and Theorem 3.4, we have

$$\begin{aligned}
&\|Z^{n+1}\|^2_{\mathcal{S}^2([0,T])} \\
&\leq C\Big(1 + \|\xi\|^2_{L^2(\Omega)} + \int_0^T \mathbb{E}[|Z^n_r|^2]dr\Big) \\
&\leq C\Big(1 + \|\xi\|^2_{L^2(\Omega)} + \int_0^T \|Z^n\|^2_{\mathcal{S}^2([0,r])}dr\Big) \\
&\leq C\Big(1 + \|\xi\|^2_{L^2(\Omega)} + \int_0^T \Big\{C(1 + \|\xi\|^2_{L^2(\Omega)} + \int_0^r \|Z^{n-1}\|^2_{\mathcal{S}^2([0,s])}ds\Big\}dr\Big) \\
&\leq \cdots \leq C(1 + \|\xi\|^2_{L^2(\Omega)})\Big(\sum_{j=0}^{n} \frac{(CT)^j}{j!}\Big)\|Z^0\|^2_{\mathcal{S}^2([0,T])} \\
&\qquad \leq C(1 + \|\xi\|^2_{L^2(\Omega)})e^{CT}\|Z^0\|^2_{\mathcal{S}^2([0,T])},
\end{aligned}$$

where we iterated the initial estimate on $[0,T]$ over small subintervals $[0,r]$ leading to a known *simplex* estimate. We conclude that

$$\sup_{n\geq 0}\{\|Z^n\|_{\mathcal{S}^2([0,T])} + \sup_{0\leq t\leq T} W_2(\delta_0, \mu^n_t)\} < \infty. \tag{6}$$

Step 1.2. Convergence of Z^n. Recall that (5) is a standard SDE, thus standard SDE stability estimation arguments apply. We sketch such argument only and invite the reader to inspect the proof of [dRST19, Proposition 3.3] for the full details. Take the SDE for the difference of $Z^{n+1} - Z^n$, i.e.

$$Z_t^{n+1} - Z_t^n = \int_0^T \left[b(s, Z_s^{n+1}, \mu_s^n) - b(s, Z_t^n, \mu_s^{n-1})\right] ds \\ + \sum_{l=1}^m \int_0^T \left[\sigma^l(s, Z_s^{n+1}, \mu_s^n) - \sigma^l(s, Z_t^n, \mu_s^{n-1})\right] dW_s^l.$$

Applying Itô's formula to $|Z_t^{n+1} - Z_t^n|^2$, using the growth assumptions on b, σ and taking the supremum over time and expectations, we use the BDG inequality and then the Grönwall inequality to obtain

$$\begin{aligned} &\|Z^{n+1} - Z^n\|^2_{\mathcal{S}^2([0,T])} \\ &\quad \leq C_T \Big(\int_0^T W_2(\mu_r^n, \mu_r^{n-1})^2 dr \Big) \leq C_T \int_0^T \mathbb{E}[|Z_r^n - Z_r^{n-1}|^2]] dr \\ &\quad \leq C_T \int_0^T \|Z^n - Z^{n-1}\|^2_{\mathcal{S}^2([0,r])} dr \leq \cdots \leq \frac{(C_T T)^n}{n!} \|Z^1 - Z^0\|^2_{\mathcal{S}^2([0,T])}, \end{aligned}$$

where we used the same *simplex* trick as in *Step 1.1* above. We conclude that Z^n converges to the solution of (1) in $\mathcal{S}^2([0,T])$ as $\|Z^1 - Z^0\|^2_{\mathcal{S}^2([0,T])}$ is bounded and independent of n.

Step 2. Malliavin differentiability for Z^n. Now, under our Assumption 3.2, [IdRS19, Corollary 3.5] holds applied to yield the Malliavin differentiability of Z^n for each fixed n; critically, due to the equation's coefficients being deterministic, that corollary does not require $p > 2$ and it holds for any $p \geq 2$ (in particular, our case here for the time being $p = 2$). We have that the Malliavin Derivative DZ^{n+1} satisfies $D_s Z_t^{n+1} = 0$ for $0 \leq t < s \leq T$, while for $0 \leq s \leq t \leq T$ it is given by the SDE dynamics

$$D_s Z_t^{n+1} = \sigma(s, Z_s^{n+1}, \mu_s^n) + \int_s^t (\nabla_x b)(r, Z_r^{n+1}, \mu_r^n) D_s Z_r^{n+1} dr \\ + \sum_{l=1}^m \int_s^t (\nabla_x \sigma^l)(r, Z_r^{n+1}, \mu_r^n) D_s Z_r^{n+1} dW_r^l.$$

Step 3. Uniform bound on DZ^n. We remark that the SDE for DZ^n is linear and satisfies [IdRS19, Assumption 2.4] with their $b(s,\omega), \sigma(s,\omega)$ set to zero, hence [IdRS19, Theorem 2.5] applies to yield

$$\begin{aligned}\|D.Z^n\|^2_{\mathcal{S}^2([0,T])} = \mathbb{E}\big[\sup_{0\le t\le T}|D.Z^n_t|^2\big] &\le C\mathbb{E}\big[\,|\sigma(\cdot,Z^n_\cdot,\mu^{n-1}_\cdot)|^2\,\big]\\ &\le C\Big(1+\|Z^n\|^2_{\mathcal{S}^2([0,T])}+W_2^2(\mu^{n-1}_\cdot,\delta_0)\Big)\\ &\le C\Big(1+\|Z^n\|^2_{\mathcal{S}^2([0,T])}+\|Z^{n-1}\|^2_{\mathcal{S}^p([0,T])}\Big)\\ &\le C(1+\|\xi\|^2_{L^2(\Omega)})<\infty, \end{aligned}\tag{7}$$

where we used the linear growth of σ and its time continuity property. The constant C depends heavily on the constants appearing in Assumption 3.1 and the upper bound on $t\mapsto\sigma(t,0,\delta_0)$ stemming from its continuity over the compact $[0,T]$. Using that (6) provides an estimate of $\|Z^n\|_{\mathcal{S}^2([0,T])}$ uniform over n and uniform over the Malliavin derivative parameter, we conclude taking supremum that

$$\sup_{n\in\mathbb{N}}\sup_{s\in[0,T]}\mathbb{E}\big[\sup_{t\in[0,T]}|D_sZ^n_t|^2\big]\le C(1+\|\xi\|^2_{L^2(\Omega)})<\infty.\tag{8}$$

We also establish the inequality for DZ^n_t that yields its interpretation as Hilbert space valued random variable. Simple manipulations and using (8) above easily yields estimate (3) when $p=2$ (when we pass to the limit in $n\to\infty$). Concretely, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\int_0^T|D_sZ^n_t|^2ds\Big] &\le C\mathbb{E}\Big[\int_0^T\{\sup_{t\in[0,T]}|D_sZ^n_t|^2\}ds\Big]\\ &\le C\int_0^T\mathbb{E}\Big[\sup_{t\in[0,T]}|D_sZ^n_t|^2\Big]ds\\ &\le C\sup_{s\in[0,T]}\mathbb{E}\Big[\sup_{t\in[0,T]}|D_sZ^n_t|^2\Big]\\ &\le C(1+\|\xi\|^p_{L^2(\Omega)})<\infty,\end{aligned}\tag{9}$$

where we firstly used Jensen's inequality and moved the supremum inside the integral, then used Fubini and dominated the integral by the supremum over the integration variable in a way that (7) can be used. Taking supremum over n on both sides and using (8) yields (3).

By applying [Nua06, Lemma 1.2.3], we conclude that the limit of Z is Malliavin differentiable and the limit of DZ^n gives its Malliavin derivative (identified by Equation (2)). The moment estimates for DZ^n, holding uniformly over n, yield the moment estimate for DZ.

Step 4. Higher-order moments on DZ and conclusion for $\mathbb{D}^{1,p}(\mathcal{S}^p)$.

Recall that $p \geq 2$. From Theorem 3.4 we have that $Z \in \mathcal{S}^p([0,T])$ and the estimates

$$\|Z\|^p_{\mathcal{S}^p([0,T])} \leq C(1 + \|\xi\|^p_{L^p(\Omega)}) \quad \text{and} \quad \sup_{0\leq t\leq T} W_2^2(\mu_t, \delta_0) \leq C(1 + \|\xi\|^2_{L^2(\Omega)}).$$

Noticing now that (2) is a standard linear SDE (in DZ) with random (time-continuous) coefficients satisfying a one-sided Lipschitz condition, i.e. the SDE for DZ is linear and satisfies [IdRS19, Assumption 2.4] with their $b(s,\omega), \sigma(s,\omega)$ set to zero, hence the moment estimate of [IdRS19, Theorem 2.5] applies with general $p \geq 2$ and yields

$$\begin{aligned}
\|D_\cdot Z\|^p_{\mathcal{S}^p([0,T])} = \mathbb{E}\big[\sup_{0\leq t\leq T} |D_\cdot Z^n_t|^p\,\big] &\leq C\mathbb{E}\big[\,|\sigma(\cdot, Z_\cdot, \mu_\cdot)|^p\,\big] \\
&\leq C\Big(1 + \|Z^n\|^p_{\mathcal{S}^p([0,T])} + \sup_{0\leq t\leq T} W_2^p(\mu_t, \delta_0)\Big) \\
&\leq C(1 + \|\xi\|^p_{L^p(\Omega)}) < \infty.
\end{aligned}$$

Thus, akin to Estimate (8), we obtain from the above inequality

$$\sup_{s\in[0,T]} \mathbb{E}[\sup_{t\in[0,T]} |D_s Z_t|^p\,] = \sup_{s\in[0,T]} \|D_s Z\|^p_{\mathcal{S}^p([0,T])} \leq C(1 + \|\xi\|^p_{L^p(\Omega)}). \tag{10}$$

Further, following the footsteps of (9) but this time in p-moment norms we have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[0,T]} \Big(\int_0^T |D_s Z_t|^2 ds\Big)^{\frac{p}{2}}\Big] &\leq C\mathbb{E}\Big[\int_0^T \{\sup_{t\in[0,T]} |D_s Z_t|^p\} ds\Big] \\
&\leq C\int_0^T \mathbb{E}\Big[\sup_{t\in[0,T]} |D_s Z_t|^p\Big] ds \\
&\leq C \sup_{s\in[0,T]} \mathbb{E}\Big[\sup_{t\in[0,T]} |D_s Z_t|^p\Big] \\
&\leq C(1 + \|\xi\|^p_{L^p(\Omega)}) < \infty,
\end{aligned}$$

where we firstly used Jensen's inequality and moved the supremum inside the integral, then used Fubini and dominated the integral by the supremum over the integration variable in a way that (10) can be used. This shows (3).

Having shown that $Z \in \mathcal{S}^p([0,T])$ and that DZ has higher-order p-moments (in the sense of (10) and (3)) we conclude via [Nua06, Proposition 1.5.5] that $Z \in \mathbb{D}^{1,p}(\mathcal{S}^p)$ (and space interpolation that $Z \in \mathbb{D}^{1,q}(\mathcal{S}^q)$ for any $q \in [2,p]$).

□

4 Malliavin differentiability via the interacting particle system

The main goal of this section is to explore, in a didactic fashion, how much of Theorem 3.5 can be recovered under the interacting particle system approach. A somewhat close approach has been taken, for instance, in [HAHT21] and [CdRSW24].

To simplify arguments we will work under the further restriction of full Lipschitz conditions on the MV-SDE's coefficients and thus abdicate the more general super-linear growth and one-sided Lipschitz assumption.

4.1 The interacting and non-interacting particle system

We introduce the interacting particle system (IPS) associated to McKean-Vlasov SDE (1). Consider the system of SDEs for $i = 1, \cdots, N$:

$$dX_t^i = b(t, X_t^i, \bar{\mu}_t^N)dt + \sum_{l=1}^{m} \sigma^l(t, X_t^i, \bar{\mu}_t^N)dW_t^{l,i}, \quad X_0^i = \xi^i, \tag{11}$$

where $\bar{\mu}_t^N(dy) = \frac{1}{N}\sum_{k=1}^{N} \delta_{X_t^i}(dy)$ and for $l = 1, \ldots, m$, $\{W^{l,i}\}_{i=1,\cdots,N}$, are independent 1-dimensional Brownian motions and $\{\xi^i\}_{i=1,\cdots,N}$ are i.i.d. copies of ξ; the $(W^{l,i}, \xi^i)_i$ in (11) are independent of W^l, ξ in (1) (and in fact, live in different probability spaces). We write $W^i = (W^{1,i}, \ldots, W^{m,i})$ to be the corresponding $m-$dimensional Brownian motions for $i = 1, \ldots, N$. The dependence on the empirical distributions in the coefficients introduces non-linearity into the system in the form of self-interaction; hence we refer to the above set of equations as an *interacting particle system* (IPS).

Since (1) and (11) live in different probability spaces, we construct an auxiliary *non-interacting particle system* (non-IPS) as living in the same probability space as (11). For $i = 1, \cdots, N$,

$$dZ_t^i = b(t, Z_t^i, \mu_t^i)dt + \sum_{l=1}^{m} \sigma^l(t, Z_t^i, \mu_t^i)dW_t^{l,i}, \qquad Z_0^i = \xi^i, \tag{12}$$

where μ^i is defined as the law of Z^i. In this case, the $\{Z^i\}_{i=1,\cdots,N}$ are independent of each other, since the $(W^i, \xi^i)_i$ are all i.i.d. and $\mu^i = \mu^j = \mu$ $\forall\, i, j = 1, \cdots, N$ where μ denotes the law of the McKean-Vlasov SDE (1). In essence, (12) is a decoupled system of N copies of (1). From direct inspection of (12), we have the following lemma regarding the cross-Malliavin derivatives $D^j Z^i$ for $i \neq j$.

Lemma 4.1. *Assume* (12) *is well-posed. Then, the cross-Malliavin derivatives of the solution* $\{Z^i\}_{i=1,\cdots,N}$ *to* (12) *are all zero. That is,*

$$D_s^j Z_t^i = 0 \quad \text{for any} \quad j \neq i, 1 \leq i, j \leq N, \quad s, t \in [0, T].$$

Proof. One sees that Z^i acts independently of any other Brownian motion W^j for $j \neq i$. The result follows immediately from the definition of the Malliavin derivative. □

4.1.1 Preliminaries

In this section, we work under stronger assumptions requiring b and σ to be globally space-measure Lipschitz. Formally we state the framework as follows:

Assumption 4.2. *Let* $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ *and for* $l = 1, \ldots, m$, $\sigma^l : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ *be progressively measurable deterministic maps and* $\exists L > 0$ *such that:*

1. *For some* $p \geq 2$, $\xi^i \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ *for* $i = 1, \cdots, N$,
2. σ^l *is continuous in time and Lipschitz in space-measure* $\forall t \in [0,T]$, $\forall x, x' \in \mathbb{R}^d$ *and* $\forall \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ *we have*
$$|\sigma^l(t,x,\mu) - \sigma^l(t,x',\mu')| \leq L\Big(|x - x'| + W_2(\mu,\mu')\Big).$$
3. b *is continuous in time and Lipschitz in space-measure* $\forall t \in [0,T]$, $\forall x, x' \in \mathbb{R}^d$ *and* $\forall \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$ *we have*
$$|b(t,x,\mu) - b(t,x',\mu')| \leq L\Big(|x - x'| + W_2(\mu,\mu')\Big).$$

A quick inspection of (11) highlights that this system of equations can be seen as a system in $(\mathbb{R}^d)^N$ as opposed to N equations valued in $\mathbb{R}^d$. The former has a few advantages and to formalize it we introduce the notion of an *empirical projection* introduced in [CD18a, Definition 5.34].

Definition 4.3 (Empirical projection of a map)**.** *Given* $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ *and* $N \in \mathbb{N}$, *define the empirical projection* u^N *of* u *via* $u^N : (\mathbb{R}^d)^N \to \mathbb{R}^d$, *such that*

$$u^N(x^1, \ldots, x^N) := u(\bar{\mu}^N), \quad \text{with} \quad \bar{\mu}^N(dy) := \frac{1}{N}\sum_{l=1}^N \delta_{x^l}(dy),$$

for $x^l \in \mathbb{R}^d, l = 1, \ldots, N$.

We can use a similar notion of mapping points onto empirical projections to express the interacting particle system as a high-dimensional SDE. Hence we can interpret system (11) as a system in $(\mathbb{R}^d)^N$ with $B^N : [0,T]\times(\mathbb{R}^d)^N \to (\mathbb{R}^d)^N$ and $\Sigma^N : [0,T] \times (\mathbb{R}^d)^N \to \mathbb{R}^{mN\times dN}$,

$$d\mathbf{X}_t = B^N(t, \mathbf{X}_t)dt + \Sigma^N(t, \mathbf{X}_t)d\mathbf{W}_t, \quad \mathbf{X}_0 = \boldsymbol{\xi}, \tag{13}$$

for $\mathbf{X} = (X^1, \cdots, X^N)$, $\mathbf{W} = (W^1, \cdots, W^N)$ and $\boldsymbol{\xi} = (\xi^1, \cdots, \xi^N)$ where for $t \in [0,T]$ and $\mathbf{x} = (x^1, \cdots, x^N) \in \mathbb{R}^{dN}$, $x^i \in \mathbb{R}^d$, $i = 1, \cdots, N$ we have

$$\begin{aligned}
B^N(t, \mathbf{x}) &= \big(b_1^N(t, \mathbf{x}), \cdots, b_N^N(t, \mathbf{x})\big) \\
&:= \big(b(t, x^1, \bar{\mu}^N), \cdots, b(t, x^N, \bar{\mu}^N)\big) \\
\Sigma^N(t, \mathbf{x}) &= \operatorname{diag}(\sigma_1^N(t, \mathbf{x}), \cdots, \sigma_N^N(t, \mathbf{x})) \\
&:= \operatorname{diag}\big(\sigma(t, x^1, \bar{\mu}^N), \cdots, \sigma(t, x^N, \bar{\mu}^N)\big),
\end{aligned}$$

with the relation between $\mathbf{x}$ and $\bar{\mu}^N$ being as highlighted in Definition 4.3. The next result shows that from the Lipschitz properties of b, σ in space and measure, one can show that $\mathbf{x} \mapsto B^N(\cdot, \mathbf{x}), \Sigma^N(\cdot, \mathbf{x})$ are uniformly Lipschitz (uniformly in time).

Lemma 4.4. *Under Assumption 3.1, the maps B^N and Σ^N in (13) are globally Lipschitz in their spatial variables.*

Proof. Let $\mathbf{x} = (x^1, \cdots, x^N), \mathbf{y} = (y^1, \cdots, y^N) \in \mathbb{R}^{dN}$, for $x^i, y^i \in \mathbb{R}^d$, $i = 1, \cdots, N$ then

$$\begin{aligned}
|B^N(t, \mathbf{x}) - B^N(t, \mathbf{y})|^2 &= \sum_{k=1}^N \big|b(t, x^k, \bar{\mu}^N(\mathbf{x}) - b(t, y^k, \bar{\mu}^N(\mathbf{y})\big|^2 \\
&\le L^2 \sum_{k=1}^N \Big(|x^k - y^k| + W_2(\bar{\mu}^N(\mathbf{x}), \bar{\mu}^N(\mathbf{y})\Big)^2 \\
&\le 2L^2 \sum_{k=1}^N |x^k - y^k|^2 + W_2^2(\bar{\mu}^N(\mathbf{x}), \bar{\mu}^N(\mathbf{y})) \le 4L^2|\mathbf{x} - \mathbf{y}|^2,
\end{aligned}$$

with the final inequality arising from the fact

$$W_2\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i}, \frac{1}{N}\sum_{i=1}^N \delta_{y^i}\right) \le \left(\frac{1}{N}\sum_{i=1}^N |x^i - y^i|^2\right)^{\frac{1}{2}} = \frac{1}{\sqrt{N}}|\mathbf{x} - \mathbf{y}|. \tag{14}$$

The proof is similar for $\mathbb{R}^{dN} \ni \mathbf{x} \mapsto \Sigma^N(\mathbf{x}) \in \mathbb{R}^{mN\times dN}$. $\square$

4.1.2 Classical results

We briefly recall classical results involving the relationship between systems described above. Well-posedness follows from classic literature, while the second result is classically known as Propagation of Chaos (PoC), which ascertains convergence of X^i to Z^i as $N \to \infty$ (as respective laws) [Szn91].

Proposition 4.5 (Well-posedness and Propagation of Chaos)**.** *Let Assumption 4.2 hold. Then, the solutions to the systems* (11) *and* (12), *given by* $\{X_t^i\}_{i=1,\cdots,N}$ *and* $\{Z_t^i\}_{i=1,\cdots,N}$ *respectively are well-posed, unique and square integrable. It holds that*

$$\sup_{N\in\mathbb{N}} \max_{1\leq i\leq N} \Big\{ \mathbb{E}\big[\sup_{0\leq t\leq T} |Z_t^i|^2\big] + \mathbb{E}\big[\sup_{0\leq t\leq T} |X_t^i|^2\big] \Big\} \leq C(1+\mathbb{E}[|\xi^{\cdot}|^2])e^{CT} < \infty, \tag{15}$$

where the involved constant C depends on d, m, L and the quantity

$$\int_0^T |b(t,0,\delta_0)|dt + \int_0^T |\sigma(t,0,\delta_0)|^2 dt,$$

but independent of N. Moreover, we have for any $i = 1, \cdots, N$,

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \mathbb{E}[W_2^2(\mu_t^i, \bar{\mu}_t^N)] = 0 \ \textit{and} \lim_{N\to\infty} \max_{1\leq i\leq N} \mathbb{E}\Big[\sup_{0\leq t\leq T} \Big|X_t^i - Z_t^i\Big|^2\Big] = 0. \tag{16}$$

Proof. Under Lipschitz conditions this result is classical. Well-posedness of (12) follows directly from Theorem 3.4 as it is a non-interacting particle system (thus well-posedness of the initial McKean-Vlasov equation suffices).

As for system (11), Lemma 4.4 ensures the coefficients are uniformly Lipschitz (as maps in $(\mathbb{R}^d)^N$) and thus well-posedness (for fixed N) follows from general SDE theory [GK80, Theorem 1]. One can conclude the uniform in N estimates of (15) for $\{X^i\}_i$ by mimicking the arguments used in the proof of Theorem 3.5.

The convergence results of (16) follows from [Car16, Lemma 1.9 and Theorem 1.10]. □

4.1.3 A primer on Lions derivatives

To consider the calculus for the mean-field setting, one requires to build a suitable differentiation operator on 2-Wasserstein space. Among the numerous notions of differentiability of a function u defined over the $\mathcal{P}_2(\mathbb{R}^d)$, we try to follow the approach introduced by Lions in his lectures at Collège de France. A comprehensive collection of recent results was done in the joint monographs of Carmona and Delarue [CD18a], [CD18b]. In line with the construction we assume our probability space to be an atomless Polish space [CD18a, Chapter 5].

We consider a canonical lifting of the function $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ to $\tilde{u} : L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \ni X \to \tilde{u}(X) = u(Law(X)) \in \mathbb{R}^d$. We can say that u is L-differentiable at μ, if $\tilde{u}$ is Fréchet differentiable at some X, such that $\mu = \mathbb{P} \circ X^{(-1)}$. Denoting the gradient by $D\tilde{u}$ and using a Hilbert structure of the L^2 space, we can identify $D\tilde{u}$ as an element of L^2. It has been shown that $D\tilde{u}$ is a $\sigma(X)$-measurable random variable and given by the function $Du(\mu)(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, depending on the law of X and satisfying $Du(\mu)(\cdot) \in L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d)$. Hereinafter the L-derivative of u at μ is the map $\partial_\mu u(\mu)(\cdot) : \mathbb{R}^d \ni v \to \partial_\mu u(\mu)(v) \in \mathbb{R}^d$, satisfying $D\tilde{u}(X) = \partial_\mu u(\mu)(X)$. We always denote $\partial_\mu u$ as the version of the L-derivative that is continuous in product topology of all components of u.

Definition 4.3 relates the spatial derivatives of u^N with the Lions derivative of the measure function u. Such is stated next; see also [CD18a, Proposition 5.35 (p.399)].

Proposition 4.6. *Let $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ be a continuously L-differentiable map, then, for any $N > 1$, the empirical projection u^N is differentiable in $(\mathbb{R}^d)^N$ and for all $x^1, \cdots, x^N \in \mathbb{R}^d$ we have the following relation:*

$$\partial_{x^j} u^N(x^1, \dots, x^N) = \frac{1}{N}\, \partial_\mu u\Big(\frac{1}{N}\sum_{l=1}^{N} \delta_{x^l}\Big)(x^j).$$

4.2 Exploring Malliavin differentiability via interacting particle system limits

The novelty in this section lies not within the results, as these are implied directly by those in our Section 3.2 or [CM18], but in the proof methodology via limits of interacting particle systems.

Assumption 4.7. *Let Assumption 4.2 hold.*

1. *The functions b, σ^l, $l = 1, \dots, m$ are continuously differentiable in their spatial variables and their spatial derivative maps are continuous in time. Further, the maps $(\nabla_x b)$, $(\nabla_x \sigma^l)(t, x, \mu)$ are uniformly bounded for $l = 1, \dots, m$ (over all variables $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$).*

2. *For any* $t \in [0,T]$ *the maps* $\mu \mapsto b(t,x,\mu)$ *and* $\mu \mapsto \sigma^l(t,x,\mu)$, $l = 1,\dots,m$, *are* $\mathbb{P}$*-a.s. continuous in topology, induced by the Wasserstein metric and L-differentiable* $\mathbb{P}$*-a.s. at every* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$*. Moreover,* $\partial_\mu b(t,x,\mu)(v)$ *and* $\partial_\mu \sigma^l(t,x,\mu)(v)$ *have* μ*-versions such that* $\partial_\mu b(t,x,\mu)(v)$ *and* $\partial_\mu \sigma^l(t,x,\mu)(v)$ *are* $\mathbb{P}$*-a.s. joint-continuous at every quadruple* (t,x,μ,v) *with* $(t,x,\mu) \in [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, $v \in Supp(\mu)$ *and uniformly bounded (in all variables).*

Focusing on the technique of limits of particle systems, we establish the following result that covers the Malliavin differentiability of the particle system, the Propagation of Chaos result and how to transfer the Malliavin regularity to the limiting McKean-Vlasov SDE.

Proposition 4.8. *Let Assumption 4.7 hold. Then the solution* $\{Z^i\}_{i=1,\cdots,N}$ *to* (12) *and the solution* $\{X^i\}_{i=1,\cdots,N}$ *to* (11) *are Malliavin differentiable with Malliavin derivatives* $\{D^jZ^i\}_{i,j=1,\cdots,N}$ *and* $\{D^jX^i\}_{i,j=1,\cdots,N}$ *respectively.*

For $\{D^jZ^i\}_{i,j=1,\cdots,N}$ *we have that:*

- *For any* $j \neq i$, $1 \le i,j \le N$ $s,t \in [0,T]$ *that* $D^j_sZ^i_t = 0$ *(Lemma 4.1).*
- *When* $j = i$ *then* D^iZ^i *satisfies for* $0 \le s \le t \le T$

$$D^i_sZ^i_t = \sigma(s,Z^i_s,\mu^i_s) + \int_s^t (\nabla_x b)(r,Z^i_r,\mu^i_r)D^i_sZ^i_r dr + \sum_{l=1}^m \int_s^t (\nabla_x\sigma^l)(r,Z^i_r,\mu^i_r)D^i_sZ^i_r dW^{l,i}_r. \tag{17}$$

  If $s > t$ *then* $D^i_sZ^i_t = 0$ $\mathbb{P}$*-almost surely.*

- *Moreover, for some* $C > 0$ *dependent on* T *and* L *but not on* N*, (c.f. Theorem 3.5)*

$$\sup_{0\le s\le T} \|D^i_sZ^i\|^2_{\mathcal{S}^2([0,T])} \le \sup_{0\le s\le T} \mathbb{E}\Big[\sup_{0\le t\le T} |D^i_sZ^i_t|^2\Big] \le C(1+\|Z^i\|^2_{\mathcal{S}^2([0,T])}) \le C(1+\|\xi^\cdot\|^2_{L^2(\Omega)}) < \infty.$$

For $\{D^jX^i\}_{i,j=1,\cdots,N}$*, the Malliavin derivative of* (11)*, we have that:*

- *If* $s > t$ *then* $D^j_sX^i_t = 0$ $\mathbb{P}$*-almost surely for any* $i,j = 1,\cdots,N$.

- *If* $0 \le s \le t$ *then* $D_s^j X_t^i$ *satisfies*

$$\begin{aligned} D_s^j X_t^i &= \sigma(s, X_s^i, \bar{\mu}_s^N) \mathbb{1}_{i=j} \\ &\quad + \int_s^t \Big\{ (\nabla_x b)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i \qquad (18) \\ &\qquad + \frac{1}{N} \sum_{k=1}^N (\partial_\mu b)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k \Big\} \, dr \\ &\quad + \sum_{l=1}^m \int_s^t \Big\{ (\nabla_x \sigma^l)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i \\ &\qquad + \frac{1}{N} \sum_{k=1}^N (\partial_\mu \sigma^l)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k \Big\} \, dW_r^i. \qquad (19) \end{aligned}$$

- *Moreover, there exists a constant* $C > 0$ *depending on* T *and* L *but not on* N *such that*

$$\sup_{0 \le s \le T} \mathbb{E} \left[\sup_{0 \le t \le T} |D_s^j X_t^i|^2 \right] \le C \Big(\mathbb{1}_{i=j} + \frac{1}{N} \Big).$$

Finally, for any $s, t \in [0, T]$,

- $D_s^j X_t^i \to 0$ *as* $N \to \infty$ *for* $j \neq i$ *in* $L^2(\Omega)$ *and almost surely, and*
- $D_s^i X_t^i \to D_s^i Z_t^i$ *as* $N \to \infty$ *in* $L^2(\Omega)$.

Proof. Rewrite the IPS (11) in integral form: for $i = 1, \dots, N$,

$$\begin{aligned} X_t^i &= \xi^i + \int_0^t b(r, X_r^i, \bar{\mu}_r^N) dr + \sum_{l=1}^m \int_0^t \sigma^l(r, X_r^i, \bar{\mu}_r^N) dW_r^{l,i}, \qquad (20) \\ &= \xi^i + \int_0^t b_i^N(r, X_r^1, \cdots, X_r^N) dr \\ &\qquad + \sum_{l=1}^m \int_0^t \sigma_i^{l,N}(r, X_r^1, \cdots, X_r^N) dW_r^{l,i}, \qquad (21) \end{aligned}$$

where (21) uses the empirical projection representation (Definition 4.3) of the coefficients in (20); that is $b_i^N(t, X^1, \cdots, X^N) := b(t, X^i, \bar{\mu}^N)$ and similarly for $\sigma_i^{l,N}$. In view of Assumption 4.7, it is easy to conclude that the coefficients $b_i^N, \sigma_i^{l,N}$ of (20) are Lipschitz continuous (via Lemma 4.4) and also differentiable in their variables (via Proposition 4.6). From the results in [Nua06] or [IdRS19] for classical SDEs we conclude immediately that the Malliavin derivatives of X^i exist, are unique and are square-integrable. Furthermore, if $s > t$ then $D_s^j X_t^i = 0$ $\mathbb{P}$-a.s. for any $i, j = 1, \cdots, N$.

From application of the chain rule, the Malliavin derivative is written for $0 \le s \le t \le T < \infty$ as

$$D_s^j X_t^i = \sigma_i^N(s, X_s^1, \cdots, X_s^N)\mathbb{1}_{i=j} + \int_s^t \sum_{k=1}^N (\partial_{x^k} b_i^N)(r, X_r^1, \cdots, X_r^N) D_s^j X_r^k dr \\ + \sum_{l=1}^m \int_s^t \sum_{k=1}^N (\partial_{x^k} \sigma_i^{l,N})(r, X_r^1, \cdots, X_r^N) D_s^j X_r^k dW_r^{l,i}.$$

Exploiting Proposition 4.6 and reverting the empirical projection maps to their original form, we rewrite this as

$$\begin{aligned} D_s^j X_t^i &= \sigma(s, X_s^i, \bar{\mu}_s^N)\mathbb{1}_{i=j} \\ &+ \int_s^t \Big\{ (\nabla_x b)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i \\ &\qquad + \frac{1}{N} \sum_{k=1}^N (\partial_\mu b)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k \Big\} dr \\ &+ \sum_{l=1}^m \int_s^t \Big\{ (\nabla_x \sigma^l)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i \\ &\qquad + \frac{1}{N} \sum_{k=1}^N (\partial_\mu \sigma^l)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k \Big\} dW_r^i. \end{aligned}$$

noting that the derivatives $\partial_{x^i} b_i^N$ and $\partial_{x^i} \sigma_i^{l,N}$ produce two components as opposed to the one produced by the cross-terms $\partial_{x^k} b_i^N$ and $\partial_{x^k} \sigma_i^{l,N}$, $k \neq i$.

Step 1. Preliminary manipulations. Applying Itô's formula to (18),

$$|D_s^j X_t^i|^2 = |\sigma(s, X_s^i, \bar{\mu}_s^N)|^2 \mathbb{1}_{i=j} \tag{22}$$

$$+ \int_s^t 2 D_s^j X_r^i : \Big\{ (\nabla_x b)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i \tag{23}$$

$$+ \frac{1}{N} \sum_{k=1}^N (\partial_\mu b)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k \Big\}$$

$$+ \sum_{l=1}^m \Big| (\nabla_x \sigma^l)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i + \frac{1}{N} \sum_{k=1}^N (\partial_\mu \sigma^l)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k \Big|^2 dr$$

$$+ M_N(s,t), \tag{24}$$

where M_N is a Hilbert space-valued local martingale term. In fact, by standard SDE well-posedness theory, e.g. [GK80, Theorem 1], for any $s \in [0,T]$, a L^2-integrable solution to Equation (22) exists, allowing us to conclude M_N is a proper martingale for finite N.[1].

Let L be the Lipschitz constant which bounds the spatial and measure derivatives of b and σ^l, $l = 1, \dots, m$. Temporarily fix $s \in [0,T]$. We have

$$
\begin{aligned}
&\mathbb{E}\big[\sup_{0\le t\le T} |D_s^j X_t^i|^2\big] \\
&\quad\le \mathbb{E}\big[|\sigma(s, X_s^i, \bar{\mu}_s^N)\mathbb{1}_{i=j}|^2\big] + \mathbb{E}\big[\sup_{0\le t\le T} |M_N(t,s)|\big] \\
&\quad+ \mathbb{E}\Big[\sup_{0\le t\le T} \int_s^t 2L|D_s^j X_r^i|^2 + 2mL^2 |D_s^j X_r^i|^2 \\
&\qquad\qquad + \frac{2L|D_s^j X_r^i|}{N} \sum_{k=1}^N |D_s^j X_r^k| + 2mL^2 \Big|\frac{1}{N}\sum_{k=1}^N |D_s^j X_r^k|\Big|^2 dr\Big] \\
&\quad\le \mathbb{E}\big[|\sigma(s, X_s^i, \bar{\mu}_s^N)\mathbb{1}_{i=j}|^2\big] + \kappa \mathbb{E}\Big[\sup_{0\le t\le T} \int_s^t |D_s^j X_r^i|^2 + \frac{1}{N}\sum_{k=1}^N |D_s^j X_r^k|^2 dr\Big],
\end{aligned}
\tag{25}
$$

for some $\kappa > 0$, by applying the Young and Jensen inequalities, where we used the BDG inequality to control the martingale term. Further, one can bound $\sup_{0\le s\le T} \mathbb{E}[\,|\sigma(s, X_s^i, \bar{\mu}_s^N)|^2]$ uniformly in N, justified by well-posedness and the uniform in N moment bounds of the SDE-IPS (and also using the continuity assumption on $t \mapsto \sigma(t, 0, \delta_0)$). That is, by linear growth of σ, properties of the Wasserstein metric and Proposition 4.5,

$$
\begin{aligned}
&\sup_{0\le s\le T} \mathbb{E}[\,|\sigma(s, X_s^i, \bar{\mu}_s^N)|^2] \\
&\qquad\qquad \le C\Big(1 + \sup_{0\le s\le T} \mathbb{E}[\,|X_s^i|^2] + \frac{1}{N}\sum_{k=1}^N \sup_{0\le s\le T} \mathbb{E}[\,|X_s^k|^2]\Big) \\
&\qquad\qquad \le C(1 + \mathbb{E}[|\xi^{\cdot}|^2])e^{CT} =: \alpha,
\end{aligned}
\tag{26}
$$

for some $C > 0$ independent of N, observing that the $\{\xi^j\}_j$ are i.i.d.

[1]We note that [GK80, Theorem 1] provides wellposedness and moment estimates for the SDE, but critically, such estimates are *dependent on* N and explode as $N \nearrow +\infty$. Nonetheless, for any fixed N the estimates suffice to ensure that $\mathbb{E}[M_N(s,t)] = 0$ for any s, t. Obtaining moment estimates uniformly in N is done in a subsequent step of the proof. This is the exact same as in the proof of Proposition 4.5.

Step 2. Controlling the empirical mean of the Malliavin derivative. We now aim to gain control over the quantity $\frac{1}{N}\sum_{i=1}^{N} D_s^j X_r^i$. Averaging (25) over the index i and once more using that the $\{\xi^j\}_j$ are i.i.d., we have

$$\begin{aligned}\mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\sup_{0\le t\le T}|D_s^j X_t^i|^2\Big]\\ &\le \frac{\alpha}{N} + \mathbb{E}\Bigg[\frac{\kappa}{N}\sum_{i=1}^{N}\int_s^T \Big\{|D_s^j X_r^i|^2 + \frac{1}{N}\sum_{m=1}^{N}|D_s^j X_r^m|^2\Big\}dr\Bigg]\\ &\le \frac{\alpha}{N} + 2\kappa\mathbb{E}\Bigg[\frac{1}{N}\int_s^T\sum_{i=1}^{N}|D_s^j X_r^i|^2 dr\Bigg]. \qquad (27)\end{aligned}$$

Taking the supremum inside the expectation means we are not able to apply Grönwall's inequality directly. However, we have that

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|D_s^j X_r^i|^2\right] \le \frac{\alpha}{N} + 2\kappa\mathbb{E}\Bigg[\frac{1}{N}\int_s^r\sum_{i=1}^{N}|D_s^j X_\rho^i|^2 d\rho\Bigg].$$

Applying Grönwall's inequality yields

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|D_s^j X_r^i|^2\right] \le \frac{\alpha}{N}\sup_{0\le r,s\le T}\Big(1 + e^{2\kappa(r-s)}(e^{-2\kappa s} - e^{-2\kappa r})\Big) \le \frac{\alpha}{N}e^{2\kappa T}. \qquad (28)$$

Hence, substituting (28) back into (27), we get

$$\sup_{0\le s\le T}\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\sup_{0\le t\le T}|D_s^j X_t^i|^2\right] \le \frac{2\alpha}{N}(1 + \kappa T e^{2\kappa T}) := \Psi_N.$$

It is immediate to see that Ψ_N is uniformly bounded over N, hence

$$\lim_{N\to\infty}\sup_{0\le s\le T}\mathbb{E}\left[\sup_{0\le t\le T}\frac{1}{N}\sum_{i=1}^{N}|D_s^j X_t^i|^2\right] = 0 \qquad \text{for any} \quad j = 1,\cdots,N.$$

Using Jensen's inequality and (28), observe that

$$\sup_{0\le s\le T}\mathbb{E}\Bigg[\sup_{0\le t\le T}\frac{1}{N}\sum_{i=1}^{N}|D_s^j X_t^i|^2\Bigg] \le \Psi_N \to 0 \quad \text{as } N\to+\infty \qquad (29)$$

$$\Rightarrow \quad \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}|D_s^j X_t^k| = 0 \ \mathbb{P}\text{-a.s.}$$

Step 3. Convergence of the individual terms. Injecting the bound Ψ_N into (25), we obtain

$$\sup_{0\leq s\leq T}\mathbb{E}\left[\sup_{0\leq t\leq T}|D_s^jX_t^i|^2\right]\leq \alpha\mathbb{1}_{i=j}+\kappa T\Psi_N+\kappa\int_0^T\mathbb{E}[|D_s^jX_t^i|^2]dr.$$

Applying Grönwall's inequality again, we bound

$$\mathbb{E}\left[|D_s^jX_t^i|^2\right]\leq\Big(\alpha\mathbb{1}_{i=j}+\kappa T\Psi_N\Big)+\kappa\int_s^t e^{\kappa(t-s-r)}\Big(\alpha\mathbb{1}_{i=j}+\kappa T\Psi_N\Big)dr$$
$$\Rightarrow \mathbb{E}[\,|D_s^jX_t^i|^2\,]\leq e^{\kappa T}\Big(\alpha\mathbb{1}_{i=j}+\kappa T\Psi_N\Big).$$

Hence,

$$\sup_{0\leq s\leq T}\mathbb{E}\left[\sup_{0\leq t\leq T}|D_s^jX_t^i|^2\right]\leq\Big(\alpha\mathbb{1}_{i=j}+\kappa T\Psi_N\Big)\Big(1+\kappa Te^{\kappa T}\Big). \tag{30}$$

Hence, applying an identical argument to Equation (9) we have:

$$\sup_{N\in\mathbb{N}}\max_{1\leq i,j\leq N}\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^T|D_s^jX_t^i|^2ds\right]<\infty.$$

In particular, we have that $X_t^i\in\mathbb{D}^{1,2}(\mathcal{S}^2)$ uniformly in N.

Aside, we obtain that $D_s^jX_t^i\to 0$ in $\mathbb{D}^{1,2}(\mathcal{S}^2)$ for $i\neq j$ as the size of interacting particle system $N\to\infty$. This is in line with Lemma 4.1, since referring back to our non-IPS analogy, particles essentially become more conditionally independent as the particle system size gets larger.

We can now apply [Nua06, Lemma 1.2.3]. Let $\{Z^i\}_{i=1,\cdots,N}$ denote the solution to the non-IPS (12). By Proposition 4.5, we have

$$\lim_{N\to\infty}\max_{1\leq i\leq N}\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|X_t^i-Z_t^i\Big|^2\Big]=0.$$

The established uniform in N upper bound on $\mathbb{E}[|D_s^jX_r^i|^2]$, given by (30), allows us to conclude that $Z^i\in\mathbb{D}^{1,2}(\mathcal{S}^2)$ and $D_s^jZ_t^i=\lim_{N\to\infty}D_s^jX_t^i$ as prescribed by [Nua06, Lemma 1.2.3].

Step 4. Recovering the limiting equation and conclusion. It remains only to identify and confirm the stochastic differential equation the limiting object of $D_s^jX_t^i$ as $N\to\infty$ satisfies.

We now consider the L^2-limit of the right hand side of (18). First, note that the MV-SDE (17) is an affine SDE with random coefficients that are space-measure Lipschitz, hence existence and uniqueness follows from [Mao08, Theorem 2.1]. Define $\delta\sigma_s^l := \sigma^l(s, X_s^i, \bar{\mu}_s^N) - \sigma^l(s, Z_s^i, \mu_s^i)$, $l = 1, \ldots, m$ and $\delta\sigma_s$ to be the $d \times m$ matrix valued process with columns $\delta\sigma_s^l$. By our Lipschitz assumption on σ^l, for all $s \in [0, T]$:

$$|\delta\sigma_s| \leq L\Big(|X_s^i - Z_s^i| + W_2(\bar{\mu}_s^N, \mu_s^i)\Big) \to 0, \tag{31}$$

as $N \to \infty$ in $L^2(\Omega)$ by Proposition 4.5. Computing the difference between the SDEs (17) and (18),

$$\begin{aligned} D_s^j X_t^i - D_s^j Z_t^i = \delta\sigma_s + \int_s^t \theta_r dr + \int_s^t \Big\{\frac{1}{N}\sum_{k=1}^N (\partial_\mu b)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k\Big\} dr \\ + \sum_{l=1}^m \int_s^t \eta_r^l dW_r^{l,i} + \sum_{l=1}^m \int_s^t \Big\{\frac{1}{N}\sum_{k=1}^N (\partial_\mu \sigma^l)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k\Big\} dW_r^{l,i}, \end{aligned}$$

where

$$\begin{aligned} \theta_r &:= (\nabla_x b)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i - (\nabla_x b)(r, Z_r^i, \mu_r^i) D_s^j Z_r^i, \\ \eta_r^l &:= (\nabla_x \sigma^l)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i - (\nabla_x \sigma^l)(r, Z_r^i, \mu_r^i) D_s^j Z_r^i. \end{aligned}$$

Squaring both sides of the equation, expanding the squares and taking expectations, we have

$$\|D_s^j X_t^i - D_s^j Z_t^i\|_{L^2(\Omega)}^2 - \Phi_N = \Big\|\int_s^t \theta_r dr\Big\|_{L^2(\Omega)}^2 + \Big\|\sum_{l=1}^m \int_s^t \eta_r^l dW_r^{l,i}\Big\|_{L^2(\Omega)}^2, \tag{32}$$

where we define

$$
\begin{aligned}
\Phi_N := \mathbb{E}\Big[|\delta\sigma_s|^2\Big] + 2\mathbb{E}\Big[\delta\sigma_s : \int_s^t \theta_r dr\Big] \\
&+ 2\mathbb{E}\Big[\delta\sigma_s : \int_s^t \frac{1}{N}\sum_{k=1}^N (\partial_\mu b)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k dr\Big] \\
&+ 2\mathbb{E}\Big[\int_s^t \theta_r dr : \int_s^t \frac{1}{N}\sum_{k=1}^N (\partial_\mu b)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k dr\Big] \\
&+ 2\mathbb{E}\Big[\sum_{l=1}^m \int_s^t \eta_r^l dr : \int_s^t \frac{1}{N}\sum_{k=1}^N (\partial_\mu \sigma^l)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k dr\Big] \\
&+ \mathbb{E}\Big[\Big|\int_s^t \frac{1}{N}\sum_{k=1}^N (\partial_\mu b)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k dr\Big|^2\Big] \\
&+ \mathbb{E}\Big[\sum_{l=1}^m \int_s^t \Big|\frac{1}{N}\sum_{k=1}^N (\partial_\mu \sigma^l)(r, X_r^i, \bar{\mu}_r^N)(X_r^k) D_s^j X_r^k\Big|^2 dr\Big].
\end{aligned}
$$

The term Φ_N is a sequence which converges to zero as $N \to \infty$. This is justified by repeated use of the Cauchy-Schwarz and Jensen inequalities, over the bounds (29) and (31); the $L^2(\Omega)$ boundedness of θ_r and η_r is justified by the Lipschitz property of σ^l and b (i.e. $\nabla_x \sigma^l$ and $\nabla_x b$ are uniformly bounded maps). The remaining term on the left hand side of (32) converges to zero by the conclusion of [Nua06, Lemma 1.2.3] (shown in Step 3 of this proof). Hence so must the right hand side of (32) and we obtain that

$$
\lim_{N\to\infty} \int_s^t (\nabla_x b)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i \, dr = \int_s^t (\nabla_x b)(r, Z_r^i, \mu_r^i) D_s^j Z_r^i \, dr,
$$

and

$$
\lim_{N\to\infty} \int_s^t (\nabla_x \sigma^l)(r, X_r^i, \bar{\mu}_r^N) D_s^j X_r^i \, dW_r^i = \int_s^t (\nabla_x \sigma^l)(r, Z_r^i, \mu_r^i) D_s^j Z_r^i dW_r^{l,i}.
$$

in the L^2-sense and the it follows that MV-SDE (17) is identified as the limit of (18).

□

Remark 4.9 (Mollification: Lifting Assumption 4.7(2.); the measure differentiability requirement)**.** *We do not carry out the analysis here, but by drawing on techniques of mollification in Wasserstein spaces it seems possible* *to remove the measure differentiability assumption of Proposition 4.8. There are several ways to carry out mollification over the space of measures, with the main difficulty being that the Wasserstein space of measures is an infinite dimensional one.*

[Inf-Sup convolution]: Motivated by the study of Hamilton-Jacobi equations in infinite dimensional spaces, Lasry-Lions [LL86] *propose inf-sup convolution (in Hilbert spaces) to show that bounded uniformly continuous scalar functions defined on a Hilbert space $\mathcal{H}$ can be uniformly approximated by functions belonging to the class of differentiable maps with Lipschitz derivatives. Critically, their methodology preserves, in the infinite dimensional space, certain good properties that other known mollifications at the time were not known to. As an example, in* [DDJ23] *and working on the torus, the the inf-sup convolution techniques of* [LL86] *are used to establish optimal rates for the convergence problem in mean field control. Notably, the mollification procedures of* [DDJ23] *rely on the properties of the Hilbert Sobolev space H^{-s} for $s > d/2 + 1$, i.e., the dual of the Hilbert space of functions with s generalized derivatives in L^2.*

[Smoothing via truncating Fourier expansions]: In [CD22]*, working over the probability measure $\mathcal{P}(\mathbb{T}^d)$ on the d-dimensional torus $\mathbb{T}^d$, a mollification procedure of real-valued functions on $\mathcal{P}(\mathbb{T}^d)$ based on the Fourier coefficients of the measure is introduced. Very roughly, the mollification is carried out by truncating higher-order coefficients of the Fourier expansion at a conveniently chosen threshold and combining with a Fejér kernel (see their Definition 3.13). It is currently open how to extend the analysis from $\mathbb{T}^d$ to $\mathbb{R}^d$.*

[Smooth approximation over L^∞ under W_1]: In [CZ21, Section 3]*, the authors construct a novel mollification technique for measure functionals in $C(\mathcal{P}_1(\mathbb{R}^d))$ under the 1-Wasserstein distance and the approximation is carried in L^∞ (the subset of bounded RVs with norm $\|X\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{\omega\in\Omega}|X(\omega)|$). Critically, their mollified map satisfies uniformly the same Lipschitz property of the original functional. Such property does not hold under W_2 (unless the functional is also W_1-Lipschitz) but a convergence result is provided (see their Theorem 3.1).*

[Smoothing the approximating particle projection]: The interacting particle system point of view allows us to benefit from a finite dimensional framework and draw from standard mollification arguments (in $(\mathbb{R}^d)^N$). The most suitable regularisation method for our IPS approach has been shown in [CCD22] *(also* [CD18a, CD18b] *and the complementary* [dRP23] *adding all the missing details of the proofs of* [CD18a, CD18b]*). There, regularization*

is applied directly to the empirical projection map u^N (of Definition 4.3 and not u) via convolution with a smooth kernel. The method allows to control the derivatives of the mollified map by the Lipschitz constant.

In [CM23]*, the authors close the two open questions left by* [CZ21]*. They show that when u is W_2-continuous there exists a sequence $\{u_k\}_k \in C^\infty(\mathscr{P}_2(\mathbb{R}^d))$ converging to u uniformly on compact subsets of $\mathscr{P}_2(\mathbb{R}^d)$. If u is additionally uniformly (resp. Lipschitz) continuous then each u_k is also uniformly (resp. Lipschitz) continuous, with the same modulus of continuity (resp. Lipschitz constant) as u – this closes* [CZ21, Remark 3.2(i)]*. Moreover, for $u \in C^1(\mathscr{P}_2(\mathbb{R}^d))$ (resp. $u \in C^2(\mathscr{P}_2(\mathbb{R}^d))$) we show that the convergence also holds for the first-order derivative (resp. second-order derivatives) – this closes* [CZ21, Remark 3.2(ii)]*. The smooth approximating sequence $\{u_k\}_k \in C^\infty(\mathscr{P}_2(\mathbb{R}^d))$ of* [CM23] *is constructed relying on the empirical distribution, similarly to what is done in* [CD18a, Theorem 5.92] *in the proof of Itô's formula along a flow of measures, although there it is not really a smoothing as both u and u_k are of class $C^2(\mathscr{P}_2(\mathbb{R}^d))$, but rather a way to approximate a function u on $\mathscr{P}_2(\mathbb{R}^d)$ by functions defined on finite-dimensional spaces.*

Acknowledgment

G. dos Reis acknowledges partial support from the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (`https://doi.org/10.54499/UIDB/00297/2020`) and UIDP/00297/2020 (`https://doi.org/10.54499/UIDP/00297/2020`) (Center for Mathematics and Applications, NOVA Math), and by the UK Research and Innovation (UKRI) under the UK government's Horizon Europe funding Guarantee [grant number UKRI343].

References

[AdRR+22] D. Adams, G. dos Reis, R. Ravaille, W. Salkeld, and J. Tugaut, *Large deviations and exit-times for reflected McKean-Vlasov equations with self-stabilising terms and superlinear drifts*, Stochastic Process. Appl. **146** (2022), 264–310. MR4374937

[BCC11] F. Bolley, J. A. Cañizo, and J. A. Carrillo, *Stochastic mean-field limit: non-Lipschitz forces and swarming*, Math Models Methods Appl Sci **21** (2011).

[BFFT12] J. Baladron, D. Fasoli, O. Faugeras, and J. Touboul, *Mean-field description and propagation of chaos in networks of Hodgkin-Huxley and FitzHugh-Nagumo neurons*, The Journal of Mathematical Neuroscience **2** (2012May), no. 1, 10.

[BFT15] M. Bossy, O. Faugeras, and D. Talay, *Clarification and complement to "mean-field description and propagation of chaos in networks of Hodgkin–Huxley and Fitzhugh–Nagumo neurons"*, The Journal of Mathematical Neuroscience (JMN) **5** (2015), no. 1, 19.

[Car16] R. Carmona, *Lectures on BSDEs, stochastic control, and stochastic differential games with financial applications*, Financial Mathematics, vol. 1, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016. MR3629171

[CCD22] J.-F. Chassagneux, D. Crisan, and F. Delarue, *A probabilistic approach to classical solutions of the master equation for large population equilibria*, Mem. Amer. Math. Soc. **280** (2022), no. 1379, v+123. MR4493576

[CD18a] R. Carmona and F. Delarue, *Probabilistic theory of mean field games with applications I*, Probability Theory and Stochastic Modelling, vol. 83, Springer-Verlag, 2018.

[CD18b] R. Carmona and F. Delarue, *Probabilistic theory of mean field games with applications II*, Probability Theory and Stochastic Modelling, vol. 84, Springer-Verlag, 2018.

[CD22] A. Cecchin and F. Delarue, *Weak solutions to the master equation of potential mean field games*, arXiv preprint arXiv:2204.04315 (2022).

[CdR22] X. Chen and G. dos Reis, *A flexible split-step scheme for solving McKean-Vlasov stochastic differential equations*, Appl. Math. Comput. **427** (2022), Paper No. 127180, 23. MR4413221

[CdR24] X. Chen and G. dos Reis, *Euler simulation of interacting particle systems and McKean-Vlasov SDEs with fully super-linear growth drifts in space and interaction*, IMA J. Numer. Anal. **44** (2024), no. 2, 751–796. MR4727110

[CdS25] X. Chen, G. dos Reis, and W. Stockinger, *Wellposedness, exponential ergodicity and numerical approximation of fully super-linear McKean–Vlasov SDEs and associated particle systems*, Electronic Journal of Probability **30** (2025), 1–50.

[CdRSW24] X. Chen, G. dos Reis, W. Stockinger, and Z. Wilde, *Improved weak convergence for the long time simulation of mean-field Langevin equations*, arXiv preprint arXiv:2405.01346 (2024). To appear in Electr. J. of Prob.

[CM18] D. Crisan and E. McMurray, *Smoothing properties of McKean-Vlasov SDEs*, Probab. Theory Related Fields **171** (2018), no. 1-2, 97–148. MR3800831

[CM23] A. Cosso and M. Martini, *On smooth approximations in the Wasserstein space*, Electron. Commun. Probab. **28** (2023), Paper No. 30, 11. MR4627412

[CZ21] M. Chenchen and J. Zhang, *Wellposedness of second order master equations for mean field games with nonsmooth data*, Memoirs of the American Mathematical Society (2021).

[DDJ23] S. Daudin, F. Delarue, and J. Jackson, *On the optimal rate for the convergence problem in mean field control*, arXiv preprint arXiv:2305.08423 (2023).

[DFG+16] W. Dreyer, P. K. Friz, P. Gajewski, C. Guhlke, and M. Maurelli, *Stochastic model for LFP-electrodes*, WIAS preprint no. 2329, 2016.

[DGG+11] W. Dreyer, M. Gaberšček, C. Guhlke, R. Huth, and J. Jamnik, *Phase transition in a rechargeable lithium battery*, European Journal of Applied Mathematics **22** (2011), no. 3, 267–290.

[dRES22] G. dos Reis, S. Engelhardt, and G. Smith, *Simulation of McKean-Vlasov SDEs with super-linear growth*, IMA J. Numer. Anal. **42** (2022), no. 1, 874–922. MR4367675

[dRP23] G. dos Reis and V. Platonov, *Itô-Wentzell-Lions Formula for Measure Dependent Random Fields under Full and Conditional Measure Flows*, Potential Anal. **59** (2023), no. 3, 1313–1344. MR4647952

[dRST19] G. dos Reis, W. Salkeld, and J. Tugaut, *Freidlin-Wentzell LDP in path space for McKean-Vlasov equations and the functional iterated logarithm law*, Ann. Appl. Probab. **29** (2019), no. 3, 1487–1540. MR3914550

[GK80] I. Gyöngy and N. V. Krylov, *On stochastic equations with respect to semimartingales. I*, Stochastics **4** (1980/81), no. 1, 1–21. MR587426

[GPV20] S. N. Gomes, G. A. Pavliotis, and U. Vaes, *Mean field limits for interacting diffusions with colored noise: phase transitions and spectral numerical methods*, Multiscale Modeling & Simulation **18** (2020), no. 3, 1343–1370.

[HAHT21] A.-L. Haji-Ali, H. Hoel, and R. Tempone, *A simple approach to proving the existence, uniqueness, and strong and weak convergence rates for a broad class of mckean–vlasov equations*, arXiv preprint arXiv:2101.00886 (2021).

[IdRS19] P. Imkeller, G. dos Reis, and W. Salkeld, *Differentiability of SDEs with drifts of super-linear growth*, Electron. J. Probab. **24** (2019), Paper No. 3, 43. MR3916323

[LL86] J.-M. Lasry and P.-L. Lions, *A remark on regularization in Hilbert spaces*, Israel J. Math. **55** (1986), no. 3, 257–266. MR876394

[Mao08] X. Mao, *Stochastic differential equations and applications*, Horwood, 2008.

[Nua06] D. Nualart, *The Malliavin calculus and related topics*, Vol. 1995, Springer, 2006.

[RW19] P. Ren and F.-Y. Wang, *Bismut formula for Lions derivative of distribution dependent SDEs and applications*, J. Differential Equations **267** (2019), no. 8, 4745–4777. MR3983053

[Szn91] A.-S. Sznitman, *Topics in Propagation of Chaos*, École d'été de probabilités de saint-flour xix - 1989, 1991.

Geometric representation theory and p-adic geometry

João Lourenço
Afiliação: Universität Münster
Endereço: Einsteinstrasse 62, Münster, Alemanha
e-mail: j.lourenco@uni-muenster.de

Resumo: Discute-se as origens do programa de Langlands em teoria dos números, a sua geometrização e categorificação sobre corpos de funções, bem como a mais recente abordagem de Fargues–Scholze para corpos p-ádicos. Conclui-se com uma descrição das próprias contribuições para este campo emergente e eventuais direcções futuras.

Abstract We discuss the number theoretic origins of the Langlands program, its geometrization and categorification over function fields, and more recently over p-adic fields by Fargues–Scholze. We conclude by describing some of our own contributions to the emerging field and possible future directions.

palavras-chave: programa de Langlands, corpos p-ádicos, campos de Hecke, Grassmannianas afins, fibrados de grupos sobre curvas

keywords: Langlands program, p-adic fields, Hecke stacks, affine Grassmannians, group bundles over curves

1 Reciprocity laws

One of the most celebrated theorems of Gauss [Gau86] is the quadratic reciprocity law. It states that for two distinct odd primes $p \neq q$, there is an equality

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}},$$

where the Legendre symbol on the left equals 1 if p has a square root modulo q, and -1 otherwise. This law has the mesmerizing effect of determining the existence of square roots of a prime p modulo another prime q, provided one understands the case of q modulo p, which would be a completely different problem a priori.

There are several non-trivial proofs of this fact, but we are going to regard it as a special case of a very general reciprocity law using the language of algebraic number theory. Consider a finite field extension $F/\mathbb{Q}$ and let

$O_F \subset F$ be the subring of algebraic integers. While this is no longer a unique factorization domain, the ideals $I \subset O_F$ do factor uniquely into a product of prime ideals $\mathfrak{p}$, i.e., such that the quotient $O_F/\mathfrak{p}$ is an integral domain. In algebraic number theory, the goal is to gather as much information as possible on the prime ideal decomposition of pO_F for any finite extension $F/\mathbb{Q}$. This can lead to the solution of certain Diophantine equations, e.g., decompose $p = m^2 + n^2$ with p prime inside $\mathbb{Z}[i]$.

We say that the prime p is unramified with respect to $F/\mathbb{Q}$ if pO_F decomposes as a product of different primes. In this case, there is a conjugacy class of Frobenius $\varphi_p \in \mathrm{Gal}_{F/\mathbb{Q}}$ whose elements reduce to $x \mapsto x^p$ modulo some $\mathfrak{p}$ above p. The Frobenius plays a decisive role in the entirety of this article. For instance, the Legendre symbols can be rewritten in terms of the value of φ_p in $\mathrm{Gal}_{\mathbb{Q}(\sqrt{\pm q})/\mathbb{Q}} \simeq \{\pm 1\}$. Emil Artin [Art27] found a formulation of reciprocity encompassing all previously known examples.

Theorem 1.1 ([Art27]) *For any abelian Galois extension $F/\mathbb{Q}$ and every character $\rho\colon \mathrm{Gal}_{F/\mathbb{Q}} \to \mathbb{C}^\times$, there exists a Dirichlet character $\chi_\rho\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ for some N such that $\rho(\varphi_p) = \chi_\rho(p)$.*

The original proof of this result was analytic in nature rather than algebraic, and had to do with the density of split primes. This is also not so surprising if we think in terms of L-functions. Dirichlet [LD69] proved the existence of infinitely many primes in arithmetic progressions by studying the L-function given by the Euler product

$$L(s,\chi) = \prod_{p \nmid N} (1 - \chi(p)p^{-s})^{-1}$$

that converges on the right half-plane $\mathrm{Re}(s) > 1$ and then meromorphically continued to the entire complex plane. In a similar fashion, we have the Artin L-function $L(s,\rho)$ for $\mathrm{Re}(s) \gg 0$ attached to a n-dimensional representation $\rho\colon \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$, whose unramified portion is given by the characteristic polynomial of $\rho(\varphi_p)$ evaluated at p^{-s}. Artin reciprocity can be formulated in terms of the equality $L(s,\chi_\rho) = L(s,\rho)$.

Let us now mention the ring of *adèles*. Hensel [Hen01] introduced the ring $\mathbb{Z}_p := \lim_n \mathbb{Z}/p^n\mathbb{Z}$ of p-adic integers, a discrete valuation ring and a profinite set. Its fraction field $\mathbb{Q}_p = \mathbb{Z}_p[p^{-1}]$ is called the field of p-adic numbers. We define the finite adèles $\mathbb{A}^\infty := \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\hat{\mathbb{Z}} := \prod_p \mathbb{Z}_p$, and the full adèles $\mathbb{A} := \mathbb{A}^\infty \oplus \mathbb{R}$ by including the real place. This is a locally compact abelian group and $\mathbb{Q}$ embeds diagonally as a discrete closed subgroup. The ring $\mathbb{A}$ plays a crucial role in number theory, as it encapsulates the *local-global* *principle*, i.e., the idea that one should compare global questions over $\mathbb{Q}$ to local questions over $\mathbb{A}$.

Chevalley [Che40] rephrased Artin reciprocity in terms of $\mathbb{A}$ and proved it algebraically. First, one defines a dense injection $\mathbb{Q}_p^\times \to \mathrm{Gal}_{\mathbb{Q}_p}^{\mathrm{ab}}$ into the profinite Galois group of the maximal abelian extension. The image of p equals φ_p on the maximal unramified extension $\mathbb{Q}_p^{\mathrm{ur}}$ and the identity on the maximal cyclotomic extension $\mathbb{Q}_p^{\mathrm{cyc}}$. The local maps can be explicitly constructed via Galois cohomology and Lubin–Tate theory [LT65], and then are assembled into a global isomorphism

$$\mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_{>0} \simeq \mathrm{Gal}_{\mathbb{Q}}^{\mathrm{ab}}$$

of topological groups. This is the adèlic formulation of Artin reciprocity.

1.1 Acknowledgements

This expository article is an expanded version of a talk given in Coimbra during the GPM conference. We thank Nuno Arala, Alexander Kovačec, and Miguel Moreira for comments and questions after the talk. We are thankful to Timo Richarz for helpful comments on a preliminary version of this text.

2 Elliptic curves and modular forms

In the previous section, we saw how the adèle units relate to the abelian Galois group of $\mathbb{Q}$. Note that $\mathbb{A}^\times$ equals the $\mathbb{A}$-valued points of the algebraic group $\mathbb{G}_m$. At the same time, $\mathrm{Gal}_{\mathbb{Q}}^{\mathrm{ab}}$ captures the information afforded by characters $\chi\colon \mathrm{Gal}_{\mathbb{Q}} \to \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C})$. In this section, we discuss what happens when $\mathbb{G}_m$ is replaced by GL_2.

Consider the upper half space $\mathscr{H} = \{\tau \in \mathbb{C} : \mathrm{im}(\mathbb{C}) > 0\}$. The real Lie group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathscr{H}$ by Möbius transformations:

$$\gamma(\tau) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

so the point i has stabilizer given by the maximal compact subgroup $\mathrm{SO}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{R})$. This identifies $\mathscr{H}$ with the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, the simplest example of a *Hermitian symmetric space*. At the same time, we still have an action of the arithmetic group $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{D} := \mathscr{H}$ and we define the *modular curve* $X_{\Gamma(1)} := \Gamma(1)\backslash\mathcal{D}$. Besides, one can write

$$X_{\Gamma(1)} = \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})/(\mathrm{SL}_2(\hat{\mathbb{Z}}) \times \mathrm{SO}_2(\mathbb{R}))$$

so the modular curve also fits into the previous framework of adelic uniformazation.

However, there is more to this story and it turns out that the points of the modular curve are *elliptic curves* themselves, i.e., genus 1 smooth curves in $\mathbb{P}^2$. This means $X_{\Gamma(1)}$ is a *moduli space* of a class of algebraic varieties, a recurrent theme in algebraic geometry. Indeed, for every τ, we associate a complex torus $\mathbb{C}/\Lambda_\tau$ with $\Lambda_\tau := (\mathbb{Z} + \tau\mathbb{Z})$. The homothety class of Λ_τ is invariant under $\mathrm{SL}_2(\mathbb{Z})$, so our assignment descends to a bijection between $X_{\Gamma(1)}$ and the set of complex tori. Now, Weierstrass defined a certain series $\wp_\tau(z)$ converging everywhere on $\mathbb{C}$ except Λ_τ, and proved that it satisfies the functional equation $\wp'_\tau(z)^2 = 4\wp_\tau(z)^3 - g_2\wp_\tau(z) - g_3$. The coefficients $g_2 = 60G_4$ and $g_3 = 140G_6$ are rescaled Eisenstein series $G_{2n}(\tau) := \sum_{0\neq\lambda\in\Lambda_\tau} \lambda^{-2n}$ converging when $n \geq 3$, some of the most famous examples of *modular forms*. We define an elliptic curve $E_\tau = \{(\wp_\tau(z), \wp'_\tau(z)), z \in \mathbb{C}/\Lambda_\tau\}$, realizing $X_{\Gamma(1)}$ as a moduli space of elliptic curves.

The advantage of realizing $X_{\Gamma(1)}$ as a moduli space of elliptic curves is that this definition extends to an algebraic variety over $\mathbb{Q}$. One can obtain variants of the modular curve by replacing $\Gamma(1)$ by deeper level $\Gamma \subset \Gamma(1)$, e.g., the congruence subgroups $\Gamma(n) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}))$. The resulting map $X_{\Gamma(n)} := \Gamma(n)\backslash\mathcal{D} \to X_{\Gamma(1)}$ is a finite étale cover and admits a $\mathbb{Q}(\zeta_n)$-realization by adding isomorphisms of the n-torsion $E[n] := \ker([n] : E \to E)$ with $(\mathbb{Z}/n\mathbb{Z})^2$. Modular curves can be compactified to X^*_Γ by adding cusps in $\mathbb{P}^1_{\mathbb{Q}}$, and Deligne–Rapoport [DR73] studied their integral models $\mathcal{X}_\Gamma$, i.e., certain schemes over $\mathbb{Z}[\zeta_n]$, whose generic fiber recovers X_Γ.

Until now, we have only described the underlying geometry of the automorphic representations of GL_2. The main players in the automorphic representation theory are certain Γ-equivariant functions investigated by Hecke [Hec27], called *modular forms*. Concretely, a modular form of weight k and level Γ is a function $f\colon \mathscr{H} \to \mathbb{C}$ such that $(c\tau + d)^k f = f \circ \gamma$ for every $\gamma \in \Gamma$, admitting a Fourier expansion near the cusps at infinity. We give some examples below so the reader can get a better feeling. We have already seen the Eisenstein series $G_n(\tau)$, which are modular forms of level $\Gamma(1)$ and weight n. Jacobi defined the theta function $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}$ converging when $\tau \in \mathscr{H}$ by the Poisson summation formula, a modular form of level $\Gamma(2)$ and weight $1/2$. Furthermore we have the discriminant of the Weierstrass equation $\Delta(\tau) = e^{2\pi i\tau} \prod_{n\geq 1}(1 - e^{2n\pi i\tau})^{24}$, a modular form of level $\Gamma(1)$ and weight 12. Its Fourier coefficients $\tau(n)$ were conjectured by Ramanujan to satisfy $|\tau(p)| < 2p^{11/2}$. The study of modular forms has lead to many remarkable arithmetic identities.

Hecke defined the L-series $L(s, f) = \sum_{n\geq 1} a_n n^{-s}$ attached to a *cusp* form f, i.e., vanishing at the cusps, of weight k and level Γ, where the sequence a_n are the Fourier coefficients of f, and proved it admits a meromorphic continuation to $\mathbb{C}$ with poles at $s = 0$ and $s = k$. Moreover, when $a_1 = 1$, then $L(s, f)$ has an Euler product with terms equal to $(1-a_p p^{-s}+p^{k-1-2s})^{-1}$ for almost all p. Deligne [Del71a] associated 2-dimensional complex Galois representations ρ_f to modular forms f of weight $k \geq 2$ by using the étale cohomology of modular curves, and proved the Ramanujan conjecture with the help of his proof in [Del74] of the Weil conjectures.

On the other hand, an elliptic curve over $\mathbb{Q}$ has an associated L-function $L(s, E)$. Indeed, consider the Galois representation $\rho_E \colon \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ deduced via scalar extension from the natural action on the Tate $\mathbb{Z}_\ell$-module $T_\ell(E) := \lim_n E[\ell^n] \simeq \mathbb{Z}_\ell^2$ for a fixed prime ℓ, and set $L(s, E) := L(s, \rho_E)$. This L-function has an Euler product expansion with terms of the form $(1 - a_p p^{-s} + p^{1-2s})^{-1}$ for almost all p, with a_p giving the trace of φ_p. This is not a coincidence:

Theorem 2.1 ([BCDT01]) *Elliptic curves over* $\mathbb{Q}$ *are modular.*

The statement means that for each such E, there exists some normalized cusp form f of weight 2 such that the associated representations $\rho_E \simeq \rho_f$ or equivalently L-functions $L(s, E) = L(s, f)$ coincide. Originally, this result was conjectured by Taniyama–Shimura [ST61]. Nowadays it is known as the *modularity theorem* and was proved by Breuil–Conrad–Diamond–Taylor [BCDT01] building on work of Wiles [Wil95] and Taylor–Wiles [TW95] for semistable curves. The work of Wiles received widespread attention because Ribet [Rib90] had previously observed it implies Fermat's last theorem.

3 Shimura varieties and Langlands

The previous two sections handled the Langlands program for GL_n with $n \leq 2$. Now, we address the much more demanding case of GL_n for arbitrary n. We being by explaining how to replace elliptic and modular curves. An *abelian variety* A is a geometrically connected projective algebraic group over a field. Elliptic curves are abelian varieties of dimension 1, with group law given by $P + Q + R = 0$ for any collinear triple. Over $\mathbb{C}$, an abelian variety is given by a n-dimensional torus $A_Z := \mathbb{C}^n/\Lambda_Z$ with $\Lambda_Z := \mathbb{Z}^n + Z\mathbb{Z}^n$, where Z belongs to the *Siegel upper half space* $\mathcal{D} := \mathscr{H}_n$. The elements of $\mathcal{D}$ are symmetric matrices $Z \in M_n(\mathbb{C})$ with positive definite imaginary part. Note that A_Z carries a polarization λ_Z (automatic for elliptic E), i.e., an alternating form on the lattice Λ_Z. Again, we have an identification $\mathcal{D} \simeq \mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}_n(\mathbb{R})$ via Möbius transformations, where $\mathrm{U}_n(\mathbb{R}) \subset \mathrm{Sp}_{2n}(\mathbb{R})$ is the real unitary subgroup of the real symplectic group. For any congruence subgroup $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$, the quotient $X_\Gamma := \Gamma\backslash\mathcal{D}$ is thus a moduli space of polarized abelian varieties with level structure, defined over a finite extension of $\mathbb{Q}$.

It is not a fluke that the symplectic group Sp_{2n} appeared above instead of SL_n: unfortunately, the automorphic quotients of SL_n for $n > 2$ are never complex manifolds. To remedy this, we need to work with more general groups. Let G be a reductive group over $\mathbb{Q}$, i.e., a linear algebraic $\mathbb{Q}$-group whose maximal smooth unipotent connected subgroup vanishes. This includes semi-simple groups like Sp_{2n} and SL_n, but also general linear groups GL_n, unitary groups $\mathrm{U}_{n,F/\mathbb{Q}}$, orthogonal groups O_n, etc. Let $\mathcal{D} = G(\mathbb{R})/Z_G(\mathbb{R})^+K_\infty$ be the Riemannian symmetric space obtained by quotienting out the connected component of the center $Z_G(\mathbb{R})$, and a compact real Lie subgroup $K_\infty \subset G(\mathbb{R})$ with maximal compact Lie subalgebra. Finally, we set

$$X_K := G(\mathbb{Q})\backslash(\mathcal{D} \times G(\mathbb{A}^\infty))/K$$

for any compact open subgroup $K \subset G(\mathbb{A}^\infty)$. One can show that this equals the disjoint union of quotients $\Gamma\backslash\mathcal{D}$ by arithmetic subgroups $\Gamma \subset G(\mathbb{Q})$, so it is a real orbifold. If K is sufficiently small, X_K is a real manifold. After Shimura [Shi63] worked out a wide variety of examples, Deligne [Del71b] introduced the notion of a Shimura datum, where $\mathcal{D}$ arises as the conjugacy class of homomorphisms $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_\mathbb{R}$, and satisfies a series of axioms to ensure that $\mathcal{D}$ is a disjoint union of *Hermitian* symmetric spaces. For Shimura data and neat K, Baily–Borel [BB66] proved that the X_K are quasi-projective smooth $\mathbb{C}$-varieties, by constructing minimal compactifications of $\Gamma\backslash\mathcal{D}$. These X_K are called *Shimura varieties* and vastly generalize moduli of polarized abelian varieties. Deligne [Del79] proved that, for almost all $(G, \mathcal{D})$ with G classical, they descend to a finite extension $E/\mathbb{Q}$, and the general case was handled independently by Borovoi [Bor83] and Milne [Mil83]. Shimura varieties play a distinguished role in the Langlands program, because their étale cohomology relates to both automorphic and Galois representations.

Finally, we are ready to discuss the notion of an automorphic representation of GL_n, or even of a general reductive group G over $\mathbb{Q}$. We consider the Hilbert space $L^2([G])$ of square-integrable functions on the automorphic space $[G] := G(\mathbb{Q})A_G(\mathbb{Q})\backslash G(\mathbb{A})$ with its natural Radon measure. Here, A_G is the maximal $\mathbb{Q}$-split central torus of G, and one kills it to ensure the finite volume of $[G]$. An *automorphic representation* of $G(\mathbb{A})$ is an irreducible unitary $G(\mathbb{A})$-representation appearing as a subquotient of $L^2([G])$. At the same time, we define *automorphic forms* as K-invariant smooth functions $\varphi : G(\mathbb{A}) \to \mathbb{C}$ of moderate growth with finite translates under K_∞ and the center of $U(\mathrm{Lie}(G_\mathbb{R}))$. For GL_2, we recover modular forms up to a twist and furthermore, automorphic representations can be described in terms of automorphic forms.

Our final ingredient for stating the global Langlands correspondence (GLC) is the notion of automorphic L-functions. Langlands [Lan70] was studying the *constant terms* of Eisenstein series, i.e., their values at the boundary of $\Gamma\backslash\mathcal{D}$, when he was led to the L-function of an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ for all n. Indeed, the Satake isomorphism encountered in the next section associates a dominant coweight $\mu(\pi_p)$ of $\mathrm{GL}_n(\mathbb{C})$ to the p-primary part π_p of the automorphic representation π. For unramified p, one sets $L(s, \pi_p)$ to be the value at p^{-s} of the characteristic polynomial of $\mu(\pi_p)$. More care is needed to define $L(s, \pi_\infty)$ and $L(s, \pi_p)$ for ramified p, and show that the Euler product $L(s, \pi) := L(s, \pi_\infty) \prod_p L(s, \pi_p)$ admits a meromorphic continuation to $\mathbb{C}$.

Conjecture 3.1 ([Lan70]) *The Artin L-function $L(s, \rho)$ of an irreducible Galois representation $\rho\colon \mathrm{Gal}_\mathbb{Q} \to \mathrm{GL}_n(\mathbb{C})$ coincides with the L-function $L(s, \pi_\rho)$ of an automorphic representation π_ρ of $\mathrm{GL}_n(\mathbb{A})$.*

This is a crude version of the GLC that does not pin down the automorphic representation π_ρ. Advances in the theory of Shimura varieties and p-adic Hodge theory have led to a better understanding of how the GLC should look like, and we refer to [BG14] for a modern treatment of the GLC in this direction. Over the p-adic field $\mathbb{Q}_p$, we get the local Langlands correspondence (LLC) with a much clearer formulation. The LLC predicts a bijection.between isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ on the automorphic side and $\mathrm{GL}_n(\mathbb{C})$-conjugacy classes of semisimple L-parameters $\mathrm{W}_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}_{\mathbb{Q}_p}$. These so-called L-parameters are certain group homomorphisms from the Weil–Deligne group $\mathrm{W}_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C})$, such that the image of the Frobenius φ is semisimple. This conjecture was proved by Harris–Taylor [HT01] using global ingredients coming from the cohomology of Shimura varieties, whose decomposition allowed them to relate automorphic and Galois representations. For general reductive groups other than GL_n, the situation is more complicated as one does not expect a bijection anymore, but rather a map with finite fibers called L-packets. This is a matter of current intense investigation.

4 Geometric Langlands

It is well known that there is an analogy between number fields and function fields, by which we mean a field $K(X)$ of rational functions on a geometrically connected proper smooth curve X over $\mathbb{F}_p$. Indeed, if one considers the affine spectrum of $\mathbb{Z}$, this is a 1-dimensional scheme and its closed points are given by its primes, just like the places of $K(X)$ correspond to closed points of X. From the point of view of algebraic geometry, it is however much easier to work with curves over $\mathbb{F}_p$. Besides, one also gets a canonical Frobenius homomorphism φ.

Let G be a split connected reductive group over the curve X in the sense of Chevalley [Che55], see also [GD63] (for the unfamiliar reader, this includes classical groups such as GL_n, Sp_{2n}, SO_n, but unitary groups like SU_n are non-split). Consider the moduli stack Bun_G classifying G-bundles on the curve X. This is a smooth Artin stack over $\mathbb{F}_p$ and we can write

$$\mathrm{Bun}_G(\mathbb{F}_p) = G(\mathbb{F}_p)\backslash G(\mathbb{A}_X)/G(\mathbb{O}_X) \tag{1}$$

where the right side resembles the automorphic side of Artin reciprocity. Indeed, in this setting, we define automorphic forms as sections of constant $\bar{\mathbb{Q}}_\ell$-sheaves on this space for some prime $\ell \neq p$. Besides the stack of G-bundles, there are other important stacks to consider, such as the *Hecke stack* denoted by Hk_G which classifies modifications $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$ of G-bundles along a point of the curve $x \in X$; or the stack of G-shtukas Sht_G that is given as the pullback of the Frobenius graph of Bun_G along the natural map $\mathrm{Hk}_G \to \mathrm{Bun}_G$. Shtuka stacks play a similar role to Shimura varieties in the number field case.

Originally, Drinfeld [Dri74] came up with the notion of vector bundles equipped with a meromorphic φ-semilinear structure, which he named shtukas (Russian for "thing, stuff") and applied it in [Dri80] to prove the GLC when GL_2 over global function fields. These techniques were then further developed by Laumon–Rapoport–Stuhler [LRS93] to prove the local Langlands correspondence (LLC) for local fields in characteristic p. After that, Laurent Lafforgue [Laf02] used shtukas to prove the GLC for GL_n over X, and finally his brother Vincent Lafforgue [Laf18] figured out the automorphic to Galois direction for arbitrary G over X.

Theorem 4.1 ([Laf18]) *There is a natural map $\pi \to \sigma_\pi$ from automorphic representations of $G(\mathbb{A}_X)$ to semisimple Langlands parameters, i.e., $G^\vee(\bar{\mathbb{Q}}_\ell)$-conjugacy classes of continuous 1-cocycles $\mathrm{Gal}_{K(X)} \to G^\vee(\bar{\mathbb{Q}}_\ell)$.*

Several ingredients go into the proof of this theorem, the most original one being the construction of the so-called excursion operators. Another fundamental ingredient that we want to address is the *geometric Satake equivalence*. Let O_x be the complete local ring of X at x and F_x be its fraction field. The classical Satake isomorphism following [Sat63] identifies the spherical Hecke algebra $\mathcal{H}_G := \mathbb{C}[G(O_x)\backslash G(F_x)/G(O_x)]$ of G with the Weyl invariants $\mathcal{H}_T^W = \mathbb{C}[X_*(T)]^W$ of the Hecke algebra of a maximal torus $T \subset G$. Up to passing to ℓ-adic coefficients, we can identify $\mathcal{H}_G$ with the Grothendieck group K_0 of étale $\bar{\mathbb{Q}}_\ell$-sheaves on the fiber $\mathrm{Hk}_{G,x}$ of the Hecke stack at the point x and $\mathcal{H}_T^W$ with that of representations of the L-group ${}^LG = G^\vee \rtimes \mathrm{Gal}_{F_x}$. This observation can be upgraded to an equivalence of categories:

Theorem 4.2 ([Gin95]) *There is a natural symmetric monoidal equivalence of abelian categories between the category $\mathcal{P}(\mathrm{Hk}_{G,x})$ of perverse $\bar{\mathbb{Q}}_\ell$-sheaves on the Hecke stack and the category $\mathrm{Rep}({}^LG)$ of representations of the L-group of G.*

Recall that for a smooth variety over $\mathbb{F}_p$, Poincaré duality holds for étale cohomology by [SGA73b]. However, if we work with non-singular varieties, then this is no longer the case and $\mathrm{Hk}_{G,x}$ is very far from smooth. It admits a pro-smooth cover by the affine Grassmannian $\mathrm{Gr}_{G,x}$ which is an ind-scheme. Its closed $G(O_x)$-equivariant subvarieties $\mathrm{Gr}_{G,x,\leq\mu}$ are called Schubert varieties and indexed by dominant coweights μ of G. These are very rarely smooth, but are always normal and Cohen–Macaulay if $\pi_1(G)$ is p-torsion free by a theorem of Faltings [Fal03] (if $\pi_1(G)$ has p-torsion, pathologies occur by [HLR18]), see [Lou23] for a new proof via distribution $\mathbb{F}_p$-algebras.

Fortunately, Goresky–MacPherson [GM83] discovered in the topological setting, later rephrased by Beilinson–Bernstein–Deligne–Gabber [BBDG18] in the algebraic setting, that constant sheaves shifted by the dimension along a smooth stratification of X can be glued to complexes on X (called a perverse *sheaf* nonetheless). One gets an abelian full subcategory $\mathcal{P}(X) \subset \mathcal{D}(X)$ of the derived category of sheaves, which satisfies a form of Poincaré duality even for non-smooth X. The geometric Satake equivalence furnishes a plethora of perverse sheaves (one calls them also *Satake sheaves*) on the Hecke stack, which are used as the convolution kernels of geometric Hecke operators. It is also known with $\bar{\mathbb{Z}}_\ell$- and $\bar{\mathbb{F}}_\ell$-coefficients thanks to the work of Mirković–Vilonen [MV07].

5 Geometrization for p-adic fields

In this section, we want to explain some of the emerging story over p-adic fields due to Fargues–Scholze [FS21], which takes a lot of inspiration from global function fields. It is already a daunting task to work in the local p-adic field situation, so from now on we will forget about global number fields.

The first problem that we encounter is that we do not really have a decent curve, or at least the curve that we would normally have, i.e. the affine spectrum of $\mathbb{Z}_p$, is not that rich geometrically. Even the number field setting is not useful because we lack a canonical Frobenius to move things around... The idea here comes in a sense from the theory of Witt vectors [Wit37]. They allow us to lift perfect $\mathbb{F}_p$-algebras to mixed characteristic. Scholze [Sch12] defined a tilting functor that passes from mixed characteristic perfectoid rings to perfect $\mathbb{F}_p$-algebras. While tilting is a functor, there is a myriad of untilts and classifying them yields the Fargues–Fontaine curve [FF18].

More precisely, a *perfectoid Tate ring* is a pair (R, R^+) consisting of a subring $R^+ \subset R = R^+[1/\varpi]$ equipped with the ϖ-adic topology such that ϖ^p divides p and the Frobenius $\varphi\colon R^+/\varpi \to R^+/\varpi^p$ is an isomorphism. The tilting functor of [Sch12] takes (R, R^+) to the perfect Tate ring $(R^\flat, R^{\flat+})$, where $R^{\flat+}$ is the limit of R^+/p along φ. Kedlaya–Liu [KL15] proved that every untilt $(S^\sharp, S^{\sharp+})$ of a perfect Tate ring (S, S^+) can be uniquely obtained as the quotient $S^{\sharp+} = W(S^+)/\xi$ with $\xi = p + [\varpi]\alpha$, where $[\varpi]$ is the Teichmüller lift. This leads us to define the absolute curve Y over $\mathrm{Spa}(\mathbb{Q}_p)$ whose (S, S^+)-valued points are given by the non-vanishing locus of $p[\varpi]$ in the affinoid adic space $\mathrm{Spa}(W(S^+))$. The adic Fargues–Fontaine curve $X := Y/\varphi^{\mathbb{Z}}$ is the quotient by the totally discontinuous action of the Frobenius φ. Here, we have to use the theory of adic spaces due to Huber [Hub96] which captures analytic features in a better fashion; this is related also to the recently developed notion of analytic rings and stacks by Clausen–Scholze [CS19].

Now, one can define Bun_G in this setting again as the stack of G-torsors on X. Its geometric points are in bijection with Kottwitz's set $B(G)$ classifying φ-conjugacy classes in $G(W(\bar{\mathbb{F}}_p)[1/p])$ by a theorem of Fargues [Far20], with topology explicitly described via the combinatorics of Newton polygons as shown by Viehmann [Vie21]. Scholze [Sch17] developed a formalism of étale cohomology for perfectoid stacks, and [FS21] proves that $\mathcal{D}(\mathrm{Bun}_G)$ captures the derived categories of smooth representations of the inner forms J_b of the Levi subgroups of G attached to $b \in B(G)$, glued in a yet mysterious way.

One can also define the Hecke stacks Hk_G, affine Grassmannians Gr_G and shtuka stacks Sht_G in this setup. These are stacks on perfectoids with a natural map to the mirror curve Div^1_X of the Fargues–Fontaine curve X. Using the concept of universally locally acyclic sheaves, [FS21] proved the geometric Satake equivalence for Hk_G, but the L-group is given by the semi-direct product $G^\vee \rtimes \mathrm{W}_{\mathbb{Q}_p}$ with the Weil group. Most of the formal arguments in [Laf18] concerning excursion operators can be repeated to yield the automorphic to Galois direction of the LLC for p-adic fields.

Besides, the Galois side of the LLC can be geometrized via the stack $Z^1(\mathrm{W}_{\mathbb{Q}_p}, G^\vee)$ of L-parameters studied by Zhu [Zhu20] and Dat–Helm–Kurinczuk–Moss [DHKM20], which classifies continuous 1-cocycles $\varphi\colon \mathrm{W}_{\mathbb{Q}_p} \to G^\vee(\bar{\mathbb{Q}}_\ell)$. The Langlands philosophy combined with the previous geometrization efforts suggests that one should find a correspondence between derived categories of sheaves on the automorphic space and the Galois space.

Conjecture 5.1 ([FS21]) *There is an equivalence* $\mathcal{D}(\mathrm{Bun}_G)^\omega \simeq \mathcal{D}^b_{\mathrm{coh}}([G^\vee \backslash Z^1(\mathrm{W}_{\mathbb{Q}_p}, G^\vee)])$ *of derived categories.*

On the left of the equivalence, we consider the full subcategory of compact objects inside $\mathcal{D}(\mathrm{Bun}_G)$, whereas on the right of the equivalence we consider the category of bounded complexes of coherent sheaves on the stack $[G^\vee \backslash Z^1(\mathrm{W}_{\mathbb{Q}_p}, G^\vee)]$. Similar versions of this conjecture have recently appeared also by Hellmann [Hel23] and Zhu [Zhu20] and it can be made much more explicit as follows. In [FS21], the Hecke action on $\mathcal{D}(\mathrm{Bun}_G)$ given by the Satake sheaves is extended to a full action by the category $\mathrm{Perf}([G^\vee \backslash Z^1(\mathrm{W}_{\mathbb{Q}_p}, G^\vee)])$ of perfect complexes on the stack of L-parameters. One can therefore ask that the equivalence above respects the spectral action. Moreover, it is expected that the inverse equivalence maps the structure sheaf $\mathcal{O}$ to a Whittaker sheaf $\mathcal{W}_\psi$, i.e., obtained via compact induction from a Whittaker datum ψ on a maximal unipotent subgroup $U(\mathbb{Q}_p)$.

6 Sheaves on integral models

In this section, we discuss further developments related to integral $\mathbb{Z}_p$-models and their reduction to $\mathbb{F}_p$. The curve Y admits an obvious integral model $\mathcal{Y}$

by including characteristic p untilts, but it is no longer natural to consider its Frobenius quotient, because the action is not free anymore. Let $\mathcal{G}$ be a parahoric $\mathbb{Z}_p$-model in the sense of Bruhat–Tits [BT84] of our connected reductive $\mathbb{Q}_p$-group G: this notion means that $\mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$ behaves for many purposes like a parabolic subgroup, e.g. take the pullback of an actual parabolic subgroup along the reduction map $G(\mathbb{Z}_p) \to G(\mathbb{F}_p)$. Then, one can still define the notions of Hecke stacks $\mathrm{Hk}_{\mathcal{G}}$ and shtuka stacks $\mathrm{Sht}_{\mathcal{G}}$, and affine Grassmannians $\mathrm{Gr}_{\mathcal{G}}$. The bounds by coweights μ extend to $\mathbb{Z}_p$ by taking closures, and one of our contributions revolved around understanding this procedure in detail.

Theorem 6.1 ([AGLR22, GL22]) *The v-sheaf* $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ *is normal with special fiber equal to the* μ*-admissible locus. If* μ *is minuscule, then it is representable by a unique normal, Cohen–Macaulay, flat* $\mathbb{Z}_p$*-scheme with reduced special fiber.*

Part of this had been previously conjectured by Scholze–Weinstein [SW20] and much of the motivation stemmed from the arithmetic of Shimura varieties, where the minuscule integral Schubert varieties appear as local models for controlling the singularities, see the book of Rapoport–Zink [RZ96]. These were studied extensively in the last decades, most notably by Pappas, Rapoport, and Zhu [PR08, PZ13], but the approach in [AGLR22] was the first to actually provide a complete and functorial theory.

One important ingredient in [AGLR22] is the formalism of kimberlites due to Gleason [Gle22], which are v-sheaves of formal nature with a scheme-theoretic reduction whose complement is a diamond, and admitting a specialization map between the underlying topological spaces. In [AGLR22] we proved that $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ can be recovered from its $\mathbb{Z}_p$-fibers and the specialization map between them. Note that the special fiber of the unbounded Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ was first defined by Zhu [Zhu17] and then Bhatt–Scholze [BS17] proved it is an ind-perfect scheme. The μ-admissible locus in the theorem goes back to Kottwitz–Rapoport [KR00] and equals the union of the $\mathcal{G}(\mathbb{Z}_p)$-orbit closures in $\mathrm{Gr}_{\mathcal{G},\mathbb{F}_p}$ of the Weyl conjugates of μ. We are able to pinpoint the specialization map for minuscule μ by using a convolution analogue of the Iwasawa decomposition.

Arguably, the most crucial task in [AGLR22] is identifying the special fiber of $\mathrm{Gr}_{\mathcal{G},\leq\mu}$. This requires working with the derived category $\mathcal{D}(\mathrm{Hk}_{\mathcal{G}})$ of étale $\bar{\mathbb{Q}}_\ell$-sheaves on the Hecke stack building on [Sch17]. The main tool for us is the functor of nearby cycles $R\Psi : \mathcal{D}(X_{\bar{\eta}}) \to \mathcal{D}(X_{\bar{s}})$ specializing between geometric fibers. Its origins lie in Morse theory: for a map $f : X \to \mathbb{D}$ with an isolated singularity at the origin, $R\Psi$ carries the cohomology classes of the non-singular fibers to the fiber at 0. In the case of schemes, Deligne defined $R\Psi$ for a map $f : X \to \mathbb{Z}_p$ in [SGA73a] by pushing sheaves forward along the absolute integral closure, and this definition of also works in the situation of [AGLR22]. Using constant terms functors, we could prove that universal locally acyclic sheaves are preserved under $R\Psi$ and their images are actually scheme-theoretic.

Theorem 6.2 ([AGLR22, ALWY23]) *The functor $R\Psi\colon \mathcal{P}(\mathrm{Hk}_{G,\mathbb{C}_p}) \to \mathcal{D}(\mathrm{Hk}_{\mathcal{G},\bar{\mathbb{F}}_p})$ lifts to the Drinfeld center and lands in $\mathcal{P}(\mathrm{Hk}_{\mathcal{G},\bar{\mathbb{F}}_p})$.*

This is the analogue of the main theorem of Gaitsgory [Gai01] in the function field case, but the p-adic setting complicates matters. For instance, nearby cycles of algebraic schemes preserve perversity by Artin vanishing, but this is no longer true for general v-sheaves. In order to prove it, we introduced Wakimoto sheaves in [ALWY23] at Iwahori level following Arkhipov–Bezrukavnikov [AB09], and used them in combination with geometric Satake to a filtration of $R\Psi$ by Wakimoto perverse sheaves. As for the centrality of $R\Psi$, we constructed the main isomorphisms in [AGLR22], but only verified the higher homotopy coherences in [ALWY23].

In [AGLR22], we applied nearby cycles and geometric Satake to show that the special fiber of $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ coincides with the μ-admissible locus. Another application of the central sheaves is the normality of $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ proved in [GL22] by a much simpler method than [Zhu14] in the function field case. We use the Wakimoto filtration to show connectedness for the analytic tubes of the closed points in the special fiber of $\mathrm{Gr}_{\mathcal{G},\mu}$ up to codimension 2. This reduces normality to a combinatorial S_2 property for the special fiber, which we verify in the function field case thanks to a dynamical argument of Le–Le Hung–Levin–Morra [LHLM22].

A natural continuation of the above is to extend the central functor to a Bezrukavnikov equivalence for p-adic fields in analogy with [Bez16] for Laurent series fields. It asserts that there is an equivalence of derived categories

$$\mathcal{D}_{\mathrm{coh}}([G^\vee \backslash \mathrm{St}_{G^\vee}]) \simeq \mathcal{D}(\mathrm{Hk}_{\mathcal{I},\bar{\mathbb{F}}_p}) \tag{2}$$

of coherent sheaves on the left and étale $\bar{\mathbb{Q}}_\ell$-sheaves on the left. Here, $\mathrm{St}_{G^\vee}$ is the dual Steinberg variety of triples and $\mathcal{I}$ is a Iwahori model of G. In [ALWY23], we followed [AB09] to construct roughly one half of the equivalence using the Springer resolution of the nilpotent cone $\mathcal{N}_{G^\vee}$ in place of the Steinberg variety. Unfortunately, there is one essential ingredient still missing in the p-adic setting beyond the GL_n case. We need to bound certain Hom spaces involving quasi-minuscule representations and the fastest way to do this is extending the monodromy operator on the image of $R\Psi$ to the entire category of perverse sheaves. This is classically done via rescaling uniformizers but cannot be performed in the p-adic setting. We hope to address this also in our future work.

The importance of the Bezrukavnikov equivalence lies in its usefulness for geometric Langlands. Recently, Zhu [Zhu20] proposed a different geometrization of local Langlands over $\mathbb{Q}_p$, where instead of Bun_G he considers the stack Isoc_G of *G-isocrystals*, which geometric points equal $B(G)$ but carries the opposite topology. In upcoming work of Hemo–Zhu, the Bezrukavnikov equivalence is used via the trace of Frobenius to produce a Langlands equivalence for tame representations with Isoc_G in place of Bun_G. We are thus led to anticipate the following.

Conjecture 6.3 *There is an equivalence* $\mathcal{D}(\mathrm{Isoc}_G) \simeq \mathcal{D}(\mathrm{Bun}_G)$ *of derived categories.*

Recently, Gleason–Ivanov [GI23] constructed a geometric correspondence between Isoc_G and Bun_G. In an ongoing project, we aim to tackle the conjecture above by using this geometric correspondence. Simultaneously, the Bezrukavnikov equivalence and our partial work towards it in [ALWY23] should be an essential ingredient in comparing the (tame) spectral action on the two sides.

Referências

[AB09] Sergey Arkhipov and Roman Bezrukavnikov. Perverse sheaves on affine flags and Langlands dual group. *Israel J. Math.*, 170:135–183, 2009. With an appendix by Bezrukavnikov and Ivan Mirković.

[AGLR22] Johannes Anschütz, Ian Gleason, João Lourenço, and Timo Richarz. On the p-adic theory of local models. *arXiv preprint arXiv:2201.01234*, 2022.

[ALWY23] Johannes Anschütz, João Lourenço, Zhiyou Wu, and Jize Yu. Gaitsgory's central functor and the Arkhipov-Bezrukavnikov equivalence in mixed characteristic. *arXiv preprint arXiv:2311.04043*, 2023.

[Art27] Emil Artin. Beweis des allgemeinen Reziprozitätsgesetzes. *Abh. Math. Sem. Univ. Hamburg*, 5(1):353–363, 1927.

[BB66] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.

[BBDG18] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, and Ofer Gabber. *Faisceaux pervers.* Société mathématique de France, 2018.

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over $\mathbf{Q}$: wild 3-adic exercises. *J. Amer. Math. Soc.*, 14(4):843–939, 2001.

[Bez16] Roman Bezrukavnikov. On two geometric realizations of an affine Hecke algebra. *Publ. Math. Inst. Hautes Études Sci.*, 123(1):1–67, 2016.

[BG14] Kevin Buzzard and Toby Gee. The conjectural connections between automorphic representations and Galois representations. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 135–187. Cambridge Univ. Press, Cambridge, 2014.

[Bor83] M. V. Borovoĭ. Langlands' conjecture concerning conjugation of connected Shimura varieties. volume 3, pages 3–39. 1983. Selected translations.

[BS17] Bhargav Bhatt and Peter Scholze. Projectivity of the Witt vector affine Grassmannian. *Invent. Math.*, 209(2):329–423, 2017.

[BT84] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Publ. Math. Inst. Hautes Études Sci.*, (60):197–376, 1984.

[Che40] C. Chevalley. La théorie du corps de classes. *Ann. of Math. (2)*, 41:394–418, 1940.

[Che55] C. Chevalley. Sur certains groupes simples. *Tohoku Math. J. (2)*, 7:14–66, 1955.

[CS19] Dustin Clausen and Peter Scholze. Condensed mathematics. 2019.

[Del71a] Pierre Deligne. Formes modulaires et représentations l-adiques. In *Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363*, volume 175 of *Lecture Notes in Math.*, pages Exp. No. 355, 139–172. Springer, Berlin, 1971.

[Del71b] Pierre Deligne. Travaux de Shimura. In *Séminaire Bourbaki vol. 1970/71 Exposés 382–399*, pages 123–165. Springer, 1971.

[Del74] Pierre Deligne. La conjecture de Weil. I. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 43(1):273–307, 1974.

[Del79] Pierre Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modeles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part*, volume 2, pages 247–289, 1979.

[DHKM20] Jean-François Dat, David Helm, Robert Kurinczuk, and Gilbert Moss. Moduli of Langlands parameters. *arXiv preprint arXiv:2009.06708*, 2020.

[DR73] Pierre Deligne and Michael Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable II*, pages 143–316. Springer, 1973.

[Dri74] V. G. Drinfeld. Elliptic modules. *Mat. Sb. (N.S.)*, 94(136):594–627, 656, 1974.

[Dri80] V. G. Drinfeld. Langlands' conjecture for GL(2) over functional fields. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 565–574. Acad. Sci. Fennica, Helsinki, 1980.

[Fal03] Gerd Faltings. Algebraic loop groups and moduli spaces of bundles. *J. Eur. Math. Soc. (JEMS)*, 5(1):41–68, 2003.

[Far20] Laurent Fargues. G-torseurs en théorie de Hodge p-adique. *Compos. Math.*, 156(10):2076–2110, 2020.

[FF18] Laurent Fargues and Jean Marc Fontaine. *Courbes et fibrés vectoriels en théorie de Hodge p-adique*. Société mathématique de France, 2018.

[FS21] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence. *preprint arXiv:2102.13459*, 2021.

[Gai01] D. Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. *Invent. Math.*, 144(2):253–280, 2001.

[Gau86] Carl Friedrich Gauss. *Disquisitiones arithmeticae.* Springer-Verlag, New York, 1986. Translated and with a preface by Arthur A. Clarke, Revised by William C. Waterhouse, Cornelius Greither and A. W. Grootendorst and with a preface by Waterhouse.

[GD63] Alexandre Grothendieck and Michel Demazure. *Schémas en groupes.* Publ. Math. Inst. Hautes Études Sci., 1963.

[GI23] Ian Gleason and Alexander B Ivanov. Meromorphic vector bundles on the Fargues–Fontaine curve. *arXiv preprint arXiv:2307.00887*, 2023.

[Gin95] Victor Ginzburg. Perverse sheaves on a loop group and Langlands' duality. *arXiv preprint alg-geom/9511007*, 1995.

[GL22] Ian Gleason and João Lourenço. Tubular neighborhoods of local models. *arXiv preprint arXiv:2204.05526, to appear in Duke Math J*, 2022.

[Gle22] Ian Gleason. Specialization maps for Scholze's category of diamonds. *preprint arXiv:2012.05483*, 2022.

[GM83] Mark Goresky and Robert MacPherson. Morse theory and intersection homology theory. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 135–192. Soc. Math. France, Paris, 1983.

[Hec27] E. Hecke. Zur Theorie der elliptischen Modulfunktionen. *Math. Ann.*, 97(1):210–242, 1927.

[Hel23] Eugen Hellmann. On the derived category of the Iwahori-Hecke algebra. *Compos. Math.*, 159(5):1042–1110, 2023.

[Hen01] Kurt Hensel. Ueber die Entwickelung der algebraischen Zahlen in Potenzreihen. *Math. Ann.*, 55(2):301–336, 1901.

[HLR18] Thomas J Haines, João Lourenço, and Timo Richarz. On the normality of Schubert varieties: remaining cases in positive characteristic. *arXiv preprint arXiv:1806.11001, accepted at Ann. Sci. Éc. Norm. Supér.*, 2018.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties.(AM-151), Volume 151.* Princeton University press, 2001.

[Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces.* Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.

[KL15] Kiran S. Kedlaya and Ruochuan Liu. Relative p-adic Hodge theory: foundations. *Astérisque*, (371):239, 2015.

[KR00] R Kottwitz and M Rapoport. Minuscule alcoves for GL_n and GSp_{2n}. *Manuscripta mathematica*, 102(4):403–428, 2000.

[Laf02] Laurent Lafforgue. Chtoucas de drinfeld et correspondance de Langlands. *Invent. Math.*, 147(1):1–241, 2002.

[Laf18] Vincent Lafforgue. Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale. *J. Amer. Math. Soc.*, 31(3):719–891, 2018.

[Lan70] R. P. Langlands. Problems in the theory of automorphic forms. In *Lectures in Modern Analysis and Applications, III*, Lecture Notes in Math., Vol. 170, pages 18–61. Springer, Berlin-New York, 1970.

[LD69] G. Lejeune Dirichlet. *Mathematische Werke. Bände I, II.* Chelsea Publishing Co., Bronx, NY, 1969. Herausgegeben auf Veranlassung der Königlich Preussischen Akademie der Wissenschaften von L. Kronecker.

[LHLM22] Daniel Le, Bao V Le Hung, Brandon Levin, and Stefano Morra. Local models for Galois deformation rings and applications. *Inventiones mathematicae*, pages 1–212, 2022.

[Lou23] João Lourenço. Distributions and normality theorems. *arXiv preprint arXiv:2312.17121*, 2023.

[LRS93] Gérard Laumon, Michael Rapoport, and Ulrich Stuhler. *D-elliptic sheaves and the Langlands correspondence.* *Invent. Math.*, 113(1):217–338, 1993.

[LT65] Jonathan Lubin and John Tate. Formal complex multiplication in local fields. *Ann. of Math. (2)*, 81:380–387, 1965.

[Mil83] J. S. Milne. The action of an automorphism of **C** on a Shimura variety and its special points. In *Arithmetic and geometry, Vol. I*, volume 35 of *Progr. Math.*, pages 239–265. Birkhäuser Boston, Boston, MA, 1983.

[MV07] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007.

[PR08] G. Pappas and M. Rapoport. Twisted loop groups and their affine flag varieties. *Adv. Math.*, 219(1):118–198, 2008. With an appendix by T. Haines and Rapoport.

[PZ13] George Pappas and Xinwen Zhu. Local models of Shimura varieties and a conjecture of Kottwitz. *Invent. Math.*, 194(1):147–254, 2013.

[Rib90] Kenneth A. Ribet. From the Taniyama-Shimura conjecture to Fermat's last theorem. *Ann. Fac. Sci. Toulouse Math. (5)*, 11(1):116–139, 1990.

[RZ96] M. Rapoport and Th. Zink. *Period spaces for p-divisible groups*, volume 141 of *Ann. of Math. Stud.* Princeton University Press, Princeton, NJ, 1996.

[Sat63] Ichirô Satake. Theory of spherical functions on reductive algebraic groups over p-adic fields. *Inst. Hautes Études Sci. Publ. Math.*, (18):5–69, 1963.

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[Sch17] Peter Scholze. Étale cohomology of diamonds. *ArXiv e-prints*, September 2017.

[SGA73a] *Groupes de monodromie en géométrie algébrique. II.* Lecture Notes in Mathematics, Vol. 340. Springer-Verlag, Berlin-New York, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz.

[SGA73b] *Théorie des topos et cohomologie étale des schémas. Tome 3.* Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin-New York, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[Shi63] Goro Shimura. On analytic families of polarized abelian varieties and automorphic functions. *Ann. of Math. (2)*, 78:149–192, 1963.

[ST61] Goro Shimura and Yutaka Taniyama. *Complex multiplication of abelian varieties and its applications to number theory*, volume 6 of *Publications of the Mathematical Society of Japan.* Mathematical Society of Japan, Tokyo, 1961.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley Lectures on p-adic Geometry*, volume 389 of *AMS-207.* Princeton University Press, 2020.

[TW95] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.

[Vie21] Eva Viehmann. On Newton strata in the B_{dr}^{+}-Grassmannian. *arXiv preprint arXiv:2101.07510*, 2021.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

[Wit37] Ernst Witt. Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p. *J. Reine Angew. Math.*, 176:126–140, 1937.

[Zhu14] Xinwen Zhu. On the coherence conjecture of Pappas and Rapoport. *Ann. of Math. (2)*, pages 1–85, 2014.

[Zhu17] Xinwen Zhu. Affine Grassmannians and the geometric Satake in mixed characteristic. *Ann. of Math. (2)*, pages 403–492, 2017.

[Zhu20] Xinwen Zhu. Coherent sheaves on the stack of Langlands parameters. *arXiv preprint arXiv:2008.02998*, 2020.

Tropicalizing Moduli Spaces and Applications

Margarida Melo
Dipartimento di Matematica e Fisica, Universita Roma Tre
Largo San Leonardo Murialdo I-00146 Roma Italy
e-mail: `margarida.melo@uniroma3.it`

Resumo:

Descrevemos relações intrínsecas entre o espaço de moduli de curvas projetivas, a sua compactificação através de curvas de Deligne-Mumford estáveis, e o espaço de moduli de curvas tropicais. O ponto chave consiste na identificação de uma categoria de objetos combinatoriais que governa a estratificação por tipo topológico em ambos os espaços de moduli. Descrevemos também algumas aplicações do uso destes instrumentos ao estudo da geometria do espaço de moduli de curvas e aplicações semelhantes a outros espaços de moduli.

Abstract

We describe intrinsic relations between the moduli space of smooth projective curves, its compactification via Deligne-Mumford stable curves, and the moduli space of tropical curves. The key point lies in describing a category of combinatorial objects governing the topological type stratification of both moduli spaces. We also illustrate remarkable applications of this interplay on understanding the geometry of the original space and explain similar applications to other moduli spaces.

palavras-chave: espaço de moduli, curvas suaves, curvas estáveis, tropicalização, estratificação topológica, curvas tropicais, cohomologia racional.

keywords: moduli spaces, smooth curves, stable curves, tropicalization, topological stratification, tropical curves, rational cohomology.

1 Introduction

Moduli spaces are parameter spaces for algebro-geometric objects with certain fixed geometric invariants. In algebraic geometry, it is often the case that such spaces are also endowed with the structure of algebraic variety, algebraic scheme or algebraic stack. Therefore, one can use again algebro-geometric tools to study such spaces and, using their modular interpretation, give answers to classification problems for the original varieties and their deformations.

The most studied moduli space in algebraic geometry is certainly the moduli space of curves of given genus. Indeed, it was already considered by Riemann, who introduced the term *Moduli* and considered the set of isomorphism classes of complex structures definable over a compact, connected, topological surface of genus $g \geq 2$ as a complex variety of dimension $3g-3$, denoted by $\mathcal{M}_g$. Along the second half of the 20'th century, the moduli space of curves was then extensively studied by algebraic geometers and generalized in many ways. For instance, one can fix a number n of distinct and ordered points on the curves: such a space is known as the moduli space of pointed/marked curves and is denoted by $\mathcal{M}_{g,n}$; or even add the information of a map from the curve to a given space with geometric constraints on the image of the fixed points, the moduli space of stable maps: this space is particularly relevant for applications to enumerative geometry and the development of Gromov-Witten theory.

An important limitation of the moduli space of curves is the fact that it is not compact, so it cannot be used properly to classify important phenomena regarding curves from the point of view of enumerative geometry, which requires techniques usually available for compact spaces. This limitation was completely solved after the breakthrough work by Deligne and Mumford, followed by Knudsen, who constructed a remarkable modular compactification for $\mathcal{M}_{g,n}$: the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ in [DM69] and [Knu83]. This compactification has many important geometrical properties, such as the fact that it is smooth (viewed as an algebraic stack), normal crossings (i.e., the boundary locus looks like the intersection of coordinate hyperplanes locally at any point). More importantly, its strength relies on its modular properties: $\overline{\mathcal{M}}_{g,n}$ itself is a moduli space parametrizing pointed stable curves, i.e., curves admitting nodal singularities and satisfying a stability condition, which ensures that there are only finitely many topological types of such curves.

Even if the moduli space of curves is one of the most studied and fascinating spaces in algebraic geometry, there are still many mysterious aspects of its geometry that keep attracting the attention of mathematicians with different points of view. A remarkable example of success is the structure of the stable cohomology ring of $\mathcal{M}_g$, which is a polynomial ring in the so-called kappa classes as conjectured by Mumford and later proved by Madsen and Weiss in [MW07]. On the other hand, the rational cohomology ring of $M_{g,n}$ is completely understood only for low values of g and n, and recently there have been lots of efforts trying to understand more deeply its topology.

It turns out that, by applying Deligne's theory of weights, part of the cohomology of $\mathcal{M}_{g,n}$ is governed by the topology of any of its normal crossings compactifications. It is then natural to expect that combinatorial methods come into the picture in order to shed some light on the understanding of the geometry of such boundaries.

Tropical geometry is a new area within algebraic geometry whose geometric objects are piecewise linear, and therefore much more combinatorial in nature. It has experienced an extraordinary development in the last 20 years and has shown to be an important tool as well when applied to describe degenerations of algebro-geometric objects. The spirit of tropicalization is that one can zoom in locally at a given point of a variety and extract the geometrical behaviour of its nearby points by representing them within a fan of polyhedral cones. When applied to normal crossings compactifications of moduli spaces, this acquires a modular meaning: one can classify all possible degenerations of a given object by looking at the *tropicalized moduli space*, which is a (generalized) cone complex parametrizing the analogous tropical objects up to tropical modification.

The construction of tropical moduli spaces was inspired by the pioneering work of Mikhalkin-Zharkov in [MZ07], who described analogous tropical notions of abstract curves, divisors, sheaves, abelian varieties, and so on. Based on these notions, one can construct tropical moduli spaces as generalized cone complexes, as in the case of curves, Jacobians and abelian varieties in [BMV11]. These spaces have also been intrinsically realized as tropicalization of both the non-archimedean analytification and the logarithmic version of the original compactified spaces in [ACP15] and in [CCUW20].

At this point, the role of tropical geometry becomes clear as it provides a modular description for the combinatorial data used to compactify the original objects. These modular descriptions are indeed quite helpful and have been recently applied with great success in a number of results concerning geometric properties of the original spaces. One of such achievements is the beautiful computation of the top-weight cohomology of the moduli space of curves in the breakthrough work of Chan, Galatius and Payne [CGP21]. We will try to explain this application in the remaining part of this text.

We start by introducing our main combinatorial tools: graphs and categories of graphs.

We then give a short description of the moduli space of projective curves, its Deligne-Mumford compactification, and its stratification by topological type.

We continue by introducing the moduli space of tropical curves, highlighting its analogy with the moduli space of stable curves, and by briefly explaining the tropicalization morphism from the Berkovich analytification of the Deligne-Mumford compactification to the moduli space of curves.

Finally, we put all our spaces together and explain Chan, Galatius and Payne's computation of the top-weight rational cohomology of the moduli space of curves by computing the homology of the moduli space of curves. We conclude by briefly mentioning other remarkable applications of tropical geometry to the geometry of classical algebro-geometric varieties.

2 Preliminaries: Graphs and categories of graphs

Given a graph G, we will indicate with $V(G)$ the set of vertices of G and with $E(G)$ the set of edges of G. Given an edge $e \in E(G)$, we will write $e = uv$, for $u, v \in V(G)$ to indicate that u and v are the ends of e. Notice that we will allow for the existence of multiple edges and loop edges as well (in that case we write $e = uu$). We will consider our graphs to be connected, unless otherwise stated: in that case, we will denote with $c(G)$ the number of connected components of G.

Our graphs will often be weighted, i.e., endowed with a weight function $w : V(G) \to \mathbb{Z}_{\geq 0}$. Given a (weighted) graph $G = (G, w)$, the genus of G is set to be

$$g(G) := \sum_{v \in V(G)} w(v) + b_1(G) = g,$$

where $b_1(G) = |E(G)| - |V(G)| + c(G)$ is the first Betti number of G.

Definition 2.1. Let g, n be non-negative integers such that $2g - 2 + n > 0$.

- A $(g, n)-$graph is a (weighted) graph G of genus g and n legs, i.e., G is a genus g graph endowed with a leg function $\mathrm{leg} : [n] := \{1, \ldots, n\} \to V(G)$.
- The $(g, n)-$graph G is said to be *stable* if $\forall v \in V(G)$ with $w(v) = 0$, we have that
$$\mathrm{val}(v) + |\,\mathrm{leg}^{-1}(v)| \geq 3,$$
where $\mathrm{val}(v)$ denotes the valence (or degree) of the vertex v.

A *morphism* of weighted graphs $G = (G, w)$ and $G' = (G', w')$ is a pair $\pi = (\pi_V, \pi_E)$, where $\pi_V : V(G) \twoheadrightarrow V(G')$ is surjective and $\pi_E : E(G) \to E(G') \cup V(G')$ is surjective onto $E(G')$ such that:

- π_E is compatible with the incidence function, i.e., if $e = uv \in E(G)$ and $\pi_E(e) = e' = u'v' \in E(G')$, then we must have that $\pi_V(u) = u'$ and $\pi_V(v) = v'$;
- π respects the weight function, i.e., for all $v' \in V(G')$, $\pi^{-1}(v') \subset G$ is a connected subgraph of G with $g(\pi^{-1}(v')) = w'(v')$;
- π induces an inclusion $i_\pi : E(G') \hookrightarrow E(G)$ such that $\pi_E \circ i_\pi = \mathrm{id}_{\mathrm{E(G')}}$.

If both G and G' are endowed with leg functions $\mathrm{leg} : [n] \to V(G)$ and $\mathrm{leg}' : [n] \to V(G')$, we further ask that π_V be compatible with induced markings, i.e., $\forall i \in [n]$, $\mathrm{leg}'(i) = \pi_V(\mathrm{leg}(i))$.

The morphism π is said to be an *isomorphism* if i_π is bijective (that is, edges of G are mapped bijectively to the edges of G'). Indeed, since we are assuming that our graphs are connected, this implies that also π_V is bijective. Given $S \subset E(G)$ inducing a connected subgraph $G[S]$ of G, the contraction of S is a particular morphism $\pi_S : G \to G/S$ which maps all the vertices and edges of S to a vertex $v_S \in G/S$ and is an isomorphism elsewhere. One can check that morphisms of weighted graphs are obtained by iterating isomorphisms and contractions.

Remark 2.2. It follows easily from the definitions that morphisms of weighted graphs preserve the genus, the markings and also the stability condition, i.e., the image of a stable graph via a morphism of graphs is still stable.

In the light of Remark 2.2, we can consider the set of (stable) (g, n)-graphs as a category, as follows.

Definition 2.3. Let g, n be non-negative integers as above. The category of stable graphs of type (g, n) is the category $\mathcal{G}_{g,n}$ such that:

- objects are stable weighted graphs of genus g and n markings;
- morphisms are morphisms of weighted stable graphs.

Remark 2.4. The stability condition implies that, for fixed g and n, there are only finitely many isomorphim classes of (g, n)-stable graphs. Moreover, it is an easy consequence of the Handshaking Lemma, that the maximal number of edges and vertices of a stable graph of genus g with n legs is, respectively, $3g - 3 + n$ and $2g - 2 + n$.

The reader might suspect that the fact that the maximum number of edges of a stable graph of given genus matches the dimension of $M_{g,n}$ is not a coincidence.

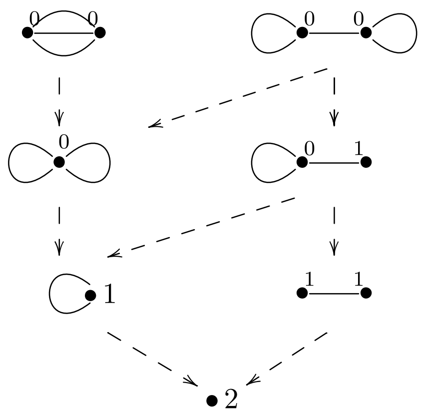


Figura 1: The category of stable graphs of genus 2 and no markings, $\mathcal{G}_{2,0}$. Dotted arrows represent morphisms in the category obtained by (weighted) contraction of edges.

3 Moduli spaces in algebraic geometry

Let $\mathcal{M}_g$ denote the space of isomorphism classes of smooth projective curves of genus g (remember that these correspond to compact Riemann surfaces of genus g if the curves are defined over the complex numbers). Recall that for $g = 0$, since all smooth rational curves are isomorphic to the projective line, $\mathcal{M}_0 \cong \{pt\}$; while for $g = 1$, one can see that isomorphism classes of elliptic curves are classified by an invariant, called the j-invariant, which corresponds to a point in the affine line $\mathbb{A}^1$, so $\mathcal{M}_1 \cong \mathbb{A}^1$. For $g \geq 2$, in the seminal paper [DM69], Deligne and Mumford showed the existence of $\mathcal{M}_g$ over $\mathbb{Z}$ and proved that it is an irreducible quasiprojective variety of dimension $3g - 3$.[1].

More generally, one can consider the moduli space $\mathcal{M}_{g,n}$ of smooth curves of genus g with n distinct markings. Then, if (g, n) is such that $2g-2+n > 0$, Knudsen in [Knu83] built on the work of Deligne-Mumford to show that $\mathcal{M}_{g,n}$ is a quasiprojetive variety of dimension $3g - 3 + n$.

Since then, the moduli space of smooth curves of given genus and distinct n markings has been deeply studied. However, it is easy to see that $\mathcal{M}_{g,n}$ is not compact, as smooth curves degenerate to singular curves, and since distinct points may collide. In the above-cited papers, the authors

[1]Deligne-Mumford actually constructed $\mathcal{M}_g$ as an algebraic stack in [DM69]

constructed modular compactifications for the spaces $\mathcal{M}_{g,n}$ by means of (Deligne-Mumford) stable curves.

Definition 3.1. A stable (marked) curve is a nodal curve X, possibly with several irreducible components (and n distinct smooth points), satisfying the following stability condition: for each irreducible component C of X isomorphic to a smooth rational curve, C contains at least 3 special points of X, i.e., the number of points where C intersects the rest of X plus the number of markings on C must be at least 3.

Remark 3.2. It is easy to see that the above stability condition is equivalent to the condition that the automorphism group of X is finite. Notice also that there exist stable curves of genus g with n markings if and only if $2g-2+n>0$.

Theorem 3.3. [[DM69], [Knu83]] Let g and n be non-negative integers such that $2g-2+n>0$. Then there are irreducible projective varieties $\overline{\mathcal{M}}_{g,n}$ parametrizing isomorphism classes of stable curves of genus g and n markings. The compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is toroidal[2] and smooth if seen as an algebraic stack, and the boundary is a normal crossings divisor.

The topological type of a nodal curve X is encoded by its dual graph, defined as follows.

Definition 3.4. Let X be a nodal curve with irreducible components $C_i, i = 1, \dots, \gamma$. The dual (weighted) graph of X is the graph G_X whose vertices v_i of G_X correspond to the irreducible components C_i of X, with weight $w(v_i)$ equal to the geometric genus of C_i, and such that to each node n meeting components C_i and C_j we associate an edge e_n with ends in v_i and v_j (in particular, internal nodes correspond to loops of G_X).

Figure 1 above contains all dual graphs of stable curves of genus 2 and no markings.

Remark 3.5. It is easy to see that the arithmetic genus of a nodal curve X coincides with the genus of its dual graph G_X. Moreover, X is stable if and only if G_X is stable.

[2]The theory of toroidal embedding was developed in [KKMSD73] and [AMRT75] as a tool to compactify locally symmetric domains. An embedding is said to be *toroidal* if locally in the étale topology it is isomorphic to the embedding of a torus in a toric variety.

3.1 Stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type

Let $G \in \mathcal{G}_{g,n}$ be a stable graph and set

$$M_G := \{X \in \overline{\mathcal{M}}_{g,n} : G_X \cong G\},$$

i.e., M_G consists of the locus of curves in $\overline{\mathcal{M}}_{g,n}$ with fixed topological type.

It is a quite important and deep fact about the geometry of $\overline{\mathcal{M}}_{g,n}$ that the category $\mathcal{G}_{g,n}$ governs the stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type, i.e. each stratum M_G is irreducible and locally closed of codimension $|E(G)|$, and its closure is made of smaller dimensional strata according to morphisms in $\mathcal{G}_{g,n}$, i.e.,

$$\overline{M_G} = \coprod_{G' \to G \in \mathrm{Mor}(\mathcal{G}g,n)} M_{G'}. \tag{1}$$

So, not only do objects in the category of stable graphs $\mathcal{G}_{g,n}$ parametrize the strata of the decomposition of $\overline{\mathcal{M}}_{g,n}$ by topological type, but the morphisms encode the topology of strata: in particular, a weighted contraction $G' \to G$ indicates an algebraic degeneration from curves with dual graph G to curves with dual graph G', while an automorphism of a graph $G \in \mathcal{G}_{g,n}$ indicates that the stratum M_G has a self-gluing.

This decomposition can be understood in terms of local affine coordinates of the variety $\overline{\mathcal{M}}_{g,n}$, showing that the boundary of the compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is a normal crossings divisor[3].

4 Moduli Space of tropical curves

4.1 Tropical curves

Definition 4.1. A (g, n) tropical curve (or tropical curve of genus g with n legs, or markings), is a pair $\Gamma = (G, l)$ where G is a (g, n) stable (connected) graph, called the *combinatorial type* of Γ, and $l : E(G) \to \mathbb{R}_{>0}$ is a length function on the edge set of G.

Notice that a tropical curve is endowed with the structure of a metric space obtained by gluing intervals in Euclidean space of length $l(e)$, for all edges $e \in E(G)$, along their ends, according to the incidence rule associated with G. We say that two tropical curves are isomorphic if their combinatorial types are isomorphic via an isomorphism of graphs which induces an isometry of the associated metric spaces.

[3]A normal crossings divisor is a divisor that can be expressed locally as the union of coordinate hyperplanes.

The moduli space of tropical curves $M_{g,n}^{\text{trop}}$ was constructed in [BMV11] for $n = 0$ and for any n in [Cap12] following the work of Mikhalkin-Zharkov in [MZ07]. Indeed, our space is a parameter space for abstract tropical curves, i.e., tropical curves up to tropical equivalence, as discussed by Mikhalkin-Zharkov in loc.cit., as opposed to a parameter space of tropical curves embedded in a certain space.

The starting point is to associate to any stable graph $G \in \mathcal{G}_{g,n}$, a rational polyhedral cone

$$\mathcal{C}_G := \mathbb{R}_{>0}^{|E(G)|}$$

parametrizing length functions on tropical curves Γ with topological type G. Then, faces of the closed cone $\overline{\mathcal{C}_G}$ should parametrize degenerations of tropical curves in $\mathcal{C}_G$ obtained from sending the lengths of some edges to zero. Formally, such identifications are associated to morphisms $\pi : G \to G' \in \mathcal{G}_{g,n}$, whose induced inclusions $i_\pi : E(G') \hookrightarrow E(G)$ naturally yield face morphisms

$$i_\pi : \overline{\mathcal{C}_{G'}} \hookrightarrow \overline{\mathcal{C}_G}, \tag{2}$$

which we will denote with the same symbol, by abuse of notation.

Notice that in the case when $\pi : G \to G$ is an isomorphism, the associated face morphism $i_\pi : \overline{\mathcal{C}_G} \hookrightarrow \overline{\mathcal{C}_G}$ is an automorphism of the cone $\overline{\mathcal{C}_G}$. Indeed, this corresponds to the fact that points in the moduli space of tropical curves with combinatorial type G should be in bijective correspondence with $\mathcal{C}_G/\text{Aut}(G)$.

In order to construct the moduli space $M_{g,n}^{\text{trop}}$ of tropical curves of genus g with n markings, one considers the union of all cones $\overline{\mathcal{C}_G}$, for $G \in \mathcal{G}_{g,n}$, glued together along face morphisms associated to graph morphisms $G \to G'$ in $\mathcal{G}_{g,n})$. We are therefore ready to define $M_{g,n}^{\text{trop}}$ in the category of generalized cone complexes.

Definition 4.2. Let g, n be non-negative integers such that $2g - 2 + n > 0$. The moduli space of tropical curves of genus g and n markings is the generalized cone complex

$$M_{g,n}^{\text{trop}} := \varinjlim_{G \in \mathcal{G}_{g,n}} \overline{\mathcal{C}_{g,n}}$$

By its very construction, the moduli space of tropical curves is a complex of polyhedral cones of real dimension $3g - 3 + n$. Since some of the cones may have some interior points identified via self-gluing morphisms, such an object is usually called a generalized cone complex.

For the sake of applications to the topology of $\mathcal{M}_{g,n}$, it is actually more convenient to consider the link of $M_{g,n}^{\mathrm{trop}}$, i.e., the locus of tropical curves with volume (or total length) equal to one, which we will denote by $L(M_{g,n}^{\mathrm{trop}})$. Indeed, one can think of $L(M_{g,n}^{\mathrm{trop}})$ as being obtained by intersecting $M_{g,n}^{\mathrm{trop}}$ with a sphere. Indeed, while $M_{g,n}^{\mathrm{trop}}$, being made of cones, is easily seen to be contractible, its link $L(M_{g,n}^{\mathrm{trop}})$ is not, and its topology is much more interesting.

The reader may also notice that $L(M_{g,n}^{\mathrm{trop}})$ is closely related to Outer Space: the classifying space for the free group on g letters constructed by Culler and Vogtmann in [CV86].

4.2 Tropicalization morphisms

The moduli space of Deligne Mumford stable curves and the moduli space of tropical curves are closely related: in fact, the category of stable graphs $\mathcal{G}_{g,n}$ governs both the toroidal stratification of the boundary $\partial(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n})$ and the topology of $M_{g,n}^{\mathrm{trop}}$ (see (1), (2) and Definition 4.2). Indeed, in both situations, objects in $\mathcal{G}_{g,n}$ parametrize topological types of stable curves and of tropical curves, while morphisms encode the topological type of strata lying in the closure of curves of a given topological type. However, the identification is order reversing: a morphism $G \to G'$ in $\mathcal{G}_{g,n}$ indicates that the locus M_G of curves with dual graph G lies in the closure of the locus of curves with dual graph G', and, on the other direction, that the locus of tropical curves with combinatorial type G' can be seen as a face of the cone parametrizing tropical curves of combinatorial type G (see [Cap13] for more details).

The relation between the Deligne-Mumford compactification and the moduli space of tropical curves turns out to be much deeper, as highlighted in the breakthrough work of Abramovich, Caporaso and Payne [ACP15]. Indeed, in loc. cit., the authors study the Berkovich analytification associated with the toroidal compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$, together with its intrinsic topological retraction to a generalized cone complex.[4] The authors then show that this generalized cone complex is isomorphic to the moduli space of tropical curves $M_{g,n}^{\mathrm{trop}}$.

As a consequence of this relation, one can also obtain the following identification:

$$\Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}) \cong L(M_{g,n}^{\mathrm{trop}}), \tag{3}$$

[4]A tropicalization morphism for the moduli space of curves has also been constructed in the category of logarithmic geometry in [CCUW20].

where $\Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n})$ is the dual complex associated with the normal crossings compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$. Recall that the dual complex of a normal crossings compactification $M \subset \overline{M}$ is a cell complex encoding the irreducible components of the boundary divisor and the way they intersect. More precisely, if $D = \overline{M} \setminus M$ is a normal crossings divisor, the boundary complex $\Delta(D)$ associated with the embedding of D in M is a Δ-complex whose geometric realization has a vertex for each irreducible component D_i of D, an edge for each irreducible component of the intersection of two D_i's, and so on. For instance, the dual complex of the union of three lines intersecting transversally in the projective plane is a triangle.

5 Applications to the topology of the moduli space of curves

As mentioned in the introduction, many aspects of the topology of the moduli space of curves remain unknown, despite the enormous efforts that algebraic geometers and mathematicians, in general, have been making to study this fascinating space. For instance, as mentioned in the Introduction, the rational cohomology ring of $\mathcal{M}_{g,n}$, $H^*(\mathcal{M}_{g,n}, \mathbb{Q})$, is completely known only for very small values of g and n.

If we see the moduli space of stable curves in the category of algebraic stacks, it turns out to be smooth. One can therefore apply Deligne's theory of weights to endow $H^*(\mathcal{M}_{g,n}, \mathbb{Q})$ with a mixed Hodge structure whose graded pieces $\mathrm{Gr}_j^W H^k(\mathcal{M}_{g,n}, \mathbb{Q})$ range between k and $\min\{2k, 2d\}$, where $d = \dim(\mathcal{M}_{g,n}) = 3g - 3 + n$. Since the graded pieces vanish for $j > 2d$, we call $\mathrm{Gr}_{2d}^W H^k(\mathcal{M}_{g,n}, \mathbb{Q})$ the top-weight rational cohomology of $\mathcal{M}_{g,n}$.

Moreover, as we discussed above, $\mathcal{M}_{g,n}$ is endowed with a normal crossings compactification: the moduli space of stable Deligne-Mumford curves $\overline{\mathcal{M}}_{g,n}$. Therefore, again by Deligne's theory of weights, the following identification holds:

$$\mathrm{Gr}_{2d}^W H^{2d-k}(\mathcal{M}_{g,n}, \mathbb{Q}) \cong H_{k-1}(\Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}), \mathbb{Q}), \tag{4}$$

where $\Delta(\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n})$ is the boundary complex of the normal crossings compactification introduced in (3).

Therefore, the identification in (4) allows us to compute part of the cohomology of $\mathcal{M}_{g,n}$ via the homology of its boundary complex, which is an object of combinatorial nature, and therefore expected to be simpler. However, as it is often the case, the combinatorial problem turned out to be quite difficult itself, so it remained unknown until quite recently, via the insight introduced by the identification of the boundary complex with the link of the moduli space of tropical curves (3). Indeed, via this identification, the combinatorial object encoding the geometry of the boundary divisor was given a modular interpretation via tropical curves, which gave a whole new insight into the problem.

The remarkable contribution of Chan, Galatius and Payne in [CGP21] consisted in identifying a graph complex, the so-called Kontsevich commutative graph complex, which computes the cohomology of $L(M_g^{\mathrm{trop}})$. Then, taking advantage of (3) and (4) and of previous results by Willwacher concerning the asymptotic behaviour of such complex, the authors were able to prove the following:

$$\dim H^{4g-6}(M_g, \mathbb{Q}) > \beta^g,$$

where β is a certain constant bigger than 1.

Notice that this result is contrary to expectations. In fact, the virtual cohomological dimension of $\mathcal{M}_g$ is equal to $4g-5$ and the cohomology groups $H^{4g-4-k}(\mathcal{M}_g, \mathbb{Q})$ were expected to be non-zero for only finitely many values of g by previous conjectures by Kontsevich and Church.

5.1 Further achievements

The remarkable results obtained by Chan, Galatius and Payne motivated many similar applications. Indeed, the fact that tropical moduli spaces can give a (at least partial) modular interpretation to the combinatorial data used to compactify algebro-geometric moduli spaces has been proved to hold in other remarkable situations, as the moduli space of spin curves in [CMP20] the moduli space of admissible covers in [CMR16], Universal Jacobians in [AP20] and [MMUV22], etc.

Even if an intrinsic tropicalization result is not known in the case of the moduli space of abelian varieties of given dimension, A_g, the machinery of Chan, Galatius and Payne was nevertheless used to compute the top weight rational cohomology of A_g in [BBCMMW24] for small values of g. Similarly, it is also being applied to a number of different moduli spaces, and the community expects that several interesting results will be proved by applying closely related strategies in the near future.

There are other remarkable applications of tropical methods in approaching problems in classical algebraic geometry, as for instance the use of tropical linear series in proving results concerned with linear series in algebraic curves, by applying Baker's specialization lemma. This line of research has been extremely successful in the last few years, starting with the tropical proof of the Brill-Noether theorem in [CDPR12] and heading to most recent achievements as the notorious computation of the Kodaira dimension of M_{22} and M_{23} in [FJP25]. We invite the reader to look at [BJ16] for a very nice survey on such applications.

Referências

[ACP15] Abramovich, D. and Caporaso, L. and Payne, S.; *The tropicalization of the moduli space of curves.* Ann. Sci. Éc. Norm. Supér. (4), 48.4 (2015), pp. 765–809.

[AP20] Abreu, A. and Pacini, M.; *The universal tropical Jacobian and the skeleton of the Esteves' universal Jacobian.* Proc. Lond. Math. Soc. (3) 120 (2020), no. 3, 328–369.

[AMRT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai; *Smooth compactification of locally symmetric varieties.* Math. Sci. Press, Brookline, Mass., 1975. Lie Groups: History, Frontiers and Applications, Vol. IV.

[BJ16] Baker, M. and Jensen, D.; *Degeneration of linear series from the tropical point of view and applications.* Simons Symp. Springer, [Cham], 2016, 365–433.

[BBCMMW24] Brandt, M. and Bruce, J. and Chan, M. and Melo, M and Moreland, G. and Wolfe, C.; *On the top-weight rational cohomology of* $\mathcal{A}_g$, Geom. Topol. 28 (2024), no. 2, 497–538.

[BMV11] Brannetti, S. and Melo, M. and Viviani, F., *On the tropical Torelli map,* Advances in Mathematics, Adv. Math., (2011) no. 3, 226, 2546–2586.

[Cap12] Caporaso, L., *Geometry of tropical moduli spaces and linkage of graphs*, (2012), no. 3, 119, 579–598.

[Cap13] Caporaso, L.; *Algebraic and tropical curves: comparing their moduli spaces.* Handbook of moduli. Vol. I, 119?160. Adv. Lect. Math. (ALM), 24, International Press, Somerville, MA, 2013.

[CMP20] Caporaso, L. and Melo, M. and Pacini, M.; *Tropicalizing the moduli space of spin curves.* Selecta Math. (N.S.) 26 (2020), no. 1, Paper No. 16, 44 pp.

[CCUW20] Cavalieri, R. and Chan, M. and Ulirsch, M. and Wise, J., *A moduli stack of tropical curves,* Forum of Mathematics, Sigma, Cambridge University Press, (2020), 8.

[CMR16] Cavalieri, R. and Markwig, H. and Ranganathan, D.; *Tropicalizing the space of admissible covers.* Math. Ann. 364 (2016), no. 3-4, 1275–1313.

[CGP21] Chan, M. and Galatius, S. and Payne, S.; *Tropical curves, graph complexes, and top weight cohomology of* $\mathcal{M}_g$, J. Amer. Math. Soc. (34), (2021), no. 2, 565–594.

[CDPR12] Cools, F. and Draisma, J. and Payne, S. and Robeva, E.; *A tropical proof of the Brill-Noether theorem.* Adv. Math. 230 (2012), no. 2, 759–776.

[CV86] Culler, M. and Vogtmann, K., *Moduli of graphs and automorphisms of free groups*, Invent. Math., (1986), no. 1, 84, 91–119.

[DM69] Deligne, P.; Mumford, D.; *The irreducibility of the space of curves of given genus.* Inst. Hautes Études Sci. Publ. Math. No. 36 (1969), 75–109.

[FJP25] Farkas, G. and Jensen, D. Payne, S.; *The Kodaira dimension of* $\mathcal{M}_{22}$ *and* $\mathcal{M}_{23}$. Camb. J. Math. 13 (2025), no. 3, 431–607.

[KKMSD73] Kempf, G. and Knudsen, F. F and Mumford, D. and Saint-Donat. B.; *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973.

[Knu83] Knudsen, F. F.; *The projectivity of the moduli space of stable curves. II. The stacks* $M_{g,n}$. Math. Scand. 52 (1983), no. 2, 161–199.

[MW07] Madsen, I. and Weiss, M; *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2), (2007), no. 3, 165, 843–941.

[MMUV22] Melo, M. and Molcho, S. and Ulirsch, M. and Viviani, F.; *Tropicalization of the universal Jacobian.* Épijournal Géom. Algébrique 6 (2022), Art. 15, 51 pp.

[MZ07] Mikhalkin, G. and Zharkov, I.; *Tropical curves, their Jacobians and theta functions*, Curves and abelian varieties. Proceedings of the international conference, Athens, GA, USA, March 30–April 2, 2007, Providence, RI: American Mathematical Society (AMS) (2008), 203–230.

A survey on the Virasoro constraints in moduli spaces of sheaves and quiver representations

Miguel Moreira
MIT, Department of Mathematics
182 Memorial Dr
Cambridge, MA 02139, USA
e-mail: miguel@mit.edu

Resumo:

As restrições de Virasoro para os invariantes de Gromov-Witten são um tópico que tem estado ligado desde o início ao desenvolvimento da teoria de Gromov-Witten. Neste artigo vamos sintetizar alguns dos desenvolvimentos recentes no estudo das restrições de Virasoro no mundo paralelo das teorias de contagem de feixes e representações de quivers. Damos uma visão histórica do tópico e explicaremos os resultados precisos no contexto de espaços moduli de representações de quivers. Vamos ainda discutir a vertex algebra que controla fenómenos de wall-crossing, que é a principal ferramenta na prova dos resultados existentes.

Abstract

Virasoro constraints for Gromov-Witten invariants is a topic that has been tied together with the development of Gromov-Witten theory. In this paper we survey the recent developments in analogous Virasoro constraints in the parallel world of sheaf and quiver representations counting theories. We give a historical overview of the subject and explain the precise statements in the setting of moduli spaces of representations of a quiver. We discuss the wall-crossing vertex algebra, which is the main tool in the existing proofs.

palavras-chave: Geometria enumerativa, espaços moduli, restrições de Virasoro.

keywords: Enumerative geometry, moduli spaces, Virasoro constraints.

1 Introduction

1.1 Enumerative geometry and moduli spaces

Enumerative geometry is a very classical subject and it has been greatly influential in the development of Algebraic Geometry since the 19th century. In its classical form, the goal is to count the number of geometric objects with given properties. One beautiful theorem from the early days of enumerative geometry is the following:

Theorem 1.1 (Cayley-Salmon)**.** Let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface. Then X contains exactly 27 lines.

In the early 90s, the field was revolutionized by the introduction of ideas coming from string theory; physicists Candelas-Ossa-Green-Parks [CdlOGP] were able to predict the answer to a classical problem (counting rational curves on a quintic 3-fold) using the idea of mirror symmetry in string theory. Their prediction was then proved mathematically by Givental [Giv1]. Witten's conjecture [Wit], which we will discuss in Section 2, was another striking mathematical conjecture inspired by physical ideas, and it is the beginning of the story of Virasoro constraints in enumerative geometry.

Along with the new physical input, mathematicians developed powerful tools to define and study new enumerative invariants. In the modern approach to enumerative geometry, moduli spaces play a crucial role. A moduli space M is simply a space that parametrizes some sort of geometric objects. Enumerative invariants are typically numbers that one can extract from a moduli space. This may be achieved using intersection theory, by integrating naturally defined cohomology[1] classes $D \in H^\bullet(M)$; it is often the case that the moduli space is not smooth but we can still integrate using a virtual fundamental class $[M]^{\mathrm{vir}} \in H_\bullet(M)$, constructed by Behrend–Fantechi [BF], and define numbers

$$\int_{[M]^{\mathrm{vir}}} D \coloneqq \deg\left(D \cap [M]^{\mathrm{vir}}\right) \in \mathbb{Q}\,.$$

The Virasoro constraints are explicit and universal relations among all these numbers when we vary D.

Example 1.2. Let $M = \mathsf{Gr}(\mathbb{C}^4, 2)$ be the Grassmannian, which parametrizes 2-dimensional subspaces of $\mathbb{C}^4$ or, equivalently, lines on $\mathbb{P}^3$. The Grassmannian has a rank 2 tautological bundle $\mathcal{F} \subseteq \mathbb{C}^4 \otimes \mathcal{O}_M$ whose fiber over a point is identified with the corresponding subspace of $\mathbb{C}^4$. Given a cubic surface $X \subseteq \mathbb{P}^3$, the loci of points in M corresponding to lines contained in X is the vanishing loci of a section on the rank 4 bundle $\mathrm{Sym}^3(\mathcal{F}^\vee)$. Hence, for a generic X, we expect that the number of lines in X is

$$\int_{\mathsf{Gr}(\mathbb{C}^4,2)} c_4(\mathrm{Sym}^3(\mathcal{F}^\vee)) = 27\,.$$

[1] Throughout this paper, cohomology is always with $\mathbb{Q}$ coefficients, i.e. $H^\bullet(M) = H^\bullet(M, \mathbb{Q})$.

The number can be calculated via Schubert calculus. The special feature of Theorem 1.1 is that it holds for every smooth X, and not only for a generic one.

1.2 Outline of the paper

This paper is a survey of the recent advances in the series of papers [MOOP, Mor, vB, BLM, Boj, LM] in conjecturing and proving Virasoro constraints for moduli spaces of sheaves and representations of quivers. In Section 2 we will discuss the origin of Virasoro constraints in enumerative geometry, which is Witten's conjecture and its generalization to Gromov-Witten theory. In Section 3 we start discussing the more recent developments in sheaf theory, the relation with Gromov-Witten invariants via the Gromov-Witten/Donaldson-Thomas correspondence, and we list the families of moduli spaces for which the Virasoro constraints have been proven.

In the last two sections we specialize the discussion to the case of moduli spaces of representations of quivers (without relations). These are a simpler analogue of moduli spaces of sheaves, and most of the features of Virasoro constraints can be explained in this setting. In Section 4 we give a precise formulation of the constraints proven in [Boj, LM].

Finally, Section 5 is about wall-crossing and Joyce's vertex algebra [Joy2, Joy1, GJT]. It was discovered in [BLM] that Joyce's vertex algebra is closely related to the Virasoro constraints. The wall-crossing formulas proven by Joyce are the main tool in proving most of the known cases of Virasoro constraints in sheaf/quiver theories. We explain the idea of the proof in the quiver setting.

2 Witten's conjecture

In 1990, Witten [Wit] made a striking prediction about integration on the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. His idea came from two dimensional quantum gravity, which roughly can be thought as a theory of integration over the (infinitely dimensional) space of Riemannian metrics on a surface. He argued that this physical theory could be modeled mathematically in two ways. On one hand, it could be modeled by approaching the space of Riemannian metrics by a space of triangulations (a triangulation gives a metric which is flat in the interior of the triangles and singular along the edges); this idea establishes a connection to matrix models. These matrix models were known at the time to produce a solution to the Korteweg–de Vries (KdV) hierarchy or, equivalently, to the Virasoro constraints.

On the other hand, supersymmetry indicates that the integral over the space of all metrics should localize to an integral over the space of conformal metrics, which is finite dimensional. Mathematically, this can be made precise by considering integration over the Deligne-Mumford moduli space of stable curves

$$\overline{\mathcal{M}}_{g,n} = \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \begin{array}{l} C \text{ nodal curve of genus } g, \\ p_1, \ldots, p_n \in C^{\text{smooth}}, \, \#\text{Aut}(C, p_1, \ldots, p_n) < \infty \end{array} \right\}.$$

These are smooth and projective Deligne-Mumford stacks (or orbifolds). Over the moduli of stable curves there are line bundles $\mathbb{L}_1, \ldots, \mathbb{L}_n$ defined by

$$(\mathbb{L}_i)_{(C,p_1,\ldots,p_n)} = T_{p_i}^{\vee} C\,.$$

The first Chern class of these line bundles $\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$ are called the psi classes. We can integrate them to define numbers:

$$\langle \tau_{k_1} \ldots \tau_{k_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \ldots \psi_n^{k_n} \in \mathbb{Q}\,. \tag{1}$$

These are the Gromov-Witten invariants of a point. Now, Witten's prediction was that if one organizes these numbers correctly we should also get a solution to the KdV hierarchy, or to the Virasoro constraints! Define the generating function

$$F = \sum_{g,n\geq 0} u^{2g-2} \sum_{k_1,\ldots,k_n\geq 0} \frac{t_{k_1}\ldots t_{k_n}}{n!} \langle \tau_{k_1} \ldots \tau_{k_n} \rangle_g$$

and let $Z = \exp(F)$. Both F and Z are generating series in the formal variables u and $t_1, t_2, \ldots$; Z is called the partition function. For $n \geq -1$, let L_n be the differential operator

$$L_n = \frac{1}{4} \sum_{k+l=2n} \frac{\partial^2}{\partial T_k \partial T_l} + \frac{1}{2} \sum_{k\geq 0} (2k+1) T_{2k+1} \frac{\partial}{\partial T_{2k+2n+1}} - \frac{1}{2u^2} \frac{\partial}{\partial T_{2n+3}} + \frac{\delta_{n,-1} T_1^2}{4} + \frac{\delta_{n,0}}{16}$$

where $T_{2k+1} = t_k/(2k+1)!!$.

Remark 2.1. These operators satisfy the Lie bracket relation

$$[L_n, L_m] = (n-m) L_{n+m} \text{ for } n, m \geq -1\,.$$

The Virasoro Lie algebra Vir is the Lie algebra spanned by $\{L_n\}_{n\in\mathbb{Z}}$ and a central element C with Lie bracket given by

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m=0}\frac{n^3-n}{12}C\,.$$

Thus, the operators L_n define a representation of the Lie subalgebra $\mathrm{Vir}_{\geq -1} \subseteq \mathrm{Vir}$ spanned by $\{L_n\}_{n\geq -1}$. It was observed in [Get] that one may extend this to a full representation of the entire Virasoro algebra with central charge 1 (i.e. C acting as the identity). However, the Virasoro operators L_n with $n < -1$ do not impose new constraints.

Witten's conjecture was proven by Kontsevich, and it says the following:

Theorem 2.2 ([Kon])**.** For every $n \geq -1$ one has

$$L_n(Z) = 0\,.$$

2.1 The string equation

The case $n = -1$ of Theorem 2.2 is known as the string equation and it was proved by Witten in [Wit]. By taking the $t_{k_1}\dots t_{k_n}$ coefficient, it is equivalent to the statement that

$$\left\langle \tau_0\tau_{k_1}\dots\tau_{k_n}\right\rangle_g = \sum_{i=1}^{n}\left\langle \tau_{k_1}\dots\tau_{k_i-1}\dots\tau_{k_n}\right\rangle_g \tag{2}$$

for $2g-2+n>0$, together with the exceptional case

$$\left\langle \tau_0\tau_0\tau_0\right\rangle_0 = 1\,.$$

This equation is much easier to prove than the general case of Witten's conjecture, and it follows from an analysis of the geometry of the map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ which forgets one of the marked points and stabilizes the resulting curve.

To make a connection with how we will later write the Virasoro constraints for sheaves and quiver representations, let us write down the string equation in the language of the descendent algebra. Define the descendent algebra $\mathbb{D}^{\mathrm{Wit}}$ to be the polynomial algebra in infinitely many variables

$$\mathbb{D}^{\mathrm{Wit}} = \mathbb{Q}[\tau_0, \tau_1, \tau_2, \dots]\,.$$

For each genus g, we have a linear map $\langle \cdot \rangle_g \colon \mathbb{D}^{\mathrm{Wit}} \to \mathbb{Q}$ defined on monomials $\tau_{k_1} \dots \tau_{k_n}$ by integration over $\overline{M}_{g,n}$, cf. (1). Then the string equation (for $g > 0$, so that we can ignore the exceptional case) can be written as

$$\big\langle (R_{-1} - \tau_0) D \big\rangle_g = 0 \text{ for every } D \in \mathbb{D}^{\mathrm{Wit}},\ g > 0\,.$$

It is possible to formulate Witten's conjecture in the same spirit for every n, see [MOOP, Proposition 5].

2.2 Generalization to Gromov-Witten theory

In 1997, Eguchi-Hori-Xiong [EHX] proposed a generalization of Witten's conjecture to the Gromov-Witten theory of an arbitrary (smooth, projective) target variety X. In Gromov-Witten theory, one considers the moduli spaces of stable maps

$$\overline{\mathcal{M}}_{g,n}(X, \beta)\,.$$

Its points parametrize (marked) curves together with a map $f \colon C \to X$ in curve class β, i.e. with $f_*[C] = \beta$ in $H_2(X; \mathbb{Z})$. Gromov-Witten invariants are obtained by integrating psi classes together with pull-backs of cohomology classes from X via the evaluation maps $\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$. The moduli spaces of stable maps are often not smooth, so the integration needs to be done in a virtual sense. The Gromov-Witten invariants are

$$\Big\langle \tau_{k_1}(\gamma_1) \dots \tau_{k_n}(\gamma_n) \Big\rangle_{g,\beta}^{\mathrm{GW}} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \in \mathbb{Q} \qquad (3)$$

where $k_i \geq 0$ and $\gamma_i \in H^\bullet(X)$. When X is a point we recover the integrals over the moduli of stable curves that we discussed before. We refer the reader to Chapters 25 and 26 in [HKK+] for a gentle and more detailed introduction to Gromov-Witten invariants.

After one organizes the Gromov-Witten invariants into a generating series Z^X, which is called the partition function, the general Virasoro constraints are the conjecture that for $n \geq -1$

$$L_n^X(Z^X) = 0$$

where L_n^X are some explicit differential operators. The reader can find the precise form of the operators in [Get].

There are two large families of varieties X for which the Gromov-Witten Virasoro conjecture is known: X with semisimple quantum cohomology (this includes any toric variety and Grassmannians, for instance) [Giv2, Tel] and curves [OP]. Apart from those, it is a widely open problem.

3 Virasoro constraints in sheaf and quiver theories: overview

Recently, there has been a proposal of Virasoro constraints in a new setting: moduli spaces of sheaves and moduli spaces of quiver representations. For now, we want to explain how the constraints were first discovered in this setting, give a rough idea of what they look like, and list the cases in which the constraints are proven.

3.1 Gromov-Witten/Donaldson-Thomas correspondence

Maulik, Nekrasov, Okounkov and Pandharipande [MNOP1, MNOP2] conjectured that the Gromov-Witten invariants of a 3-fold X should be related to the Donaldson-Thomas invariants of the same 3-fold; we refer to this as the GW/DT correspondence. Donaldson-Thomas invariants are a different kind of enumerative invariants, defined by integration over moduli spaces of ideal sheaves, instead of stable maps. Given a curve $C \subseteq X$, we denote by I_C its ideal sheaf. For each $m \in \mathbb{Z}$ and $\beta \in H_2(X;\mathbb{Z})$ we have a moduli space

$$I_m(X,\beta) = \{I_C : C \subseteq X \text{ a 1-dimensional subscheme}, [C] = \beta, \chi(\mathcal{O}_C) = m\}\,.$$

There is a universal curve $\mathcal{C} \subseteq I_m(X,\beta) \times X$ and a universal ideal sheaf $I_{\mathcal{C}}$ on $I_m(X,\beta) \times X$. The universal ideal sheaf may be used to construct natural cohomology classes

$$\mathrm{ch}_k(\gamma) = \pi_{I*}(\mathrm{ch}_k(I_{\mathcal{C}})\pi_X^*\gamma) \in H^\bullet(I_m(X,\beta))$$

where $k \geq 0$ is an integer, $\mathrm{ch}_k(I_{\mathcal{C}}) \in H^\bullet(I_m(X,\beta) \times X)$ denotes the k-th Chern character of the sheaf $I_{\mathcal{C}}$, $\gamma \in H^\bullet(X)$ and π_I, π_X are the projections of $I_m(X,\beta) \times X$ onto $I_m(X,\beta)$ and X, respectively. The moduli spaces $I_m(X,\beta)$ have a virtual fundamental class and the Donaldson-Thomas invariants are defined to be

$$\int_{[I_m(X,\beta)]^{\mathrm{vir}}} \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_n}(\gamma_n) \in \mathbb{Q}\,. \tag{4}$$

The correspondence proposed in [MNOP1, MNOP2] states that there is a universal way to determine all the Gromov-Witten invariants (3) from the Donaldson-Thomas invariants (4), and vice-versa. The conjectured correspondence is quite complicated. In involves taking a fairly strange change of variables $q = e^{-iu}$ to relate two generating series and it involves a complicated transformation to convert the Gromov-Witten descendents $\tau_{k_1}(\gamma_1)\dots\tau_{k_n}(\gamma_n)$ into Donaldson-Thomas descendents $\mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_n}(\gamma_n)$.

In light of this, there is a very natural question: can the Gromov-Witten Virasoro constraints be moved to the Donaldson-Thomas side? What do they say? Oblomkov, Okounkov and Pandharipande were able to guess a precise conjecture that first appeared[2] in [Pan].

The understanding of the GW/DT correspondence has improved a lot since it was first proposed, and this allowed to partially establish the connection between Virasoro constraints for Gromov-Witten and for Donaldson-Thomas invariants.

Theorem 3.1 ([MOOP, Theorem 5])**.** Suppose that the GW/DT correspondence holds. Then the Gromov-Witten Virasoro constraints in the stationary regime[3] and the Donaldson-Thomas constraints in the stationary regime are equivalent.[4]

A consequence of this are the stationary Virasoro constraints for Donaldson-Thomas invariants of toric 3-folds, since both the GW/DT correspondence and the Virasoro on the Gromov-Witten side are known.

3.2 Universal constraints

It turns out that Virasoro constraints are present (or at least expected to be) in a much more general family of moduli spaces, which includes the moduli of ideal sheaves used to define Donaldson-Thomas invariants. It was noted in [MOOP, Mor] that the Virasoro constraints for Pandharipande-Thomas invariants (a closely related cousin to Donaldson-Thomas invariants) imply similar constraints for Hilbert schemes of points on surfaces. Work of van Bree [vB] conjectured and gave strong numerical evidence that moduli spaces of stable torsion-free sheaves on surfaces should also be constrained.

This lead to the general conjectures appearing in [BLM], which apply to a large family of moduli spaces. It applies to moduli spaces of sheaves (on curves, surfaces, ideal sheaves on 3-folds, etc.) but also to many variations:

[2]The author explains that they made this conjecture roughly 10 years before the paper, so around 2007.

[3]Stationary regime means essentially that we only consider invariants (3) and (4) with $\gamma_i \in H^{\geq 2}(X)$. Away from the stationary regime, the GW/DT correspondence is poorly understood.

[4]The results in [MOOP] are stated for Pandharipande-Thomas invariants rather than Donaldson-Thomas invariants, but in the stationary case these are essentially the same by [OOP, Theorem 22].

sheaves with or without fixed determinant, moduli spaces of pairs, moduli spaces of quiver representations, moduli spaces of sheaves with Oh-Thomas virtual fundamental classes, moduli spaces of Bridgeland stable objects, and moduli spaces of parabolic bundles on curves. We remark that this general setting does not apply to Gromov-Witten theory, in which Virasoro constraints have a fundamentally different flavour.

In the next section we will explain the precise conjecture in the setting of quivers. For now, we shall only describe its general shape. Given a moduli space M of sheaves or quiver representations, there are certain natural cohomology classes we can construct on M; for example in the Donaldson-Thomas case those were $\mathrm{ch}_k(\gamma)$. We write this in terms of a descendent algebra $\mathbb{D}$ which admits a realization map $\mathbb{D} \to H^\bullet(M)$. If M has a virtual fundamental class, we get numerical invariants $\int_{[M]^{\mathrm{vir}}} D \in \mathbb{Q}$ by realizing D and integrating it. The Virasoro constraints are universal and explicit linear relations among such numbers. We always describe them analogously to how we wrote the string equation in Section 2.1. We define some operator (or operators) $\mathsf{L}\colon \mathbb{D} \to \mathbb{D}$ and formulate the constraints as

$$\int_{[M]^{\mathrm{vir}}} \mathsf{L}(D) = 0 \quad \text{for every } D \in \mathbb{D}\,.$$

The operators L will be defined in terms of some canonical representation $\{\mathsf{L}_n\}_{n\geq -1}$ of Vir_{-1} on $\mathbb{D}$. Depending on the context, L might be precisely L_n, but it might also be a small perturbation or a combination of all the L_n operators.

Remark 3.2. The sheaf/quiver Virasoro constraints relate invariants defined by integration on a single moduli space. The same is not true for Gromov-Witten theory: for instance in the string equation (2) the left hand side is defined by integration in $\overline{\mathcal{M}}_{g,n+1}$ and the right hand side by integration in $\overline{\mathcal{M}}_{g,n}$. This is a hint that the sheaf/quiver Virasoro constraints are in some sense simpler.

We list now the cases in which the Virasoro constraints have been proven in the past few years:

1. Donaldson-Thomas and Pandharipande-Thomas invariants on toric 3-fold X in the stationary regime [MOOP].
2. Hilbert scheme of points on a surface S with $h^1(S) = 0$ [Mor].[5]

[5] Joyce announced that he is able to remove the assumption $h^1(S) = 0$ in upcoming work [Joy3].

3. Moduli of stable bundles on a curve [BLM].

4. Moduli of Bradlow pairs on a curve [BLM]. This was conjectured by van Bree in [vB].

5. Moduli of torsion free stable sheaves on a surface S with $h^{0,1}(S) = h^{0,2}(S) = 0$ [BLM].

6. Moduli of Bradlow pairs on a surface S with $h^{0,1}(S) = h^{0,2}(S) = 0$ [BLM]. In particular they hold for the nested Hilbert schemes $S_\beta^{[0,m]}$ on such surfaces.

7. Moduli of 1-dimensional sheaves on a surface S with $h^{0,1}(S) = h^{0,2}(S) = 0$ [BLM] provided the technical condition [BLM, Assumption 5.7] holds. When $S = \mathbb{P}^2, S = \mathbb{P}^1 \times \mathbb{P}^1$ or $S = \mathsf{Bl}_{\mathsf{pt}}(\mathbb{P}^2)$ this is shown unconditionally in [LM].

8. Punctual Quot schemes on a curve or on a surface with $h^{0,2}(S) = 0$ [BLM] holds.

9. Moduli spaces of quiver representations, possibly with relations [Boj, LM].

In Section 5 we will sketch the proof of (9). The ideas explained there are the main tool used in the proof of results (3) – (9).

4 Virasoro constraints for moduli of quiver representations

In this section we define moduli spaces parametrizing representations of a quiver and give a precise formulation of the Virasoro constraints in this setting.

4.1 Moduli spaces of quiver representations

A quiver is a directed graph, which we write as a 4-tuple $Q = (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are the sets of vertices and arrows, respectively, and $s, t\colon Q_1 \to Q_0$ are the functions assigning to an arrow its source and target, respectively.

A representation V of a quiver is an assignement of a vector space V_v to each node $v \in Q_0$ and linear maps $f_e\colon V_{s(e)} \to V_{t(e)}$ to each arrow $e \in Q_1$.

Given a representation V, we say that $(\dim(V_v))_{v\in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$ is the dimension vector of V. A morphism bewteen representations V and W consists in a collection of maps $V_v \to W_v$ for each $v \in Q_0$ that makes all the maps assign to the edges commute. Thus, representations of a quiver Q form an abelian category Rep_Q.

The construction of well-behaved moduli spaces of quiver representations requires the choice of a stability condition. Given a choice of weights $\theta = (\theta_v)_{v\in Q_0} \in \mathbb{R}^{Q_0}$ one defines the slope of a representation by

$$\mu_\theta(V) = \frac{\sum_{v\in Q_0} \theta_v \dim(V_v)}{\sum_{v\in Q_0} \dim(V_v)} \,.$$

Note that this only depends on the dimension vector of V, so we will also write $\mu_\theta(d)$ for $d \in \mathbb{Z}^{Q_0}$ with the obvious meaning.

Definition 4.1. A representation of a quiver V is said to be θ-semistable if

$$\mu_\theta(W) \leq \mu_\theta(V) \text{ for every } 0 \neq W \subsetneq V \,.$$

It is said to be θ-stable if the inequality above is strict for every $W \subsetneq V$.

Given a dimension vector $d \in \mathbb{Z}_{\geq 0}^{Q_0}$ let

$$Z_d^{\theta-\mathrm{st}} \subseteq Z_d^{\theta-\mathrm{ss}} \subseteq \prod_{e\in Q_1} \mathrm{Hom}\big(\mathbb{C}^{d_{s(e)}}, \mathbb{C}^{d_{t(e)}}\big)$$

be the subsets of θ-stable and θ-semistable representations. Note that some of these representations are isomorphic, so we can construct a moduli space parametrizing (S-equivalence classes of) θ-semistable representations by taking the quotient (in the sense of geometric invariant theory [MFK])

$$M_d^{\theta-\mathrm{ss}} = Z_d^{\theta-\mathrm{ss}} /\!\!/ G_d$$

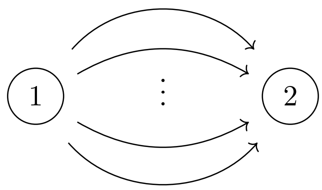
by the natural action of

$$G_d = \prod_{v\in Q_0} \mathrm{Hom}(\mathbb{C}^{d_v}, \mathbb{C}^{d_v}) \,.$$

We refer to [Kin] for further details on the construction. Similarly we may define $M_d^{\theta-\mathrm{st}} \subseteq M_d^{\theta-\mathrm{ss}}$; when $M_d^{\theta-\mathrm{ss}} = M_d^{\theta-\mathrm{st}}$ we will sometimes abbreviate to M_d^θ. When the quiver Q is acyclic (i.e. it does not contain oriented cycles), the moduli spaces $M_d^{\theta-\mathrm{ss}}$ are projective. When θ and d are such that there are no strictly semistable objects (i.e. $Z_d^{\theta-\mathrm{st}} = Z_d^{\theta-\mathrm{ss}}$) the moduli space M_d^θ is smooth of dimension

$$1 + \sum_{e\in Q_1} d_{s(e)} d_{t(e)} - \sum_{v\in Q_0} d_v^2 \,.$$

Example 4.2. Let K_N be the Kronecker quiver, with two vertices $\{1, 2\}$ and N arrows from 1 to 2:



Let $\theta = (\theta_1, \theta_2)$ with $\theta_1 > \theta_2$. Then the moduli of quiver representations

$$M^{\theta}_{(k,1)}(K_N) \simeq \mathsf{Gr}(\mathbb{C}^N, k)\,,$$

is the Grassmannian parametrizing k dimensional subspaces of $\mathbb{C}^N$.

Example 4.3. Let Q be an acyclic quiver. Then it is possible to choose a stability condition θ which is increasing along the edges, meaning that $\theta_{t(e)} > \theta_{s(e)}$ for any $e \in Q_1$. As shown in [GJT, Proposition 5.6], for such stability conditions, $M_d^{\theta-\mathrm{st}}$ is a point when there is a $v \in Q_0$ such that $d_v = 1$ and $d_w = 0$ for $w \neq v$; otherwise, $M_d^{\theta-\mathrm{st}}$ is the empty set.

4.2 Descendent algebra

Let $M = M_d^{\theta}$ be a moduli space of θ-stable quiver representations on Q and assume that $M_d^{\theta\text{-st}} = M_d^{\theta\text{-ss}}$. We have a universal representation $\mathcal{V}$ on M, which consists of a collection of vector bundles $\mathcal{F}_v$ of rank d_v for each $v \in Q_0$ and maps $\varphi_e \colon \mathcal{V}_{s(e)} \to \mathcal{V}_{t(e)}$ for each $e \in Q_1$. We can produce cohomology classes in $H^\bullet(M)$ by taking Chern classes of the bundles $\mathcal{V}_v$. This motivates the definition of the descendent algebra.

Definition 4.4. Let Q be a quiver. The descendent algebra of Q, denoted by $\mathbb{D}^Q$, is the free commutative $\mathbb{Q}$-algebra generated by symbols

$$\left\{ \mathrm{ch}_k(v) \,\middle|\, k \in \mathbb{Z}_{\geq 0},\ v \in Q_0 \right\}.$$

The universal representation $\mathcal{V}$ defines a realization homomorphism

$$\xi_{\mathcal{V}} \colon \mathbb{D}^Q \to H^\bullet(M)$$

defined on generators by

$$\mathrm{ch}_k(v) \mapsto \mathrm{ch}_k(\mathcal{V}_v)\,.$$

An important subtlety is that the universal representation $\mathcal{V}$ is not unique. If L is any line bundle on M, then $(\mathcal{V}_v \otimes L)_{v\in Q_0}$ defines a new universal representation. This issue can be addressed in two ways: either we fix a choice of universal representation in some way, or we consider the weight 0 descendent subalgebra on which $\xi_{\mathcal{V}}$ does not depend on the choice of $\mathcal{V}$. This subalgebra is defined concretely as follows:

$$\mathbb{D}^Q_{\mathrm{wt}_0} = \ker\left(\mathsf{R}_{-1} \colon \mathbb{D}^Q \to \mathbb{D}^Q\right) \subseteq \mathbb{D}^Q$$

where R_{-1} is a derivation that we define later in Definition 4.5. This subalgebra has the property that $\xi_{\mathcal{V}}(D)$ does not depend on $\mathcal{V}$ for $D \in \mathbb{D}^Q_{\mathrm{wt}_0}$. Hence we get a well-defined homomorphism

$$\xi \colon \mathbb{D}^Q_{\mathrm{wt}_0} \to H^\bullet(M)\,.$$

4.3 Virasoro constraints

To formulate the constraints we define a representation of Vir_{-1} in $\mathbb{D}^Q$ as follows:

Definition 4.5. Define the Virasoro operators $\{\mathsf{L}_n \,|\, n \geq -1\}$ on the descendent algebra $\mathbb{D}^Q$ as a sum of two operators $\mathsf{L}_n = \mathsf{R}_n + \mathsf{T}_n$. First, R_n is a derivation operator such that

$$\mathsf{R}_n(\mathrm{ch}_k(v)) = k(k+1)\cdots(k+n)\,\mathrm{ch}_{k+n}(v).$$

Second, T_n is a multiplication operator by the element

$$\mathsf{T}_n = \sum_{a+b=n} a!b!\left(\sum_{i\in Q_0}\mathrm{ch}_a(v)\mathrm{ch}_b(v) - \sum_{e\in Q_1}\mathrm{ch}_a(s(e))\mathrm{ch}_b(t(e))\right)\,.$$

We also define the weight 0 Virasoro operator

$$\mathsf{L}_{\mathrm{wt}_0} = \sum_{n\geq -1}\frac{(-1)^n}{(n+1)!}\mathsf{L}_n \circ \mathsf{R}_{-1}^{n+1}\,.$$

It is a nice exercise to check that the commutator of these operators is

$$[\mathsf{L}_n, \mathsf{L}_m] = (m-n)\mathsf{L}_{n+m}\,.$$

This is not quite a Virasoro representation as we defined earlier due to a sign difference, but it is very close; indeed, it means that the dual operators

$\mathsf{L}_n^\vee$ define a representation of Vir_{-1} on $(\mathbb{D}^Q)^\vee$. It also follows from those commutator relations that $\mathsf{R}_{-1} \circ \mathsf{L}_{\mathrm{wt}_0} = 0$; hence, the image of $\mathsf{L}_{\mathrm{wt}_0}$ is contained in $\mathbb{D}^Q_{\mathrm{wt}_0}$, so the realization $\xi(\mathsf{L}_{\mathrm{wt}_0}(D)) \in H^\bullet(M)$ is well defined. We are ready to formulate the Virasoro constraints.

Theorem 4.6 ([Boj, LM])**.** Let Q be an acyclic quiver, $d \in \mathbb{Z}^{Q_0}_{\geq 0}$ a dimension vector and $\theta \in \mathbb{R}^{Q_0}$ a stability condition such that $M_d^{\theta-\mathrm{ss}} = M_d^{\theta-\mathrm{st}}$. Then

$$\int_{M_d^\theta} \xi(\mathsf{L}_{\mathrm{wt}_0}(D)) = 0 \quad \text{for every } D \in \mathbb{D}^Q .$$

When there is some vertex v_0 such that $d_{v_0} = 1$, we may give an equivalent formulation by fixing a normalized universal sheaf.

Theorem 4.7. Let Q be an acyclic quiver, $d \in \mathbb{Z}^{Q_0}_{\geq 0}$ a dimension vector and $\theta \in \mathbb{R}^{Q_0}$ a stability condition such that $M_d^{\theta-\mathrm{ss}} = M_d^{\theta-\mathrm{st}}$. Suppose that there is $v_0 \in Q_0$ such that $d_{v_0} = 1$ and let $\mathcal{F}$ be the (unique) universal representation on M_d^θ such that $\mathcal{V}_{v_0}$ is the trivial line bundle.

Then

$$\int_{M_d^\theta} \xi_{\mathcal{V}}(\mathsf{L}_n(D)) = 0 \quad \text{for every } D \in \mathbb{D}^Q \text{ and } n \geq 0 .$$

Example 4.8. We illustrate how the Virasoro constraints look like for $M = \mathsf{Gr}(\mathbb{C}^4, 2)$, see Examples 1.2 and 4.2. Let $\mathcal{V}$ be the universal representation over $M = M^\theta_{(2,1)}(K_4)$, where K_4 is the Kronecker quiver in Example 4.2, normalized by asking that $\mathcal{V}_2$ is the trivial line bundle. Then $\mathcal{F} = \mathcal{V}_1$ is the tautological vector bundle (recall Example 1.2) and the 4 universal maps are obtained by composing $\mathcal{F} \subseteq \mathbb{C}^4 \otimes \mathcal{O}_{\mathsf{Gr}(\mathbb{C}^4,2)}$ with the 4 projections onto the coordinate axis $\mathbb{C}^4 \otimes \mathcal{O}_{\mathsf{Gr}(\mathbb{C}^4,2)} \to \mathcal{O}_{\mathsf{Gr}(\mathbb{C}^4,2)}$.

For convenience denote $p_k = k!\mathrm{ch}_k(1) \in \mathbb{D}^{K_4}$. We will omit the realization morphism $\xi_{\mathcal{V}}$, so p_k also denotes $k!\mathrm{ch}_k(\mathcal{V}_1) = k!\mathrm{ch}_k(\mathcal{F}) \in H^{2k}(\mathsf{Gr}(\mathbb{C}^4, 2))$. We have

$$\begin{aligned}
0 &= \int_{\mathsf{Gr}(\mathbb{C}^4,2)} \mathsf{L}_1(p_1^3) = 3\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^2 p_2 \\
0 &= \int_{\mathsf{Gr}(\mathbb{C}^4,2)} \mathsf{L}_2(p_1^2) = 2\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1 p_3 + \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^4 \\
0 &= \int_{\mathsf{Gr}(\mathbb{C}^4,2)} \mathsf{L}_1(p_1 p_2) = \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_2^2 + 2\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1 p_3 \\
0 &= \int_{\mathsf{Gr}(\mathbb{C}^4,2)} \mathsf{L}_3(p_1) = \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_4 + 2\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^2 p_2 .
\end{aligned}$$

By further using that $\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^4 = 2$ [EH, Exercise 4.38] we determine all the integrals of descendents using the equations above:

$$\int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^4 = 2 = \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_2^2\,, \quad \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1 p_3 = -1\,, \quad \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_4 = 0 = \int_{\mathsf{Gr}(\mathbb{C}^4,2)} p_1^2 p_2\,.$$

It is shown in [LM] that the Virasoro constraints for the Grassmannian recover all the descendent integrals up to a constant, as illustrated above in the case of $\mathsf{Gr}(\mathbb{C}^4,2)$. The same is not true for other quivers. Indeed, the descendent integrals on M_d^θ will depend heavily on the choice of stability condition θ, but the Virasoro constraints are completely independent of θ!

5 Wall-crossing and the vertex algebra

5.1 Wall-crossing and flips

We have seen that the construction of moduli spaces of representations $M_d^{\theta-\mathrm{ss}}$ depends on a choice of a stability condition $\theta \in \mathbb{R}^{Q_0}$. Wall-crossing is essentially the study of how the moduli spaces and corresponding descendent integrals change when θ changes.

Example 5.1. Consider the Kronecker quiver and Examples 4.2 and 4.3: they say that $M_{(k,1)}^\theta(K_N)$ is the Grassmannian when $\theta_1 > \theta_2$ but empty if $\theta_1 < \theta_2$. There is a drastic change when we cross the wall

$$\{(\theta_1,\theta_2)\colon \theta_1 = \theta_2\} \subseteq \mathbb{R}^{Q_0}$$

in the space of stability conditions. We call the regions $\{\theta\colon \theta_1 > \theta_2\}$ and $\{\theta\colon \theta_1 < \theta_2\}$ chambers. For θ on the wall, μ_θ is constant so every representation is semistable, but only the representations with total dimension 1 are stable.

More generally, the space of stability conditions $\mathbb{R}^{Q_0}$ is divided into chambers by walls of the form

$$W(d_1,d_2) = \{\theta \in \mathbb{R}^{Q_0}\colon \mu_\theta(d_1) = \mu_\theta(d_2)\}\,.$$

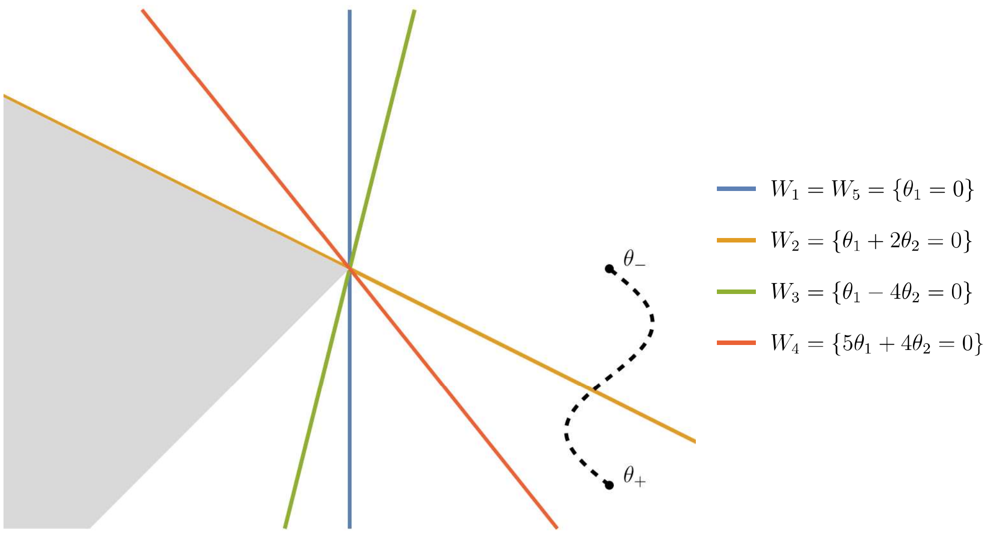
The moduli spaces M_d^θ do not change unless we cross a wall $W(d_1,d_2)$ with $d = d_1 + d_2$. Studying what changes when we cross a wall is a great tool to understand moduli spaces since sometimes we can wall-cross to a much simpler moduli space.

Example 5.2. Consider a quiver with 3 vertices $\{1,2,3\}$ with any number of arrows $1 \mapsto 2$ and $2 \mapsto 3$, and let $d = (2,1,1)$. There are 5 possible walls for the moduli spaces M_d^θ:

$$W_1 = W\big((1,0,0),(1,1,1)\big)\,,\; W_2 = W\big((1,1,0),(1,0,1)\big)\,,\; W_3 = W\big((0,1,0),(2,0,1)\big)$$
$$W_4 = W\big((0,0,1),(2,1,0)\big)\,,\; W_5 = W\big((2,0,0),(0,1,1)\big)$$

In the figure below we represent the wall and chamber structure on the two dimensional subspace $\theta_1+\theta_2+\theta_3 = 0$ of the space of stability conditions $\mathbb{R}^{Q_0}$ (note that the stability of representations is unaffected by adding a constant to each entry of θ, so any stability condition is equivalent to one of these).

The shadowed region is the region of increasing stability conditions $\theta_1 < \theta_2 < \theta_3$, see Example 4.3; the moduli spaces M_d^θ are empty in this region. Note that the wall $W_1 = W_5$ is a double wall. The remaining ones are simple walls in the sense that we explain below, and the dashed path between θ_- and θ_+ is a simple wall-crossing path; as we will explain now, $M_d^{\theta_-}$ and $M_d^{\theta_+}$ are related by a flip.



We describe now a situation in which wall-crossing can be understood very geometrically, which is that of simple wall-crossing. Let $d \in \mathbb{Z}_{\geq 0}^{Q_0}$ and θ_- and θ_+ two stability conditions. We make the following assumption:

(A) θ_- and θ_+ are not on a wall, so there are no strictly $\theta_\pm$-semistable representations.

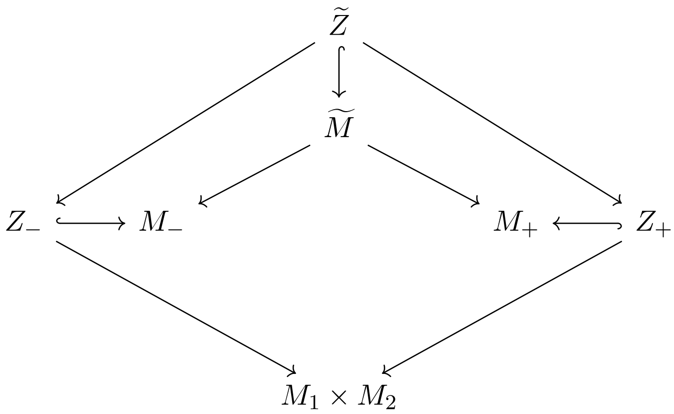
(B) There is a continuous path of stability conditions θ_t from $\theta_0 = \theta_-$ to $\theta_1 = \theta_+$ which crosses a unique wall $W(d_1, d_2)$ with $d_1 + d_2 = d$. We let θ be the stability condition at the intersection with the wall.

(C) We have
$$\mu_{\theta_-}(d_1) < \mu_{\theta_-}(d_2) \text{ and } \mu_{\theta_+}(d_1) > \mu_{\theta_+}(d_2)\,.$$

(D) The path θ_t does not cross any walls $W(d_3, d_4)$ with $d_3 + d_4 = d_1$ or $d_3 + d_4 = d_2$.

Denote
$$M_- = M_d^{\theta_-}\,,\; M_+ = M_d^{\theta_+}\,,\; M_1 = M_{d_1}^{\theta_-} \text{ and } M_2 = M_{d_1}^{\theta_-}\,.$$

Note that M_1 and M_2 are the same whether they are defined with θ_- or θ_+ stability by condition (D). How do M_- and M_+ differ? The representations which are θ_--stable but not θ_+-stable are the non-split extensions of the form
$$0 \to V_1 \to V \to V_2 \to 0 \tag{5}$$
where V_1, V_2 are $\theta_\pm$-stable representations with dimension vectors d_1, d_2, respectively. Note that $\mu_{\theta_+}(d_1) > \mu_{\theta_+}(d_2)$ is equivalent to $\mu_{\theta_+}(d_1) > \mu_{\theta_+}(d)$, so V_1 destabilizes V with respect to θ_+-stability. On the other hand, representations which are θ_+-stable but not θ_--stable are the non-split extensions of the form
$$0 \to V_2 \to V \to V_1 \to 0\,. \tag{6}$$
Both of these families of representations are θ-stable but not θ-semistable for the stability condition θ on the wall. The representations of the form (5) and (6) define loci $Z_- \subseteq M_-$ and $Z_+ \subseteq M_+$, respectively, and $M_- \setminus Z_- = M_+ \setminus Z_+$. Both $Z_\pm$ are projective bundles over $M_1 \times M_2$; their fibers over (V_1, V_2) are $\mathbb{P}\mathrm{Ext}^1(V_2, V_1)$ and $\mathbb{P}\mathrm{Ext}^1(V_1, V_2)$, respectively. So essentially we can control the "difference" between M_- and M_+ by the "smaller" moduli spaces M_1 and M_2. Indeed, Thaddeus proves a general theorem [Tha] implying that M_- and M_+ are related by a flip. This means that the blow-ups of M_- at Z_- and M_+ at Z_+ are the same space $\widetilde{M}$, and we have a flip diagram as follows:



The curved arrows are embeddings, the arrows from $\widetilde{M}$ to $M_\pm$ are blow-ups and the exterior arrows are all projective bundles; $\widetilde{Z}$ is the common exceptional divisor of the two blow-ups.

The flip diagram allows us to compare integrals on M_- and M_+ by pulling back to $\widetilde{M}$. The difference, of course, is related to integrals on M_1 and M_2. However, sometimes it is not possible to connect two stability conditions by simple wall-crossing paths. In Example 5.2 crossing the wall $W_1 = W_5$ is not simple wall-crossing because it is a double wall; successive extensions of V_1, V_2, V_3 with dimension vectors $(1,0,0)$, $(1,0,0)$ and $(0,1,1)$, respectively, will change stability.

5.2 The wall-crossing vertex algebra

In [Moc], Mochizuki described a way to understand the non-simple wall-crossing behavior of descendent integrals in the context of moduli spaces of sheaves on surfaces. More recently, Joyce [Joy1, Joy2, GJT] has proposed a formalism to state and prove more general wall-crossing formulas. Joyce constructs a vertex algebra to write down such formulas in a conceptual way. This vertex algebra is very closely related to the descendent algebra introduced before by results from [Gro, BLM, LM], and this is the perspective we take here.

For $d \in \mathbb{Z}^{Q_0}$ define

$$\mathbb{D}_d^Q = \mathbb{D}^Q / \langle \mathrm{ch}_0(v) = d_v \mid v \in Q_0 \rangle \, .$$

Given a moduli space M_d^θ and a fixed universal representation $\mathcal{V}$, the realization morphism $\xi_{\mathcal{V}}$ factors through $\mathbb{D}_d^Q$, so it defines a linear functional

$$\mathbb{D}_d^Q \xrightarrow{\xi_{\mathcal{V}}} H^\bullet(M_d^\theta) \xrightarrow{\int} \mathbb{Q}\,.$$

Define the space of functionals

$$V = \bigoplus_{d\in\mathbb{Z}^{Q_0}} (\mathbb{D}_d^Q)^\vee\,.$$

Joyce constructs a natural vertex algebra structure on V. We refer to [Kac] for an introduction to vertex algebras, but we can mention that a vertex algebra structure on a vector space V consists of the data

$$|0\rangle \in V\,,\quad T\colon V\to V\,,\quad Y\colon V\to \mathrm{End}(V)[\![z^{-1},z]\!]$$

satisfying some complicated axioms. In our case, $|0\rangle$ corresponds to the functional in $\mathbb{D}_0^Q$ sending the algebra unit to 1 and any (non-empty) product of descendents to 0. The operator T is the dual of the operator R_{-1} that we defined earlier. The state-field correspondence Y is the most complicated and interesting part of the construction, but we will not discuss it here.

A moduli space together with a universal representation defines an element of V. However, as explained in Section 4.2, there is no canonical choice of universal representation, so a moduli space only defines canonically a functional on

$$\mathbb{D}_{d,\mathrm{wt}_0}^Q = \ker\left(\mathsf{R}_{-1}\colon \mathbb{D}_d^Q\to\mathbb{D}_d^Q\right)\,.$$

Since T is defined to be the dual of R_{-1}, we have an isomorphism

$$\check{V} := V/T(V) \simeq \bigoplus_{d\in\mathbb{Z}^{Q_0}} (\mathbb{D}_{d,\mathrm{wt}_0}^Q)^\vee\,.$$

It is a general fact from vertex algebras, due to Borcherds [Bor], that $\check{V} = V/T(V)$ inherits a Lie algebra structure from the vertex algebra structure on V. Given a moduli space M, its class $[M]\in\check{V}$ contains information about all the descendent integrals. Joyce shows that wall-crossing formulas can be written using the Lie bracket on $\check{V}$! For example, in the setting of simple wall-crossing that we described before we have an identity

$$[M_+] = [M_-] + [[M_2],[M_1]]$$

in $\check{V}$. But Joyce's formalism is also able to deal with non-simple wall-crossing, in which case we get more complicated formulas with iterated brackets.

Example 5.3. We go back to Example 5.2 and consider $\theta_- = (1,5,-6)$ and $\theta_+ = (-1,5,-4)$; these are separated only by the double wall $W_1 = W_5$. Joyce's wall crossing formula in this case gives

$$[M^{\theta_+}_{(2,1,1)}] = [M^{\theta_-}_{(2,1,1)}] + [[M^{\theta_-}_{(1,0,0)}],[M^{\theta_-}_{(1,1,1)}]] + [[M^{\theta_-}_{(2,0,0)}],[M^{\theta_-}_{(0,1,1)}]] \\ + \frac{1}{2}[M^{\theta_-}_{(1,0,0)},[[M^{\theta_-}_{(1,0,0)}],[M^{\theta_-}_{(0,1,1)}]]]\,.$$

This can be used effectively since we understand explicitly the vertex algebra V.

Theorem 5.4. The vertex algebra V is isomorphic to the lattice vertex algebra associated to $\mathbb{Z}^{Q_0}$ and the pairing $\chi_Q^{\mathrm{sym}}\colon \mathbb{Z}^{Q_0}\times\mathbb{Z}^{Q_0}\to\mathbb{Z}$ which is the symmetrization of the Euler pairing χ_Q of Q given by

$$\chi_Q(d,d') = \sum_{v\in Q_0} d_v d'_v - \sum_{e\in Q_0} d_{s(e)}d'_{t(e)}\,.$$

5.3 Vertex algebra and Virasoro

Vertex algebras are a natural source of representations of the Virasoro algebra. Any lattice vertex algebra associated to a lattice (Λ, B) with $B\colon \Lambda\times\Lambda\to\mathbb{Z}$ a non-degenerate pairing comes equipped with a so called conformal element $\omega\in V$ which induces a representation of Vir in V. Even if B is degenerate (which is sometimes the case with χ_Q^{sym}), there is still a representation of Vir_{-1} on V.

It was shown in [BLM, Boj, LM] that this canonical representation of Vir_{-1} coming from the fact that V is a lattice vertex algebra is precisely the dual of the representation of Vir_{-1} on the descendent algebra that we defined earlier in Section 4.3. A moduli space M satisfying Virasoro constraints can be translated to a vertex algebra language: it means that the class $[M]\in\check{V}$ is a physical state, as defined in [Bor]! This is not only very interesting because it gives a conceptual meaning to the constraints, but it also has an important consequence: this point of view shows that the Virasoro constraints are compatible with wall-crossing in the sense that the subspace of physical states is a Lie subalgebra.

Proposition 5.5 ([Bor])**.** Suppose that $[M_1],[M_2]\in\check{V}$ are physical states (read as "M_1 and M_2 satisfy the Virasoro constraints"). Then $[[M_1],[M_2]]\in\check{V}$ is a physical state as well.

From this compatibility, the proof of Theorem 4.6 is straightforward. The Virasoro constraints hold trivially for the increasing stability conditions (see Example 4.3), so we can deduce them for any stability θ from the compatibility with wall-crossing!

Acknowledgement

The author wishes to thank Y. Bae, A. Bojko, I. Karpov, W. Lim, D. Maulik, A. Oblomkov, A. Okounkov, R. Pandharipande and W. Pi for numerous conversations about the Virasoro constraints. This paper surveys results from collaborations with some of the above. I thank the organization of the Global Portuguese Mathematicians Conference 2023, in CMUC Coimbra, for a very pleasant event and for the invitation to both speak and write this survey. I thank the referee for a few suggestions to improve the exposition. The work surveyed was done while the author was supported by ERC-2017-AdG-786580-MACI. The project received funding from the European Research Council (ERC) under the European Union Horizon 2020 research and innovation programme (grant agreement 786580).

Referências

[BF] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–99.

[BLM] A. Bojko, W. Lim, and M. Moreira, *Virasoro constraints for moduli of sheaves and vertex algebras*, Invent. Math. **236** (2024), no. 1, 387–476.

[Boj] A. Bojko, *Universal Virasoro Constraints for Quivers with Relations*, 2023, preprint.

[Bor] R. E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proceedings of the National Academy of Sciences **83** (1986), no. 10, 3068–3071.

[CdlOGP] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), no. 1, 21–74.

[EH] D. Eisenbud and J. Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016.

[EHX] T. Eguchi, K. Hori, and C.-S. Xiong, *Quantum cohomology and Virasoro algebra*, Phys. Lett. B **402** (1997), no. 1-2, 71–80.

[Get] E. Getzler, *The Virasoro conjecture for Gromov-Witten invariants*, Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math., vol. 241, Amer. Math. Soc., Providence, RI, 1999, pp. 147–176.

[Giv1] A. Givental, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141–175.

[Giv2] A. B. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, vol. 1, 2001, Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary, pp. 551–568, 645.

[GJT] J. Gross, D. Joyce, and Y. Tanaka, *Universal structures in $\mathbb{C}$-linear enumerative invariant theories*, SIGMA Symmetry Integrability Geom. Methods Appl. **18** (2022), Paper No. 068, 61.

[Gro] J. Gross, *The homology of moduli stacks of complexes*, 2019, preprint.

[HKK$^+$] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, Clay Mathematics Monographs, vol. 1, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2003.

[Joy1] D. Joyce, *Ringel–Hall style Lie algebra structures on the homology of moduli spaces*, 2019, preprint.

[Joy2] D. Joyce, *Enumerative invariants and wall-crossing formulae in abelian categories*, 2021, preprint.

[Joy3] D. Joyce, *Enumerative invariants of projective curves and surfaces*, in preparation.

[Kac] V. Kac, *Vertex algebras for beginners*, second ed., University Lecture Series, vol. 10, American Mathematical Society, Providence, RI, 1998.

[Kin] A. D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1992), 515–530.

[Kon] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23.

[LM] W. Lim and M. Moreira, *Virasoro constraints and representations for quiver moduli spaces*, 2024, preprint.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994.

[MNOP1] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory. I*, Compos. Math. **142** (2006), no. 5, 1263–1285.

[MNOP2] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory. II*, Compos. Math. **142** (2006), no. 5, 1286–1304.

[Moc] T. Mochizuki, *Donaldson type invariants for algebraic surfaces*, Lecture Notes in Mathematics, vol. 1972, Springer-Verlag, Berlin, 2009, Transition of moduli stacks.

[MOOP] M. Moreira, A. Oblomkov, A. Okounkov, and R. Pandharipande, *Virasoro constraints for stable pairs on toric threefolds*, Forum Math. Pi **10** (2022), no. 20, 1–62.

[Mor] M. Moreira, *Virasoro conjecture for the stable pairs descendent theory of simply connected 3-folds (with applications to the Hilbert scheme of points of a surface)*, J. Lond. Math. Soc. (2) **106** (2022), no. 1, 154–191.

[OOP] A. Oblomkov, A. Okounkov, and R. Pandharipande, *GW/PT descendent correspondence via vertex operators*, Comm. Math. Phys. **374** (2020), no. 3, 1321–1359.

[OP] A. Okounkov and R. Pandharipande, *Virasoro constraints for target curves*, Invent. Math. **163** (2006), no. 1, 47–108.

[Pan] R. Pandharipande, *Descendents for stable pairs on 3-folds*, Modern geometry: a celebration of the work of Simon Donaldson, Proc. Sympos. Pure Math.

[Tel] C. Teleman, *The structure of 2D semi-simple field theories*, Invent. Math. **188** (2012), no. 3, 525–588.

[Tha] M. Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), no. 3, 691–723.

[vB] D. van Bree, *Virasoro constraints for moduli spaces of sheaves on surfaces*, Forum Math. Sigma **11** (2023), no. 4, 1–35.

[Wit] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310.

Topology of vortices with toric targets

Marcel Bökstedt
Institut for Matematik, Aarhus Universitet
Ny Munkegade 118, bygn. 1530, Aarhus C, Denmark
e-mail: marcel@math.au.dk

Nuno M. Romão
Institut des Hautes Études Scientifiques
35 route de Chartres, 91440 Bures-sur-Yvette, France
e-mail: nromao@ihes.fr

Resumo: Equações de vórtices são EDPs que descrevem configurações de campos BPS para modelos sigma de gauge em fibrados. No trabalho que aqui expomos, consideramos a situação em que a base é uma superfície de Riemann e a fibra (ou alvo) é uma variedade de Kähler tórica, ambas compactas. Os espaços de módulo que parametrizam as soluções (a menos de transformações de gauge unitárias), neste caso, são espaços de configurações generalizados associados a um complexo simplicial que pode ser extraído dos dados combinatórios do alvo. Os grupos fundamentais destes espaços podem ser interpretados como um novo tipo de grupos de tranças (coloridas) em superfícies, cuja descrição apresentamos de forma bastante concreta.

Abstract: The vortex equations are PDEs describing BPS field configurations for gauged sigma models in fibre bundles. In the work surveyed here, we focus on the situation where the base is a Riemann surface and the fibre/target is a toric Kähler manifold, both assumed compact. Then the moduli spaces of solutions (up to unitary gauge) are generalised configuration spaces associated to a certain simplicial complex that can be extracted from the toric data. The fundamental groups of such spaces can be understood as a novel type of surface braid groups with coloured strands, which we shall describe in very concrete terms.

palavras-chave: teoria de gauge, vórtice, espaços de módulo, geometria tórica, grupos de tranças, grafo, sistemas diofantinos

keywords: gauge theory, vortex, moduli spaces, toric geometry, braid groups, graph, Diophantine systems

1 Introduction

In elementary particle physics and condensed-matter theory, quantum field theories (as well as approximations or extensions to them) are studied with a wide variety of tools. The concept of *particle*, which is sometimes rather elusive, is captured by 'charges' modelled on invariants provided e.g. via algebraic topology or algebraic geometry, or from the representation theory of symmetry groups. Such invariants appear naturally in classification problems in pure mathematics, where in many situations one needs to supplement them by constructing *moduli spaces* parametrising different isomorphism classes of objects with the same invariants arising in families. Prime examples of such spaces feature in the surveys by Margarida Melo [26] and Miguel Moreira [27] also included in this Special Issue.

Moduli spaces keep fascinating mathematicians [30] from a variety of backgrounds. They have been used crucially to tackle (sometimes unexpectedly) very hard problems — from the geometry of 4-manifolds to quantum gravity. In this review, we explore moduli spaces parametrising solitonic configurations called *vortices* [24] in gauge theory. Vortices are like particles in the sense of having a point core, though their energy density is extended in space, forming peaks around the cores that superpose nonlinearly. In Fig. 1 we plot the energy density distribution of two vortex configurations of topological charge 2 on the hyperbolic disc. Here the charge corresponds to the degree of a map $S^1 \to S^1$ from the boundary of the disc to a circle in the target $\mathbb{C}$ parametrising the degenerate minima of a Higgs potential. One reason we chose the hyperbolic disc is that it leads to the integrability of the vortex equations [41]; but the vortices we will concentrate on live on compact Riemann surfaces and almost never enjoy this property. Vortex moduli spaces (with their intrinsic geometry [3]) can been employed to study vortex dynamics in the classical field theory [37, 33, 31, 38]. Quantisation of moduli spaces is a more ambitious venture [33, 18, 40], but one may hope to uncover nonperturbative aspects of quantum field theories in this way.

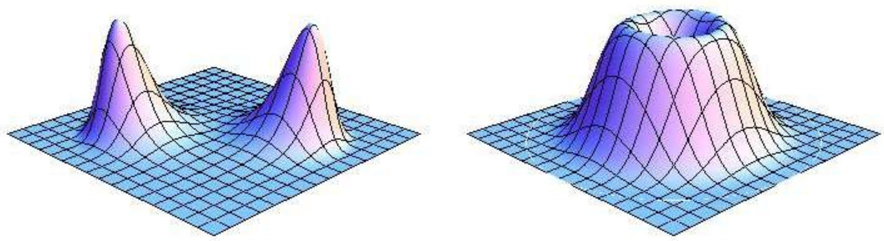


Figure 1: Plots of energy density functions for configurations of 2-vortices on the hyperbolic disc, with well-separated cores (left) and coalescing cores (right)

The topology of vortices advertised in our title refers concretely to the moduli spaces — more specifically, we shall look at their fundamental groups. It turns out that they are computable and interesting objects (providing examples of a novel type of braid group); in addition, they hint at how the solitonic particles interact in the quantum theory.

2 Gauged sigma models, the vortex equations and their moduli

We start by defining vortices more precisely and in higher generality. For that, one needs to choose:

- $(\Sigma, j_\Sigma, \omega_\Sigma)$ a riemannian, oriented surface (the *base*)
- (X, j_X, ω_X) another Kähler manifold (the *target*)

The word *gauged* refers to an internal symmetry that corresponds to a Hamiltonian and holomorphic action of a compact Lie group G on X. Letting $\mathfrak{g} := \mathrm{Lie}(G)$, we denote by $\mu^\sharp$ the composition of a moment map $\mu : X \to \mathfrak{g}^*$ for this action (satisfying $\mathrm{d}\mu(\xi) = \iota_\xi \omega_X$ for all $\xi \in \mathfrak{g}$) with the 'musical' isomorphism $\sharp : \mathfrak{g}^* \to \mathfrak{g}$ associated to an invariant metric on G.

We shall assign a distribution of energy to pairs of *fields*

$$(A, \phi) \in \mathcal{A}(P) \times \Gamma(\Sigma, P^X)$$

where A is a connection on a principal G-bundle $P \to \Sigma$ and ϕ a section of the associated bundle $P^X := P \times_G X \to \Sigma$. To each such pair we associate a topological charge

$$[\phi]_2^G := ((\tilde{f} \times \phi)/G)_*[\Sigma] \ \in \ H_2^G(X; \mathbb{Z})$$

in the G-equivariant 2-homology of X; here, $\tilde{f} : P \to \mathrm{E}G$ is the lift of a classifying map $f : \Sigma \to \mathrm{B}G$ for P with $P = f^*\mathrm{E}G$.

Given all these ingredients, the potential energy of a *gauged sigma model* is specified by the Yang–Mills–Higgs functional

$$E(A, \phi) := \frac{1}{2} \int_\Sigma \left(|F_A|^2 + |\mathrm{d}^A \phi|^2 + |\mu \circ \phi|^2 \right).$$

To find its minima, one could in principle try to solve its Euler–Lagrange equations (second-order PDEs on Σ). A more convenient strategy is to perform the so-called Bogomol'nyĭ's trick (taking advantage of the decomposition $\mathrm{d}^A = \partial^A + \bar{\partial}^A$ for the covariant derivative; see e.g. [29] for more details)

$$E(A, \phi) = \langle [\omega_X - \mu]^2_G, [\phi]^G_2 \rangle + \frac{1}{2} \int_\Sigma \left(\left| *F_A + \mu^\sharp \circ \phi \right|^2 + 2 |\bar{\partial}^A \phi|^2 \right) \tag{1}$$

and observe that the minima in each topological class, whenever the first term (which is constant) is nonnegative, satisfy the first-order PDEs

$$\bar{\partial}^A \phi = 0 \tag{2}$$
$$*F_A + \mu^\sharp \circ \phi = 0 \tag{3}$$

known as the *vortex equations*; solutions to these are examples of what are called (classical) BPS configurations in field theory. Equation (2) says that ϕ is a holomorphic section, whereas (3) relates the curvature F_A of A to the moment map evaluated at ϕ. Note that the metric on Σ intervenes in equation (3) via its Hodge operator $*$. Both equations are invariant under the infinite dimensional group $\mathcal{G}(P) := \mathrm{Aut}_\Sigma(P)$ of gauge transformations.

Remark 2.1. There is a completely analogous manipulation for the case $\langle [\omega_X - \mu]^2_G, [\phi]^G_2 \rangle \leq 0$ yielding *antivortex equations*, where $\bar{\partial}^A$ is replaced by ∂^A in (2) and the $(+)$ sign flips to $(-)$ in (3); in this sense, it is clear that a pair (A, ϕ) on a connected surface Σ cannot be of both vortex and antivortex type unless A is flat (i.e. $F_A = 0$) and ϕ covariantly constant, which corresponds to a *vacuum* configuration. We will see in a moment (cf. Example 2.1) how to implement coexistence of vortices and antivortices within a BPS solution in another sense.

Vortex *moduli spaces* are defined by fixing $\mathbf{h} \in H^G_2(X; \mathbb{Z})$ and taking

$$\mathcal{M}^X_{\mathbf{h}}(\Sigma) := \left\{ (A, \phi) \;\middle|\; \begin{array}{c} (2), (3) \text{ are satisfied} \\ \text{and } [\phi]^G_2 = \mathbf{h} \end{array} \right\} / \mathcal{G}(P)$$

(and similarly for antivortices). Such a space (of gauge orbits) can be formally understood as a Kähler quotient [28], and more rigorously one can show [29] that it receives a Kähler structure ω_{L^2} from the natural L^2 inner product on the space of fields $\mathcal{A}(P) \times \Gamma(\Sigma, P^X)$. Usually it is straightforward to understand the underlying complex structure, but it is much harder to describe or compute the Kähler form ω_{L^2} (or the corresponding Kähler metric g_{L^2}) [3, 6].

Though we have sketched how vortex moduli spaces are motivated by physics, we would like to point out that these objects have also found many applications in pure mathematics. Just to mention a few: they have been useful to compute Gromov–Witten invariants of symplectic quotients [4], to define invariants for Hamiltonian actions in analogy to Gromov–Witten theory [29, 25, 14], as a tool to study the topology of other interesting moduli spaces [21], or in a proof of the celebrated Verlinde formula [39].

In what follows, we shall restrict attention to gauged sigma models where

- X is a Kähler toric manifold (see [12, 1]);
- $G = T \subset \mathbb{T} := T^{\mathbb{C}} \subset X$ is its (real) torus.

Assuming (for simplicity) that X, Σ are compact, we obtain a very neat description of $\mathcal{M}^X_{\mathbf{h}}(\Sigma)$, which we now explain.

Toric manifolds can be described via certain combinatorial data. One possibility (favoured by algebraic geometers [15]) is in terms of a *fan* F consisting of simplicial real *cones* in all dimensions from 0 up to $\dim_{\mathbb{C}} X$. In particular, the 1-dimensional cones $\rho \in F(1)$ are called *rays* and determine T-invariant divisors D_ρ in X. Another possibility, with more of a symplectic flavour, is through the *Delzant polytope* [11]

$$\Delta = \mu(X) \subset \mathfrak{g}^*$$

(the image of the moment map for the T-action). This Δ determines a *normal fan* $F = \mathrm{Fan}_\Delta$, whose rays consist of the inner normals to the facets of Δ.

The following notation will be very useful. We denote by

$$S^k\Sigma \equiv \mathrm{Sym}^k(\Sigma) := \Sigma^k/\mathfrak{S}_k$$

the k-th symmetric product of the Riemann surface Σ; this is a smooth complex manifold, whose points are interpreted as effective divisors in Σ in classical algebraic geometry of curves. Let us now suppose that we are given a simplicial complex Λ together with a function $\mathbf{k} : \mathrm{Sk}^0(\Lambda) \to \mathbb{N}$ on its 0-skeleton (whose points/vertices we shall sometimes refer to as *colours*). For convenience, we set

$$S^{\mathbf{k}}\Sigma := \prod_{\lambda \in \mathrm{Sk}^0(\Lambda)} S^{\mathbf{k}(\lambda)}\Sigma.$$

Let $[\lambda_i, \cdots, \lambda_j]$ denote the simplex with vertices $\lambda_i, \ldots, \lambda_j \in \mathrm{Sk}^0(\Lambda)$. We consider the following space of effective divisors in Σ *braided* by Λ:

$$\mathrm{Div}^{\mathbf{k}}_{+}(\Sigma;\Lambda) := \left\{ \mathbf{d} \in S^{\mathbf{k}}\Sigma : [\lambda_0, \cdots, \lambda_\ell] \notin \Lambda \Rightarrow \bigcap_{i=0}^{\ell} \mathrm{supp}(d_{\lambda_i}) = \emptyset \right\} \qquad (4)$$

We can think of each component d_λ of $\mathbf{d}$ as a set of $\mathbf{k}(\lambda)$ points (counting multiplicities) of colour λ in Σ. The space (4) is a considerable generalisation of the usual notion of *configuration space* for Σ; we shall refer to particular realisations of it in what follows.

Now we go back to our toric targets. In that context, we shall take as simplicial complex

$$\Lambda = (\partial\Delta)^\vee. \qquad (5)$$

when a Delzant polytope $\Delta = \mu(X)$ is given for a target X. Let us spell out the notation in (5): by $\partial\Delta$ we mean its boundary, which we may interpret as spherical polytope, and $(\cdot)^\vee$ means the dual polytope in that sense. For instance: if $\partial\Delta$ is the (outer shell of a) cube, then its dual $(\partial\Delta)^\vee$ is an octahedron; this example corresponds to $X = (\mathbb{P}^1)^3$.

Under an appropriate (open) stability condition, the moduli spaces $\mathcal{M}^X_{\mathbf{h}}(\Sigma)$ can be described quite neatly. To spell out this condition precisely, we need a little more detail (see [9] for a full discussion, in a more general setting). In the commutative diagram of Abelian groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\alpha} & \mathrm{Div}_{\mathbb{T}}(X) & \xrightarrow{\beta} & \mathrm{Cl}(X) & \longrightarrow & 0 \\
 & & \cong\big\downarrow c & & \cong\big\downarrow c_1^{\mathbb{T}} & & \cong\big\downarrow c_1 & & \\
0 & \longrightarrow & H^2(\mathrm{B}\mathbb{T};\mathbb{Z}) & \xrightarrow{a} & H^2_{\mathbb{T}}(X;\mathbb{Z}) & \xrightarrow{b} & H^2(X;\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

the top row is the standard short exact sequence computing the divisor class group $\mathrm{Cl}(X)$ of a toric variety X, M denoting the character lattice of $\mathbb{T}$ (see [15]); we interpret the map $c_1^{\mathbb{T}}$ as constructing equivariant first Chern classes [22] from $\mathbb{T}$-equivariant divisors on X. Let $a^*_{\mathbb{R}}$ be the dual map to the extension of the morphism a to real coefficients, and $\mathrm{Vol}(\Sigma) = \int_\Sigma \omega_\Sigma$. Assume that $\mathbf{h} \in H_2^T(X;\mathbb{Z})$ is a BPS charge [9]: in our setting, this simply means that it is an element of the lattice cone dual to the image of the cone of equivariant effective divisors $\mathrm{Div}^+_{\mathbb{T}}(X)$ under $c_1^{\mathbb{T}}$. Then the stability condition reads

$$\frac{a^*_{\mathbb{R}}(\mathbf{h})}{\mathrm{Vol}(\Sigma)} \in \mathrm{int}\,\Delta$$

under identifications $H_2(\mathrm{B}\mathbb{T};\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(M,\mathbb{R}) = \mathrm{Lie}(T)$ (see [15], pp. 597 and 574, respectively). We have the following result.

Theorem 2.2. *Suppose that X is constructed from a Delzant polytope Δ and that*

$$k_\rho = \langle c_1^T(D_\rho), \mathbf{h}\rangle \qquad \textit{for } \rho \in \mathrm{Fan}_\Delta(1).$$

Then $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$ is nonempty and there is a diffeomorphism

$$\mathcal{M}_{\mathbf{h}}^X(\Sigma) \cong \mathrm{Div}_+^{\mathbf{k}}(\Sigma;(\partial\Delta)^\vee) \subset \prod_{\rho\in\mathrm{Fan}_\Delta(1)} S^{k_\rho}\Sigma. \tag{6}$$

Proof. The result (6) is a particular case of a more general theorem in [9] for vortices in toric fibre bundles over Kähler manifolds of any dimension. The statement $\mathcal{M}_{\mathbf{h}}^X(\Sigma) \neq \emptyset$ follows from the projectivity of Σ (implying that positive line bundles admit holomorphic sections). □

Example 2.1 (The gauged $\mathbb{P}^1$-model)**.** The simplest type of nonlinear vortices (with X compact) are obtained for $X = \mathbb{P}^1 \cong S^2$ and $T = \mathrm{U}(1) \cong S^1$, with a fan of two rays $\rho_\pm$ corresponding to the North and South poles. Essentially, the case $\Sigma = S^2$ was first considered in [34]. Theorem 2.2 gives the description

$$\mathcal{M}_{(k_+,k_-)}^{\mathbb{P}^1}(\Sigma) = S^{k_+}\Sigma \times S^{k_-}\Sigma \setminus D_{(k_+,k_-)},$$

where $D_{(k_+,k_-)}$ is the diagonal in the Cartesian product; this agrees with previous work in the literature [35, 29, 2]. In this simple example we see that the solutions of the vortex equations (2) and (3), up to gauge equivalence, are completely specified by the positions of zeros and poles of the section ϕ (taking multiplicities into account). These "vortex cores" can be totally arbitrary on the surface provided zeroes and poles do not coalesce — even though zeroes and poles can coalesce among themselves. One such configuration is suggested in Fig 2, where the larger dots convey zeroes and poles of higher multiplicity, around which the energy density will have a shape somewhat similar to the graph on right half of Fig 1. The main difference between zeroes and poles is that the magnetic field F_A has (respectively) the same or opposite sign to the orientation of Σ; thus the shape of the fields near a pole is rather similar to that of an antivortex in the sense of Remark 2.1.

We note that the moduli spaces $\mathcal{M}_{\mathbf{h}}^X(\Sigma)$ are complex manifolds (their natural complex structure being induced on $S^{\mathbf{k}}\Sigma$ from j_Σ) with boundary normal-crossing divisors. For instance, in Example 2.1 we have

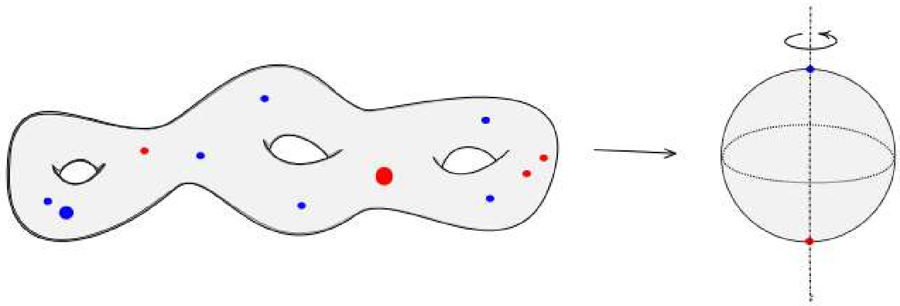


Figure 2: A configuration of vortices and antivortices in the gauged $\mathbb{P}^1$-model

$\partial\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma) = D_{(k_+,k_-)}$. The Kähler metric g_{L^2} has been studied close to this boundary in [32].

3 Fundamental groups of vortex moduli spaces with toric targets

Very concretely, the topology of vortices (in the title) that we propose to discuss is the fundamental group of the moduli spaces $\mathcal{M}^X_{\mathbf{h}}(\Sigma)$ in Theorem 2.2. Before we get there, we make a short interlude to advertise a useful variation on the braid group on surfaces [5].

Let Σ be an orientable surface. We introduce groups $DB_{\mathbf{k}}(\Sigma,\Gamma)$ of *divisor braids* on Σ that depend on a graph Γ, undirected and not necessarily connected, whose vertices are decorated with integers through a map $\mathbf{k} : \mathrm{Sk}^0(\Gamma) \to \mathbb{N}$. Here is an informal definition:

- Divisor braids have *coloured* strands, and the colours used are in bijection with the vertices $\mathrm{Sk}^0(\Gamma)$ of Γ. To allow for composition, we must demand that they are *colour-pure*.
- We extend the set of isotopies to allow strands of the same colour to pass through each other transversally, unlike ordinary braids.
- Strands of different colour are also allowed to pass through each other *unless* the vertices corresponding to their colours are connected by an edge in Γ. Clearly, we can assume that Γ has no multiple edges (connecting two given vertices), and we forbid self-loops (i.e. edges starting and ending at the same vertex).

- As subscript we use a map $\mathbf{k} : \mathrm{Sk}^0(\Gamma) \to \mathbb{N}$ (or decoration of vertices) recording how many strands $\mathbf{k}(\lambda) = k_\lambda$ there are of a given colour λ.

For instance, we could take $\Gamma = \Gamma_{(r)}$ to be the *complete graph* with r vertices (any pair of vertices is connected by an edge) and the constant function $\mathbf{k} = 1$; then it is easy to check that $DB_1(\Sigma, \Gamma_{(r)})$ is the usual pure braid group $PB_r(\Sigma)$. But we will also be interested in more general graphs with the properties listed above (see e.g. Fig. 3). Given such a graph, we denote by $\neg\Gamma$ its negative: it has the same set of vertices as Γ and the complementary set of edges. We define

$$\mathrm{Conf}_{\mathbf{k}}(\Sigma, \Gamma) := \mathrm{Div}_+^{\mathbf{k}}(\Sigma; \neg\Gamma) \quad \subset \quad S^{\mathbf{k}}\Sigma. \tag{7}$$

This generalises the usual notion of configuration space $\mathrm{Conf}_k(\Sigma)$ of k points on Σ, which is obtained as

$$\mathrm{Conf}_k(\Sigma) = \mathrm{Conf}_1(\Sigma, \Gamma_{(k)});$$

it also extends nontrivially the notion of configuration space defined from a graph without decorations in [13]. The definition (7) can be used to give an alternative (and rigorous) definition of divisor braid groups as

$$DB_{\mathbf{k}}(\Sigma, \Gamma) := \pi_1\left(\mathrm{Conf}_{\mathbf{k}}(\Sigma, \Gamma)\right).$$

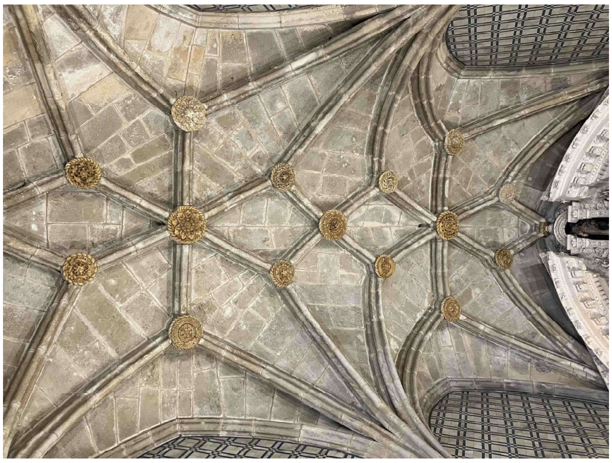


Figure 3: Sightseeing in Coimbra provides opportunities for graph theorists

Let us now go back to vortices. Recall that Theorem 2.2 gave a description of the moduli spaces $\mathcal{M}_{\mathbf{h}}^{X_\Delta}(\Sigma)$ under certain assumptions on the geometric ingredients defining the gauged sigma model. These were interpreted as spaces of effective divisors $\mathrm{Div}_+^{\mathbf{k}}(\Sigma, \Lambda)$ braided by $\Lambda = (\partial\Delta)^\vee$ that are actually of a more general sort than the configuration spaces (7), since they involve conditions where multiple intersections of points in Σ may occur. However, we have the following result from [8]:

Theorem 3.1. *There is an isomorphism of fundamental groups*

$$\pi_1 \mathrm{Div}_+^{\mathbf{k}}(\Sigma, \Lambda) \cong \pi_1 \mathrm{Div}_+^{\mathbf{k}}(\Sigma, \mathrm{Sk}^1(\Lambda)).$$

In particular, $\pi_1 \left(\mathcal{M}_{\mathbf{h}}^{X_\Delta}(\Sigma) \right)$ *is a divisor braid group.*

Thus the study of divisor braid groups is well motivated by physics. In the rest of this article, our focus will be on describing presentations for these groups.

The first obvious task is to search for a convenient set of generators. If $k = |\mathbf{k}| := \sum_{\lambda \in \mathrm{Sk}^0(\Gamma)} k_\lambda$, then $DB_{\mathbf{k}}(\Sigma, \Gamma)$ is obviously a quotient of the usual braid group $B_k(\Sigma)$ on k strands. Let

$$\mathfrak{S}_{\mathbf{k}} := \prod_{\lambda \in \mathrm{Sk}^0(\Gamma)} \mathfrak{S}_{k_\lambda}, \qquad B_{\mathbf{k}}(\Sigma) := \sigma^{-1}(\mathfrak{S}_{\mathbf{k}}) \subset B_k(\Sigma).$$

Then we have a diagram of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & PB_k(\Sigma) & \overset{\text{colouring}}{\longrightarrow} & B_{\mathbf{k}}(\Sigma) & \longrightarrow & \mathfrak{S}_{\mathbf{k}} & \longrightarrow & 1 \\ & & & & \downarrow & & & & \\ & & & & DB_{\mathbf{k}}(\Sigma) & & & & \end{array}$$

where the row is short exact, and the map $\downarrow$ is surjective. We also have

Lemma 3.2. The composite $\Psi : PB_k(\Sigma) \to B_{\mathbf{k}}(\Sigma) \to DB_{\mathbf{k}}(\Sigma)$ is surjective.

In what follows, we shall work under two simplifying assumptions:

(**A**) Σ is compact of genus g.
If $g > 0$, it is convenient to regard Σ as resulting from identifying *opposite* sides/edges of a $4g$-gon, labelled e_ℓ, and respecting mirror orientations. For $g = 0$, we may start with any such polygon and collapse its entire boundary to a point.

(**B**) $\mathbf{k}$ is such that $k_\lambda \geq 2$ for any colour λ.

Definition 3.3. For the pure braid group $PB_k(\Sigma)$, fix k distinct basepoints z_i in Σ, labelled by $1 \leq i \leq k$. Consider a closed path $\gamma_i : [0,1] \to \Sigma$ with $\gamma_i(0) = \gamma_i(1) = z_i$ and $\gamma_i(t) \neq z_j$ for all $j \neq i, t \in [0,1]$. Let $\Phi(\gamma_i)$ be a braid defined by a path in Σ^k with components

$$\Phi(\gamma_i)_j(t) = \begin{cases} z_j & \text{if } j \neq i, \\ \gamma_i(t) & \text{if } j = i, \end{cases} \qquad (j = 1, \ldots, k).$$

A pure braid of this type is called *monic*.

Our next goal is to show that divisor braid groups are generated by (images under Ψ of) certain monic braids. For convenience, let us fix distinct basepoints z_i along a bisector of the $4g$-gon used to construct Σ. We define the following monic braids:

- $a_{i,\ell} = \Phi(\gamma_{i,\ell}), \quad 1 \leq i \leq k,\ 1 \leq \ell \leq 2g$

  The path $\gamma_{i,\ell}$ runs from z_i straight to the midpoint of edge e_ℓ, crosses to the opposite side, and finally returns straight to z_i; see Fig, 4.

- $b_{i,j} = \Phi(\gamma_{i,j}), \quad 1 \leq i < j \leq k$

  The path $\gamma_{i,j}$ starts at z_i, encircles the point z_j positively, and then traces back its way to point z_i; see Fig, 5 (left).

- $t_{i,j} = \Phi(\tilde{\gamma}_{i,j}), \quad 1 \leq i < j \leq k$

  The path $\tilde{\gamma}_{i,j}$ starts at z_i, encircles the points $z_{i+1}, \ldots, z_j$, and goes back to point z_i without any further zigzagging; see Fig, 5 (right).

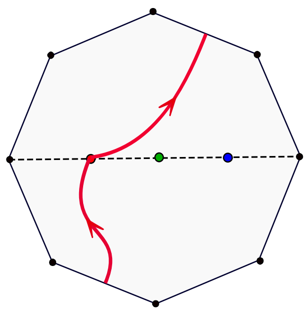


Figure 4: A divisor braid of type $a_{i,\ell}$ on a surface of genus $g > 0$

We can use these braids to construct convenient sets of generators for pure braid groups.

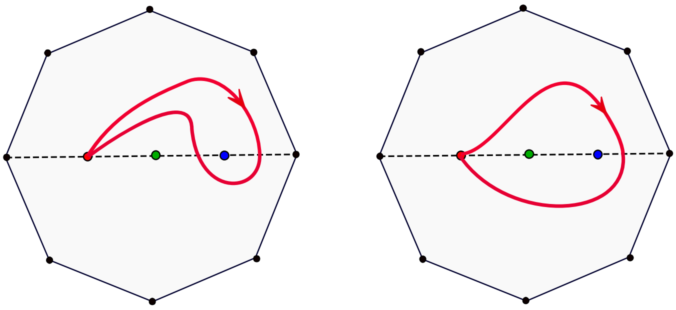


Figure 5: Divisor braids of types $b_{i,j}$ (left) and $t_{i,j}$ (right)

Lemma 3.4. Suppose Σ is a compact and oriented surface of genus g.
If $g \geq 1$, $PB_k(\Sigma)$ is generated by the classes $a_{i,\ell}$ and $b_{i,j}$.
If $g = 0$, $PB_k(\Sigma)$ is generated by the classes $t_{i,j}$.

Proof. For $g \geq 1$: the $a_{i,\ell}$ and $t_{i,j}$ are well-known generators of $PB_k(\Sigma)$ (see e.g. [20] for a presentation), and one can see that the $t_{i,j}$ are products of $b_{i,j}$ for $i < j \leq k$.
For $g = 0$: the $t_{i,j}$ correspond to generators of PB_n given by Artin, as products of his generators for B_n. □

As a consequence of Lemmas 3.2 and 3.4, we obtain a first set of generators for our divisor braid groups:

Corollary 3.5. For any $(\Gamma, \mathbf{k})$ as above, the $\Psi(a_{i,\ell}), \Psi(b_{i,j})$ generate $DB_{\mathbf{k}}(\Sigma, \Gamma)$ if $g \geq 1$, and the $\Psi(t_{i,j})$ generate $DB_{\mathbf{k}}(S^2, \Gamma)$.

These generators satisfy relations that we need to understand; as it might be expected, the sets of generators themselves contain some degree of redundancy. The following result will be useful to simplify our presentation.

Lemma 3.6. Let γ, γ' be two paths in Σ such that $\gamma(0) = \gamma(1) = z_i$, $\gamma'(0) = \gamma'(1) = z_{i'}$ and $z_i \neq z_{i'}$. Suppose further that

(i) the images of γ, γ' do not intersect in Σ; or that

(ii) $z_i, z_{i'}$ belong either to strands of the same colour or of different colours not connected by an edge in Γ.

Then $\Psi(\Phi(\gamma))$ and $\Psi(\Phi(\gamma'))$ are commuting divisor braids.

An interesting consequence of Lemma 3.6, which will bring a drastic simplification to the sets of generators in Corollary 3.5, is that

$\Psi(\Phi(a_{i,\ell})), \Psi(\Phi(b_{i,j}))$ only depend on the *colour* of their basepoints; see [8] for the complete argument. So for each λ we may pick an arbitrary z_{i_λ} of this colour and restrict the set of generators to

$$\alpha_{\lambda,\ell} := \Psi(a_{i_\lambda,\ell}), \quad \beta_{\lambda,\mu} := \Psi(b_{i_\lambda,i_\mu}).$$

In fact, we can do a little better.

Lemma 3.7. The classes $\alpha_{\lambda,\ell}$ and $\beta_{\lambda,\mu}$ generate $DB_{\mathbf{k}}(\Sigma,\Gamma)$; moreover, the classes $\beta_{\lambda,\mu}$ satisfy

$$\beta_{\lambda,\lambda} = e \text{ and } \beta_{\lambda,\lambda'} = \beta_{\lambda',\lambda}.$$

Proof. The first assertion is now clear. The relation $\beta_{\lambda,\lambda} = e$ is also clear, whereas $\beta_{\lambda,\lambda'} = \beta_{\lambda',\lambda}$ is better verified by hands-on manipulation; we try to convey this in Fig. 6. □

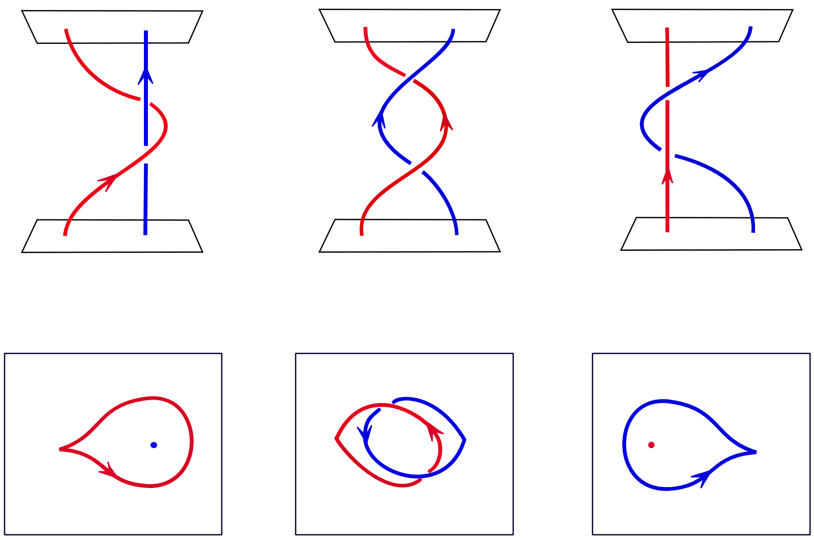


Figure 6: A pictorial check of the relation $\beta_{\lambda,\lambda'} = \beta_{\lambda',\lambda}$: all these pictures represent the same divisor braid (actually just two strands thereof, with the other strands kept straight) in two colours; the bottom row is obtained from the top row by vertical projection

We still expect further relations.

Lemma 3.8. Let λ, λ' be two different colours, and $1 \leq \ell, \ell' \leq 2g$. Then

$$[\alpha_{\lambda,\ell}, \alpha_{\lambda',\ell'}] = \beta^{-1}_{\lambda,\lambda'}.$$

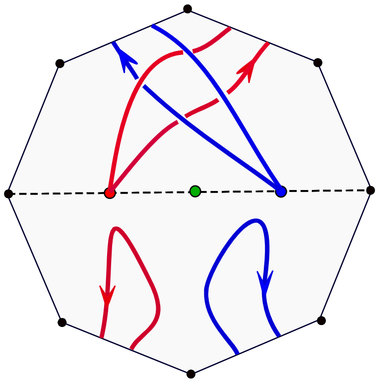


Figure 7: Checking that the relation $[\alpha_{\lambda,\ell}, \alpha_{\lambda',\ell'}] = \beta^{-1}_{\lambda,\lambda'}$ holds: one can disentangle the divisor braid depicted to obtain the inverse of Fig. 6

Proof. Again, this is best verified by drawing pictures — see Fig. 7. □

Lemmas 3.7 and 3.8 can be used to slim down further our set of generators, as follows.

Corollary 3.9. The classes $\alpha_{\lambda,\ell}$ generate $DB_{\mathbf{k}}(\Sigma,\Gamma)$ if Σ has positive genus, whereas $DB_{\mathbf{k}}(S^2,\Gamma)$ is generated by the classes $\beta_{\lambda,\lambda'}$.

Let us continue our search for relations. The following assertion depends crucially on assumption (**B**).

Lemma 3.10. The elements $\beta_{\lambda,\lambda'} \in DB_{\mathbf{k}}(\Sigma,\Gamma)$ are central.

Proof. It is required to prove that the $\alpha_{\lambda,\ell}$ commute with the $\beta_{\mu,\nu}$ (for all possible labels). Since $k_\mu \geq 2$, there is at least another basepoint $z_e \neq z_{i_\lambda}$ of colour μ. Now represent $\beta_{\mu,\nu}$ with a monic braid got from a path γ in $\Sigma \setminus \cup_{j\neq i_\mu}\{z_j\}$ starting from this z_e and avoiding the path used to define $\alpha_{\lambda,\ell}$; this commutes with $\alpha_{\lambda,\ell}$ by Lemma 3.6–(i). □

Note that the classes $\beta_{\lambda,\mu}$ were constructed from strands projecting onto a disc in Σ, so they do not depend on (the genus g of) Σ. Let $E_{\mathbf{k}}(\Gamma)$ denote the group they generate.

Theorem 3.11. $DB_{\mathbf{k}}(\Sigma,\Gamma)$ *sits in a central extension*

$$0 \longrightarrow E_{\mathbf{k}}(\Gamma) \longrightarrow DB_{\mathbf{k}}(\Sigma,\Gamma) \xrightarrow{h} H_1(\Sigma;\mathbb{Z})^{\oplus r} \longrightarrow 0 \tag{8}$$

where component λ of h sums the 1-cycles on Σ in colour λ.

Proof. Without loss of generality, we take $g \geq 1$. Certainly $E_{\mathbf{k}}(\Gamma) \subset \ker(h)$, and h factors through the quotient $DB_{\mathbf{k}}(\Sigma, \Gamma) \to DB_{\mathbf{k}}(\Sigma, \Gamma)/E_{\mathbf{k}}(\Gamma)$; so it is only required to prove that the induced map $\bar{h}$ from this quotient to $H_1(\Sigma; \mathbb{Z})^{\oplus r}$ is an isomorphism. It is clearly surjective, since the $h(\alpha_{\lambda,\ell})$ with λ fixed generate $H_1(\Sigma; \mathbb{Z})$; note also $H_1(\Sigma; \mathbb{Z})^{\oplus r}$ has rank greater or equal to that of the quotient. □

The next result summarises what we can already tell about the classes $\beta_{\lambda,\mu}$ with $1 \leq \lambda, \mu \leq k$.

Theorem 3.12. *The $\beta_{\lambda,\mu}$ satisfy:*

(i) $\beta_{\lambda,\mu} = 0$ *if there is no edge in* Γ *connecting* λ *and* μ*;*

(ii) $\beta_{\lambda,\lambda} = 0$*;*

(iii) $\sum_{\mu \neq \lambda} k_\mu \beta_{\lambda,\mu} = 0$.

Proof of (iii): Consider a path γ starting at z_{i_λ}, going straight to the boundary of the $4g$-gon, then around that boundary, and back to z_{i_λ}; it is a product of commutators in $\pi_1(\Sigma \setminus U_{j \neq i_\lambda}\{z_j\}, z_{i_\lambda})$, hence $\Phi(\gamma)$ is trivial. On the other hand γ encircles each z_j with $j \neq i_\lambda$ exactly once, so $\Phi(\gamma)$ represents the sum given. □

For concreteness, it is useful to consider a simple example in two colours; so we take the complete graph $\Gamma_{(2)} = \bullet\!\!-\!\!\!-\!\!\bullet$ (in other words, we go back to $\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma)$ in Example 2.1), and assume $k_+, k_- > 1$. As generators for $DB_{(k_+,k_-)}(\Sigma, \Gamma_{(2)})$, we want to streamline our previous notation and rather take

$$a_1, a_1', \ldots a_g, a_g' \qquad \text{and} \qquad a_1, a_1', \ldots a_g, a_g';$$

think of these as images of monic braids for generators of $\pi_1(\Sigma)$ based at far-enough points $*, * \in \Sigma$ around which the blue and red strand basepoints cluster. Here, subscript labels refer to *handles* of Σ, and the primed and unprimed generators for a given handle refer to symplectic conjugates in a canonical basis for $H_1(\Sigma; \mathbb{Z})$. We have also chosen to drop the colour labels and instead paint the generators explicitly. So the images of these generators under abelianisation (i.e. under the components of the map h in 8) yield generators of the two copies $H_1(\Sigma; \mathbb{Z}) \oplus H_1(\Sigma; \mathbb{Z})$. Let us now make a list of all the relations that can be written from the results obtained so far:

$$\begin{array}{ll} [a_i, a_j] = [a_i, a'_j] = [a'_i, a'_j] = e & \forall\, i, j \text{ by Lemma 3.6–(ii)}, \\ [a_i, a_j] = [a_i, a'_j] = [a'_i, a'_j] = e & \forall\, i, j \text{ by Lemma 3.6–(ii)}, \\ [a_i, a_j] = [a'_i, a'_j] = e & \forall\, i, j \text{ by Lemma 3.6–(i)}, \\ [a_i, a'_j] = e & \forall\, i \neq j \text{ by Lemma 3.6–(i)}, \\ [a_i, a'_i] = c & \forall\, i \text{ by Lemma 3.8.} \end{array}$$

To be precise: in the last line, we labelled the commutator of pre-images of symplectic conjugates (in the singular 1-homology of Σ) of different colours by c, and noted that it does not depend on the handle label. Recall that c corresponds to the divisor braid depicted in Fig 6. What else can we say about this element c? From what we have learned until now,

- c is central, by Lemma 3.10;
- $\mathrm{ord}(c)$ divides both k_+ and k_-, by Theorem 3.12–(iii).

However, we still need to know something very crucial about c, namely:

- is $c = e$?

We invite our readers to convince themselves that the answer to this question is, in fact, impossible to deduce from the results in this section.

Remark 3.13. If assumption (**B**) is relaxed in this example, i.e. $k_\lambda = 1$ for some $\lambda \in \mathrm{Sk}^0(\Gamma_{(2)})$, then the corresponding divisor braid group is trivial — and in particular $c = e$. If $\mathbf{k}$ is the constant 1, this statement follows from $PB_2(\Sigma)$ being the trivial group; otherwise, we need to combine the usual proof of this fact with the move on two strands depicted in Fig. 6.

4 A link invariant and metabelian presentations

We will argue that $c \neq e$ is the most general answer to the question asked before Remark 3.13; in fact, we will compute the order of the element c. For this, we shall construct a link invariant for divisor braids with graph $\Gamma = \Gamma_{(2)} = \bullet\!-\!\bullet$ and then calculate its value on the commutator c.

Let us consider the oriented 3-manifold $M := S^1 \times \Sigma$, with natural projections $p_1 : M \to S^1$ and $p_2 : M \to \Sigma$. Suppose that a pair (ℓ_+, ℓ_-) of closed braids of degree (k_+, k_-) is given; this means that $[\ell_\pm] \in H_1(M; \mathbb{Z})$ satisfy $p_{1*}[\ell_\pm] = k_\pm [S^1]$. Let us set

$$\bar{k} := \gcd(k_+, k_-)$$

and assume that:

- the images $\bar{\ell}_+, \bar{\ell}_- \subset M$ are disjoint;
- $p_{2*}[\ell_\pm] = 0 \in H_1(\Sigma; \mathbb{Z})$.

We consider $\bar{\ell}_- \subset M$ and the homology long exact sequence [23] for the pair $(M, M \setminus \bar{\ell}_-)$ with coefficients in $\mathbb{Z}_{\bar{k}} := \mathbb{Z}/\bar{k}\mathbb{Z}$,

$$\cdots \to H_2(M; \mathbb{Z}_{\bar{k}}) \xrightarrow{\psi} H_2(M, M \setminus \bar{\ell}_-; \mathbb{Z}_{\bar{k}}) \xrightarrow{\partial} H_1(M \setminus \bar{\ell}_-; \mathbb{Z}_{\bar{k}}) \xrightarrow{\varphi} H_1(M; \mathbb{Z}_{\bar{k}}) \to \cdots$$

Since $p_{1*}[\ell_+] = k_+[S^1]$ and $p_{2*}[\ell_+] = 0$, Künneth's formula implies that φ maps $[\ell_+] \in H_1(M \setminus \bar{\ell}_-; \mathbb{Z})$ as

$$[\ell_+] \overset{\varphi}{\longmapsto} (k_+ (\mathrm{mod}\, \bar{k}), 0 (\mathrm{mod}\, \bar{k})) = (0, 0) \in H_1(S^1; \mathbb{Z}_{\bar{k}}) \oplus H_1(\Sigma; \mathbb{Z}_{\bar{k}}).$$

Exactness at $H_1(M \setminus \bar{\ell}_-; \mathbb{Z}_{\bar{k}})$ now ensures that

$$[\ell_+] \in \text{coker } \psi = H_2(M, M \setminus \bar{\ell}_-; \mathbb{Z}_{\bar{k}})/\mathrm{im}(\psi). \tag{9}$$

We now invoke a somewhat exotic form of Poincaré duality known as Poincaré–Lefschetz duality, see Corollary VI.8.4 in [10]. This yields a commutative diagram

$$\begin{array}{ccccc} H^1(M; \mathbb{Z}_{\bar{k}}) & \xrightarrow{\psi'} & H^1(\bar{\ell}_-; \mathbb{Z}_{\bar{k}}) & \xrightarrow{\delta} & H^2(M, \bar{\ell}_-; \mathbb{Z}_{\bar{k}}) \\ \cong \downarrow D_M & & \cong \downarrow D_{\bar{\ell}_-} & & \cong \downarrow D_{M, \bar{\ell}_-} \\ H_2(M; \mathbb{Z}_{\bar{k}}) & \xrightarrow{\psi} & H_2(M, M \setminus \bar{\ell}_-; \mathbb{Z}_{\bar{k}}) & \xrightarrow{\partial} & H_1(M \setminus \bar{\ell}_-; \mathbb{Z}_{\bar{k}}) \end{array} \tag{10}$$

The lower row is the part of the homology long exact sequence for the pair $(M, M \setminus \bar{\ell}_-)$ above, while the upper row is part of the cohomology exact sequence [23] (cf. 3.1, p 199) corresponding to the pair $(M, \bar{\ell}_-)$. The vertical maps are isomorphisms, the leftmost being the usual Poincaré duality map.

Let us now consider the map

$$\langle \cdot, [\ell_-] \rangle : H^1(\bar{\ell}_-; \mathbb{Z}_{\bar{k}}) \to \mathbb{Z}_{\bar{k}}$$

evaluating the pairing at the generator $[\ell_-] \in H_1(\bar{\ell}_-; \mathbb{Z}_{\bar{k}})$. This vanishes on $\mathrm{im}(H^1(M; \mathbb{Z}_{\bar{k}}) \to H^1(\bar{\ell}_-; \mathbb{Z}_{\bar{k}}))$, since (once again by Künneth) it vanishes on $[S^1]$ (as $k_- \equiv 0 (\mathrm{mod}\, \bar{k})$), and thus it is well defined on the cokernel of ψ'. Therefore we can evaluate it on the class $D_{\bar{\ell}_-}^{-1}[\ell_+]$ interpreted as in (9), and conclude that

$$\langle (\ell_+, \ell_-) \rangle := \langle D_{\bar{\ell}_-}^{-1}[\ell_+], [\ell_-] \rangle \in \mathbb{Z}_{\bar{k}}$$

is a well-defined *link invariant*, which is useful in our context.

Considering the link representing the divisor braid c depicted in Fig 6, and taking into account that we are evaluating at generators in each colour, we have that

$$\langle(\ell_+, \ell_-)\rangle \in \mathbb{Z}_{\bar{k}}$$

is in fact a generator of the group $\mathbb{Z}_{\bar{k}}$. It follows that c is an element of $DB_{(k_+,k_-)}(\Sigma, \Gamma_{(2)})$ of order $\gcd(k_+, k_-)$, generating its centre. More precisely, we can characterise the divisor braid group of the two-colour example at the end of section 3, and connecting to Example 2.1, as follows.

Theorem 4.1. *The divisor braid group*

$$DB_{(k_+,k_-)}(\Sigma, \Gamma_{(2)}) \cong \pi_1 \left(\mathcal{M}^{\mathbb{P}^1}_{(k_+,k_-)}(\Sigma) \right)$$

sits in a central extension (or abelianisation short exact sequence)

$$0 \longrightarrow \mathbb{Z}_{\gcd(k_+,k_-)} \longrightarrow DB_{(k_+,k_-)}(\Sigma, \Gamma_{(2)}) \longrightarrow H_1(\Sigma; \mathbb{Z})^{\oplus 2} \longrightarrow 0.$$

A group whose commutator is Abelian is sometimes called *metabelian*; thus a short exact sequence like the one in Theorem 3.11 is called a metabelian presentation. It is not hard to classify the representations of these groups. For instance, In the concrete example of Theorem 4.1, the representation varieties have $\gcd(k_+, k_-)$ connected components, which are copies of a $2g$-torus if g is the genus of Σ. Interestingly: in the non-coprime case, there are representations that do not factor through the abelianisation if $g > 0$. This simple fact has a remarkable physical consequence: the $\mathbb{P}^1$-model, an Abelian gauge theory which can be regarded as the simplest nonlinear extension of the familiar Abelian Higgs model, may support nonabelian anyons [36] in positive genus.

At this point one might wonder what can be said of divisor braid groups in general; by Theorem 3.11, under the simplification assumptions (**A**) and (**B**), they are still metabelian, but to which extent are their commutator subgroups nontrivial? Is it possible to construct, for more general graphs Γ, enough link invariants to rescue the elements $\beta_{\lambda,\mu}$ of the group $E_{\mathbf{k}}(\Gamma)$ in Theorem 3.11 from being trivial?

Remarkably, the answer is yes! The construction of a relevant Γ-*linking number* is more technical than the link invariant constructed above, and involves a tool from homological algebra called the Eilenberg–Zilber functor [17]. We refer the reader to [8] for an account of the general construction. The upshot is that there is a presentation of the group $E_{\mathbf{k}}(\Gamma)$ with generators $\beta_{\lambda,\mu}$ and relations

(i) $\beta_{\lambda,\mu} - \beta_{\mu,\lambda}$ for $1 \leq \lambda < \mu \leq r$;

(ii) $\beta_{\lambda,\mu}$ if there is no edge between λ and μ in Γ;

(iii) $P_\mu := \sum_{\lambda \neq \mu} k_\lambda \beta_{\lambda,\mu}$ for $1 \leq \mu \leq r$.

We conclude this section by precisely stating a presentation of divisor braid groups extending the results above, and also proved in [8]. Assuming $k_\lambda \geq 2$ for all $\lambda \in \mathrm{Sk}^0(\Gamma)$, and Σ oriented compact of genus $g \geq 0$, let us suppose that $a_{\lambda,\ell}$ are λ-coloured copies of elements a_ℓ in a basis of $H_1(\Sigma;\mathbb{Z})$.

Theorem 4.2. *The divisor braid group $DB_{\mathbf{k}}(\Sigma,\Gamma)$ is isomorphic to the group generated by $a_{\lambda,\ell}, b_{\lambda,\mu}$ $(1 \leq \ell \leq 2g,\ 1 \leq \lambda, \mu \leq r)$ and relations*

(i) $b_{\lambda,\mu} b_{\lambda',\mu'} = b_{\lambda',\mu'} b_{\lambda,\mu}$;

(ii) $b_{\lambda,\mu} = e$ *if no edge in* Γ *connects* λ *and* μ;

(iii) $b_{\lambda,\mu} = b_{\mu,\lambda}$;

(iv) $\prod_{\mu \neq \lambda} b_{\lambda,\mu}^{k_\mu} = e$;

(v) $b_{\lambda,\mu} a_{\nu,\ell} = a_{\nu,\ell} b_{\lambda,\mu}$;

(vi) $a_{\lambda,\ell} a_{\mu,\ell'} a_{\lambda,\ell}^{-1} a_{\mu,\ell'}^{-1} = b_{\lambda,\mu}^{\sharp(a_\ell, a_{\ell'})}$.

In (vi), $\sharp(\cdot,\cdot)$ *denotes the intersection pairing in* $H_1(\Sigma;\mathbb{Z})$.

5 Describing the centre of divisor braid groups

One can state more precisely how the finitely generated abelian group $E_{\mathbf{k}}(\Gamma)$ depends on both the graph Γ and its decoration $\mathbf{k}$. Recall that a graph is *bipartite* if its vertices can be consistently assigned opposite signs across all edges. For instance, the graph $\Gamma_{(2)}$ is bipartite but neither $\Gamma_{(r)}$ with $r \geq 3$ nor the graph illustrated by Fig. 3 (attending to the triangles formed by the edges attaching to the walls) are. The cleanest result on $E_{\mathbf{k}}(\Gamma)$ is about its rank; it depends on Γ alone.

Theorem 5.1. *Suppose Γ has connected components Γ_i, and that:*

- r *is the total number of vertices;*
- s *is the total number of edges;*

- *t is the number of components Γ_i that are bipartite.*

Then the rank of $E_{\mathbf{k}}(\Gamma)$ is $s - r + t$.

Sketch of proof. This can be neatly rephrased as a calculation of the dimension of the cokernel of a map $d_\Gamma : \mathcal{C}^0(\Gamma) \to \mathcal{C}^1(\Gamma)$ between linear spaces spanned by vertices and edges, assigning to each vertex the sum of incident edges in Γ,

$$d_\Gamma(\lambda) := \sum_{\varepsilon \text{ incident at } \lambda} \varepsilon. \tag{11}$$

See [8] for all the details. □

Can we also write down a formula for the torsion of $E_{\mathbf{k}}(\Gamma)$ in general? In other words: given a prime number p and $n \in \mathbb{N}$, can we say how many times the cyclic group $\mathbb{Z}_{p^n}$ appears in the primary decomposition of the finitely generated Abelian group $\operatorname{Tor} E_{\mathbf{k}}(\Gamma)$? As it turns out, this is a highly nontrivial problem.

One can recast this problem in purely arithmetic terms, as follows. Let us set $\mathbf{c} = (c_\lambda)_{\lambda \in \mathrm{Sk}^0(\Gamma)} = (c_1, \ldots, c_r)$ with components $c_\lambda \in \mathbb{Q}/\mathbb{Z}$, and define

$$C_{\mathbf{k}}(\Gamma) := \{\mathbf{c} \in (\mathbb{Q}/\mathbb{Z})^{\oplus r} \,|\, k_\lambda c_\mu + k_\mu c_\lambda \equiv 0 \text{ if } \exists \text{ edge between } \lambda \text{ and } \mu\}.$$

This is a group formed by the solutions of a linear system of rational congruences, one for each edge in Γ and with as many unknowns as vertices.

The linear system in the definition of $C_{\mathbf{k}}(\Gamma)$ is equivalent to the collection of (more conventional, Diophantine) linear systems

$$k_\lambda x_\mu + k_\mu x_\lambda \equiv 0 \,(\operatorname{mod} p^n) \quad \text{if } \exists \text{ edge between } \lambda \text{ and } \mu \tag{12}$$

for all primes p and integers $n \in \mathbb{N}$. Solving (12) is an elementary problem[1], yielding a collection of Abelian groups (indexed by p and n) that can then be assembled into $C_{\mathbf{k}}(\Gamma)$ via standard homological algebra on the map (11) and the short exact sequence $\mathbb{Z} \hookrightarrow \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. For instance, let us assume that an odd prime p does not divide any of the integers k_λ. Then the solutions of the system (12) form a group $\mathbb{Z}_{p^n}^{\oplus t}$, where t is again the number of bipartite components of Γ. Each bipartite component Γ_i contributes with a copy of

[1]As historical aside, we note that linear Diophantine systems such as (12) were first discussed systematically in the mid 19th century (in work [16] published in Portuguese that was largely ignored) by Daniel Augusto da Silva, a pioneer of discrete mathematics [19] at a time when Portuguese mathematicians circulated less globally.

$\mathbb{Z}_{p^n}$ generated by the solution obtained from extending a local solution of the form

$$(x_\lambda, x_\mu) \equiv (k_\mu^{\phi(p^n)-1}, -k_\lambda^{\phi(p^n)-1}) = (k_\mu^{p^n-p^{n-1}-1}, -k_\lambda^{p^n-p^{n-1}-1}) \pmod{p^n}$$

(on the edge of Γ_i connecting the vertices λ and μ) across all edges of Γ_i using appropriate $\pm$ signs determined by the bipartitioning, and setting $x_\nu = 0$ for vertices ν in other components. When p appears as factor in one of the k_λ, the problem becomes more complicated. In particular, it is not easy to understand how the properties of the graph Γ are reflected in the order of the group of solutions to (12). An effort to address this problem (keeping Γ fixed but allowing $\mathbf{k}$ to vary) has led to a novel type of graph cohomology [7].

If Γ is bipartite and connected, we note that there is a cyclic subgroup $\Delta_{\mathbf{k}}(\Gamma) \subset C_{\mathbf{k}}(\Gamma)$ generated by a solution of the form

$$\left(\left[\pm\frac{1}{k_\lambda}\right]\right)_{\lambda\in \mathrm{Sk}^0(\Gamma)},$$

where the bipartitioning is again employed to distribute the $\pm$ signs along the whole graph. In [8] we prove

Theorem 5.2. *If Γ is connected, the linear system above determines*

$$\mathrm{Tor}\, E_{\mathbf{k}}(\Gamma) \cong \begin{cases} C_{\mathbf{k}}(\Gamma)/\Delta_{\mathbf{k}}(\Gamma) & \textit{if } \Gamma \textit{ is bipartite;} \\ C_{\mathbf{k}}(\Gamma) & \textit{if } \Gamma \textit{ is not bipartite.} \end{cases}$$

A few examples illustrating of how wildly the groups $\mathrm{Tor}\, E_{\mathbf{k}}(\Gamma)$ can vary when the charges k_λ are changed, for a given graph Γ e.g. coming from toric data as in section 2 (see Theorem 3.1), may also be found in the paper [8].

Acknowledgements

NMR would like to thank the organisers of the stimulating meeting *Global Portuguese Mathematicians III* at the University of Coimbra in July 2023 for their very kind invitation to speak. He also thanks Carlos Florentino for drawing his attention to reference [19]. This review paper is partly a result of the ERC-SyG project Recursive and Exact New Quantum Theory (ReNewQuantum), which received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement N° 810573.

References

[1] M. Abreu, “Kähler geometry of toric manifolds in symplectic coordinates”, in: Ya. Eliashberg, B. Khesin and F. Lalonde (Eds.), *Symplectic and Contact Topology: Interactions and Perspectives*, American Mathematical Society, 2003

[2] J.M. Baptista, “Vortex equations in abelian gauged σ-models”, *Commun. Math. Phys.* **261** (2006), pp. 161–194

[3] J.M. Baptista, “On the L^2 metric of vortex moduli spaces”, *Nucl. Phys. B* **844** (2011), pp. 308–333

[4] A. Bertram, G. Daskalopolous and R. Wentworth, “Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians”, *J. Amer. Math. Soc.* **5** (1994), pp. 861–875

[5] J. Birman, *Braids, Links and Mapping Class Groups*, Princeton University Press, 1975

[6] I. Biswas and N.M. Romão, “Moduli of vortices and Grassmann manifolds”, *Commun. Math. Phys.* **320** (2013), pp. 1–20

[7] M. Bökstedt, “Configuration graph cohomology”, `arXiv:1711.04624`

[8] M. Bökstedt and N.M. Romão, “Divisor braids”, `arXiv:1605.07921`

[9] M. Bökstedt and N.M. Romão, “Moduli spaces of vortices in toric fibre bundles”, preprint

[10] G.E. Bredon, *Topology and Geometry*, Springer, 1993

[11] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Springer, 2001

[12] A. Cannas da Silva, “An invitation to symplectic toric manifolds”, *Bol. Soc. Port. Mat.* **77** (2019), pp. 119–132

[13] Ö. Ceyhan and M. Marcolli, “Feynman integrals and motives on configuration spaces”, *Commun. Math. Phys.* 313 (2012), pp. 35–70

[14] K. Cieliebak, A.R. Gaio, I. Mundet i Riera and D.A. Salamon, “The symplectic vortex equations and invariants of Hamiltonian group actions”, *J. Symplectic Geom.* **1** (2002), pp. 543–645

[15] D.A. Cox, J.B. Little and H.K. Schenck, *Toric Varieties*, American Mathematical Society, 2011

[16] D.A. da Silva, "Propriedades geraes e resolução directa das congruencias binomias: introducção ao estudo da theoria dos numeros", *Memorias da Real Academia das Sciencias de Lisboa* (1ª Classe, Nova Serie) Tomo I (1854), pp. 1–168

[17] S. Eilenberg and J.A. Zilber, "On products of complexes", *Amer. J. Math. Phys.* **75** (1953), pp. 200–204

[18] D. Eriksson and N.M. Romão, "Kähler quantization of vortex moduli", *Lett. Math. Phys.* **110** (2020), pp. 659–693

[19] C.A.A. Florentino, "Daniel Augusto da Silva: Poet of Mathematics", *CIM Bulletin* **40** (2018), pp. 11–18; `arXiv:1812.06267`

[20] J. González-Meneses, "New presentations of surface braid groups", *J. Knot Theory Ram.* **10** (2001), pp. 431–451

[21] P.B. Gothen, "The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface", *Internat. J. Math.* **5** (1994), pp. 861–875

[22] V. Guillemin, V. Ginzburg and Y. Karshon, *Moment Maps, Cobordisms, and Hamiltonian Group Actions*, American Mathematical Society, 2002

[23] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2009

[24] A. Jaffe and C. Taubes, *Vortices and Monopoles: Structure of Static Gauge Theories*, Birkhäuser, 1980

[25] D. McDuff and D. Salamon, *J-holomorphic Curves and Symplectic Topology*, American Mathematical Society, 2004

[26] M. Melo, "Tropicalizing moduli spaces and applications", *Bol. Soc. Port. Mat.* **83** (2025), pp. 109–123

[27] M. Moreira, "A survey on the Virasoro constraints in moduli spaces of sheaves and quiver representations", *Bol. Soc. Port. Mat.* **83** (2025), pp. 125–143

[28] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, 3rd edition, Springer, 1994

[29] I. Mundet i Riera, *Yang–Mills–Higgs Theory for Symplectic Fibrations*, PhD Thesis, Universidad Autónoma de Madrid, 1999

[30] R. Pandharipande, "Reflections on moduli space", handout to the Hirzebruch Lecture in May 2023 at the University of Bonn, available from the link `https://www.mpim-bonn.mpg.de/webfm_send/786`

[31] N.M. Romão and J.M. Speight, "Slow Schrödinger dynamics of gauged vortices", *Nonlinearity* **17** (2004), pp. 1337–1355

[32] N.M. Romão and J.M. Speight, "The geometry of the space of BPS vortex-antivortex pairs", *Commun. Math. Phys.* **379** (2020), pp. 723–772

[33] T. Samols, "Vortex scattering", *Commun. Math. Phys.* **145** (1992), pp. 149–179

[34] B.J. Schroers, "Bogomol'nyi solitons in a gauged $O(3)$ model", *Phys. Lett. B* **356** (1995), pp. 291–296

[35] L. Sibner, R. Sibner and Y. Yang, "Abelian gauge theory on Riemann surfaces and new topological invariants", *Proc. Royal Soc. London A* **456** (2000), pp. 593–613

[36] S. Simon, *Topological Quantum*, Oxford University Press, 2023

[37] I.A.B. Strachan, "Low-velocity scattering of vortices in a modified Abelian Higgs model", *J. Math. Phys.* **33** (1992), pp. 102–110

[38] D.M.A. Stuart, "Analysis of the adiabatic limit for solitons in classical field theory", *Proc. R. Soc. A* **463** (2020), pp. 723–772

[39] M. Thaddeus, "Stable pairs, linear systems and the Verlinde formula", *Invent. Math.* **117** (1994), pp. 317–353

[40] E. Walton, "Vortex counting and the quantum Hall effect", *JHEP* **02** (2022), 155

[41] E. Witten, "Some exact multipseudoparticle solutions of classical Yang–Mills theory", *Phys. Rev. Lett.* **38** (1977), pp. 121–124