EDUCAÇÃO E DESENVOLVIMENTO SOCIAL
EDUCATION AND SOCIAL DEVELOPMENT
EDUCACIÓN Y DESAROLLO SOCIAL

millenium

Millenium, 2(Edição Especial Nº18)

en

EXPLORANDO O LÓCUS DAS ENVOLTÓRIAS: UMA ABORDAGEM INICIAL COM GEOGEBRA
EXPLORING THE LOCUS OF ENVELOPES: AN INITIAL APPROACH WITH GEOGEBRA
EXPLORANDO EL LUGAR GEOMÉTRICO DE LAS ENVOLVENTES: UNA APROXIMACIÓN INICIAL CON GEOGEBRA

Ana Paiva[1] *https://orcid.org/0000-0001-5801-9562*
Francisco Alves[1] *https://orcid.org/0000-0003-3710-1561*
Helena Campos[2,3] *https://orcid.org/0000-0003-2767-0998*

[1] Instituto Federal de Educação, Ciência e Tecnologia do Ceará – IFCE, Fortaleza, Brasil
[2] Universidade de Trás-os-Montes e Alto Douro- UTAD, Vila Real, Portugal
[3] Centro de Investigação em Didática e Tecnologia na Formação de Formadores, Vila Real, Portugal

Ana Paiva - carlapimentel00@gmail.com | Francisco Alves - fregis@ifce.edu.br | Helena Campos - hcampos@utad.pt

Corresponding Author:
Ana Paiva
Rua José Candido 327
60325490 – Fortaleza - Brasil
carlapimentel00@gmail.com

RECEIVED: 08th December, 2024
REVIEWED: 06th June, 2025
ACCEPTED: 12th June, 2025
PUBLISHED: 30th July, 2025

DOI: https://doi.org/10.29352/mill0218e.39468

RESUMO

Introdução: O ensino de Equações Diferenciais Ordinárias (EDOs) apresenta desafios relevantes, especialmente devido à predominância de métodos algébricos que frequentemente limitam a exploração de comportamentos qualitativos. O conceito matemático de envoltória, entendido como o lugar geométrico descrito por uma família de curvas, oferece uma abordagem visual e estrutural que pode enriquecer o ensino de EDOs de forma mais intuitiva e exploratória.
Objetivo: Este estudo tem como objetivo desenvolver uma proposta de ensino estruturada, centrada na integração do conceito de envoltória ao contexto das EDOs. Busca-se examinar de que maneira esse conceito, aliado às representações dinâmicas proporcionadas pelo software GeoGebra, pode apoiar e ampliar o ensino dos aspectos qualitativos das equações diferenciais.
Métodos: A metodologia adotada é a Engenharia Didática, com ênfase nas duas primeiras fases: Análises Preliminares e Concepção com Análise a Priori. O estudo inclui a revisão de fundamentos teóricos, a análise do contexto de ensino e a elaboração de uma sequência didática. O GeoGebra é utilizado como ferramenta digital para a construção de representações gráficas dinâmicas que auxiliam na abordagem conceitual do tema.
Conclusão: O resultado esperado é a elaboração de uma sequência de ensino fundamentada teoricamente, que articule o rigor conceitual a estratégias visuais. Embora a implementação ainda não tenha sido realizada, a proposta visa servir como base para futuras em sala de aula.

Palavras-chave: envoltórias; Equações Diferenciais Ordinárias; Engenharia Didática (ED); GeoGebra

ABSTRACT

Introduction: The teaching of Ordinary Differential Equations (ODEs) presents significant challenges, particularly due to the predominance of algebraic methods that often limit the exploration of qualitative behaviors. The mathematical concept of an envelope, understood as the geometric locus formed by a family of curves, offers a visual and structural approach that may enrich the teaching of ODEs in a more intuitive and exploratory way.
Objective: This study aims to develop a structured teaching proposal focused on integrating the concept of envelopes into the context of ODEs. It seeks to examine how this concept, combined with the dynamic representations provided by the GeoGebra software, can support and enhance the teaching of the qualitative aspects of differential equations.
Methods: The adopted methodology is Didactical Engineering, with emphasis on the first two phases: Preliminary Analysis and Conception with A Priori Analysis. The study includes a review of theoretical foundations, an analysis of the instructional context, and the design of a didactic sequence. GeoGebra is employed as a digital tool for constructing dynamic graphical representations that assist in the conceptual development of the topic.
Conclusion: The expected result is the development of a theoretically grounded teaching sequence that integrates conceptual rigor with visual strategies. Although implementation has not yet been carried out, the proposal is intended to serve as a foundation for future classroom applications.

Keywords: envelopes; Ordinary Differential Equations; Didactical Engineering; GeoGebra

RESUMEN

Introducción: La enseñanza de Ecuaciones Diferenciales Ordinarias (EDO) presenta desafíos relevantes, especialmente debido a la predominancia de métodos algebraicos que frecuentemente limitan la exploración de comportamientos cualitativos. El concepto matemático de envolvente, entendido como el lugar geométrico descrito por una familia de curvas, ofrece un enfoque visual y estructural que puede enriquecer la enseñanza de EDO de forma más intuitiva y exploratoria.
Objetivo: Este estudio tiene como objetivo desarrollar una propuesta de enseñanza estructurada, centrada en la integración del concepto de envolvente al contexto de las EDO. Se busca examinar de qué manera este concepto, aliado a las representaciones dinámicas proporcionadas por el software Geogebra, puede apoyar y ampliar la enseñanza de los aspectos cualitativos de las ecuaciones diferenciales.
Métodos: La metodología adoptada es la Ingeniería Didáctica, con énfasis en las dos primeras fases: Análisis Preliminares y Concepción con Análisis a Priori. El estudio incluye la revisión de fundamentos teóricos, el análisis del contexto de enseñanza y la elaboración de una secuencia didáctica. GeoGebra se utiliza como herramienta digital para la construcción de representaciones gráficas dinámicas que auxilian en el abordaje conceptual del tema.
Conclusion: El resultado esperado es la elaboración de una secuencia de enseñanza fundamentada teóricamente, que articule el rigor conceptual con estrategias visuales. Aunque la implementación aún no se ha realizado, la propuesta busca servir como base para futuras aplicaciones en el aula.

Palabras Clave: envolventes; Ecuaciones Diferenciales Ordinarias; ingeniería didáctica; GeoGebra

DOI: https://doi.org/10.29352/mill0218e.39468

INTRODUCTION

The teaching of Ordinary Differential Equations (ODEs) presents significant challenges in the education of students in Mathematics and related disciplines. Although ODEs play a fundamental role in modeling phenomena across various fields of knowledge, such as Physics, Engineering, Biology, and Economics, the traditional approach—primarily algebraic and formal—often limits the development of a qualitative understanding of the subject. Therefore, it is necessary to propose didactic alternatives that incorporate visual representations and qualitative strategies in the teaching process.

This study is part of the development of a teaching proposal focused on introducing the concept of the envelope in the context of ODEs. The emphasis is placed on the graphical and structural analysis of solutions. Envelopes, understood as the geometric locus generated by a family of curves, enable the visualization of patterns and trends emerging from the solutions of a differential equation. They offer teachers a conceptual and graphical tool to approach the subject in a more intuitive and exploratory manner.

The teaching proposal under development is based on the Didactical Engineering methodology, which provides a systematic framework for designing and analyzing pedagogical interventions in mathematics education. This study addresses only the first two phases of the methodological process. The Preliminary Analysis phase involves reviewing theoretical foundations and investigating the teaching context in order to identify common difficulties and potential didactic strategies. The Conception and A Priori Analysis phase consists of designing a teaching sequence based on the initial analysis and anticipating possible student responses and the mathematical knowledge that may be mobilized during its implementation.

The proposal is developed using the GeoGebra software, which offers a graphical environment and dynamic tools for constructing visual representations that support the exploration of ODE solutions. The objective is to structure a teaching approach that integrates conceptual rigor with didactic strategies grounded in visualization and qualitative analysis, thus enhancing the methodological repertoire for teaching ODEs. The implementation utilized GeoGebra software, version 6.0, which provided a robust and versatile environment for the development of simulations and visual aids.

1. AN ANALYSIS OF THE ALGORITHM TO DETERMINE ENVELOPE DISCOVERY

The equations that describe the envelopes arise as solutions of ordinary differential equations that are related to the family of curves. However, to derive the envelope equation, it is crucial to identify the condition under which the family of curves has a singular solution.

Definition: A curve $y = \varphi(x)$, whose points (x,ϕ(x)) are on the uniparametric family f(x,y,α)=0 and whose tangent at each point is tangent to a curve of this family is called the envelope (De Figueiredo and Neves, 2018). According to Vilches (2009, p.29), the envelope can be identified as a system solution:

$$\begin{cases} f(x,y,\lambda) = 0 & eq.(1) \\ \dfrac{df}{d\lambda}(x,y,\lambda) = 0 & eq.(2) \end{cases}$$

equation (1) consists of the family of curves through the function f, while equation (2) corresponds to the condition under which the family of solution curves has a singular solution.

It may be that the relative envelope for a given uniparametric family of curves does not exist.

1ºExample: The family of straight $f(x,y,\lambda) = xsen(\lambda) + y\cos(\lambda) - dcos(\lambda)sen(\lambda)$ with $cos(\lambda)sen(\lambda) \neq 0$, and the differential function $\frac{df}{d\lambda} = xcos(\lambda) - ysen(\lambda) - dcos(2\lambda)$, solving the system:

$$\begin{cases} f(x,y,\lambda) = xsen(\lambda) + y\cos(\lambda) - dcos(\lambda)sen(\lambda) = 0 & eq.(3) \\ \dfrac{df}{d\lambda} = xcos(\lambda) - ysen(\lambda) - dcos(2\lambda) = 0 & eq.(4) \end{cases}$$

The solution of system is $x = dcos^3(\lambda)$ and $y = dsen^3(\lambda)$, that the family envelope describes the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = d^{\frac{2}{3}}$. To better understand this family of curves, and what the envelope would be, we will use GeoGebra to visualize these concepts, as explained in figure 1.

DOI: https://doi.org/10.29352/mill0218e.39468

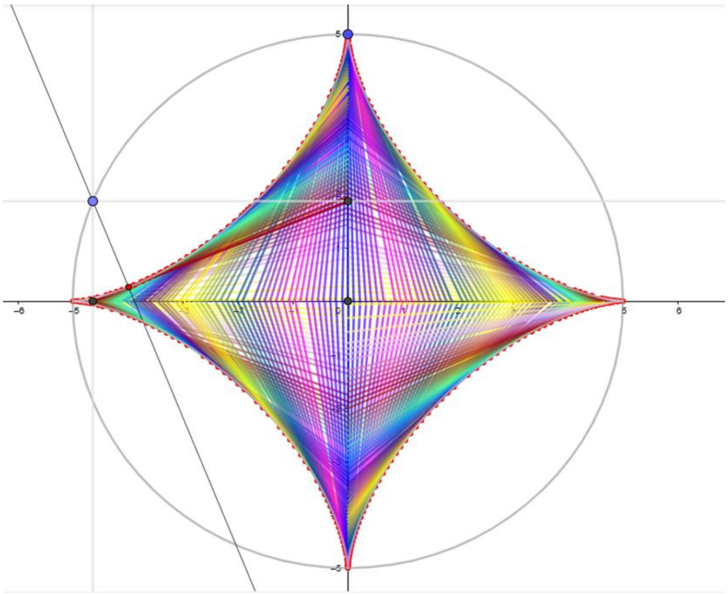

Figure 1- Representation of the envelope of the curve family the $f(x,y,\lambda) = xsen(\lambda) + y\,cos(\lambda) - dcos(\lambda)sen(\lambda)$ developed by the authors

Through this example, the geometric meaning of these solutions is illustrated, which establishes that if we have a uniparametric family of curves that associates a curve $f(x,y,\lambda) = 0$ to each $\lambda \in \mathbb{R}$, we have a real application in the space of functions: $\lambda \mapsto f(x,y,\lambda) = 0$. In which this application and each curve $f(x,y,\lambda) = 0$ will be held by those *involved* (De Figueiredo and Neves, 2018).

2ºExample: To support the understanding of the algorithm to determine the envelope equation, given the family of curves $x^2 + y^2 + 2(\alpha+2)y + \alpha^2$, we will show how to find the equation.

$$\begin{cases} x^2 + y^2 + 2(\alpha+2)y + \alpha^2 = 0 & eq.05 \\ \dfrac{d(x^2 + y^2 + 2(\alpha+2)y + \alpha^2)}{d\alpha} = 0 & eq.06 \end{cases}$$

Deriving the expression: $\frac{d(x^2+y^2+2(\alpha+2)y+\alpha^2)}{d\alpha} = 2\alpha + 2y = 0 \Rightarrow y = -\alpha$.
Replacing the identity found in equation 5, $x^2 + y^2 + 2(-y+2)y + (-y)^2 = 0 \Leftrightarrow x^2 + 4\,y = 0$, is the envelope equation.

DOI: https://doi.org/10.29352/mill0218e.39468

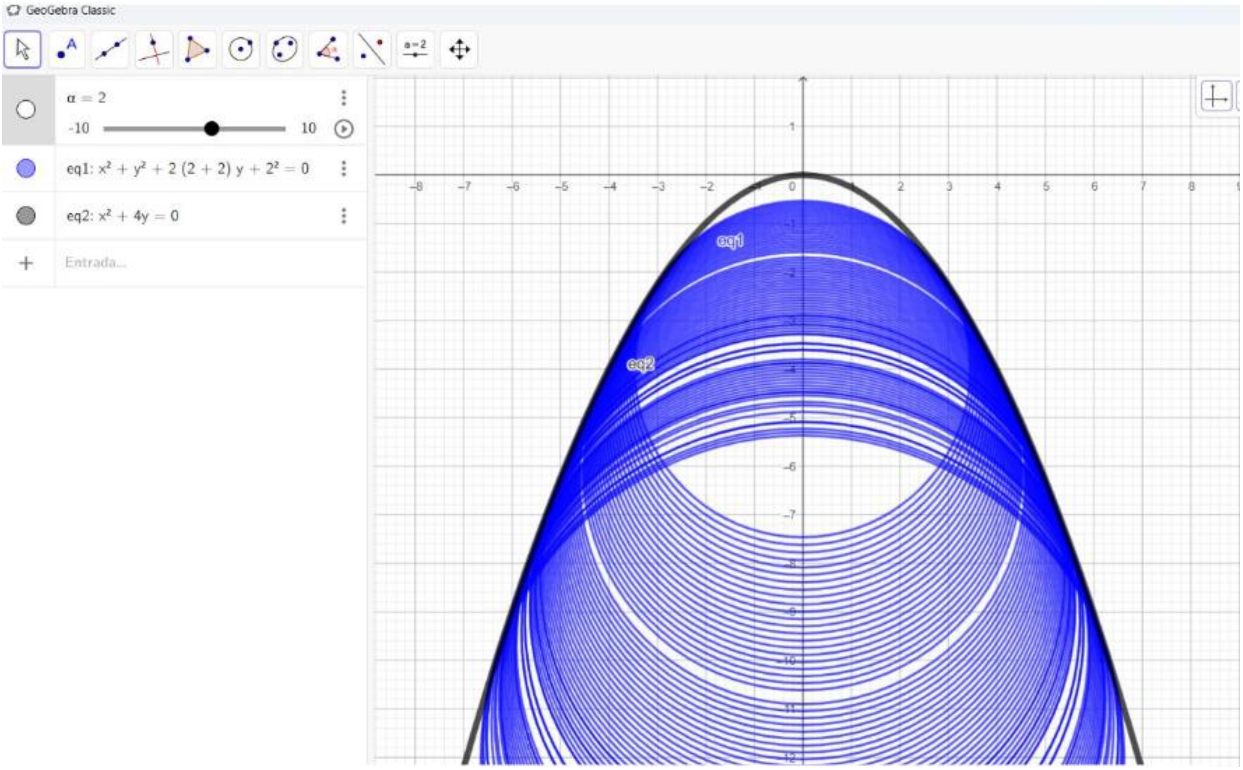

Figure 2 - Family of curves $x^2 + y^2 + 2(\alpha + 2)y + \alpha^2$, represented in blue with α varying from -10 to 10, generating a family of circles, and envelope $x^2 + 4\,y = 0$ represented in black developed by Paiva *et al* (2024).

In Figure 2, the graph of the envelope function is presented, in which it is observed that the function is tangent to all the curves generated by the expression $x^2 + y^2 + 2(\alpha + 2)y + \alpha^2$, this visualization is presented to better understand the wraps concept.

2. METHODS

Didactic Engineering (DE) emerged in France in the late 1980s as a research methodology to operationalize the principles of Mathematics Didactics (DM). DM, in turn, is a broad field of study dedicated to the analysis and improvement of mathematics teaching and learning processes (Alves, Dias, and De Lima, 2018).

In the methodological scheme proposed by ED, the researcher assumes a role similar to that of an engineer who designs and builds solutions for teaching Mathematics. ED focuses on understanding how mathematical concepts are constructed by students and teachers, investigating the challenges and opportunities that arise in this process (Alves, Catarino & Borges, 2020).

According to Figure 3, for the teacher to understand the problems related to teaching, there must be an integration between theory, research, and practice in training. Artigue (2020). This research methodology is composed of four dialectical phases: preliminary analysis phase, phase of conception and a priori analysis of teaching situations, experimentation, and, finally, a posteriori analysis and validation of all theoretical apparatus constructed with the aim of obtaining scientific knowledge about the teaching of a subject.

DOI: https://doi.org/10.29352/mill0218e.39468

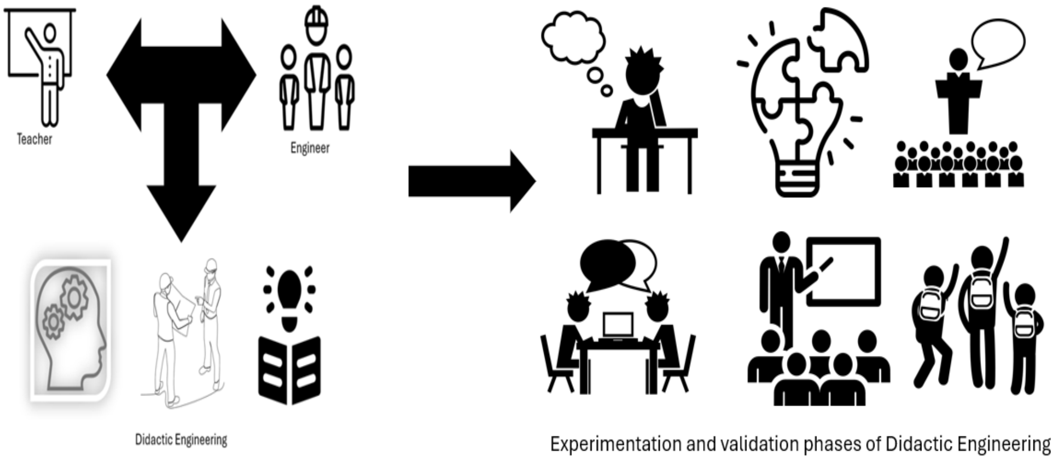

Figure 3 - A description of the representation of how the didactic engineering methodology

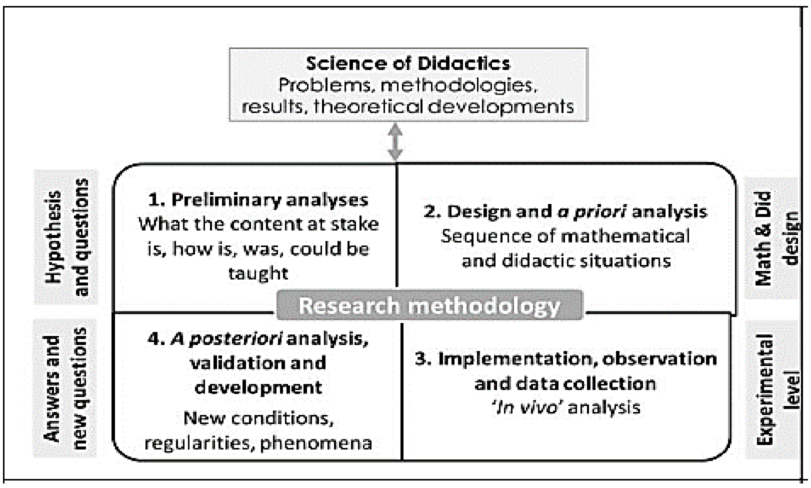

Figure 4 - Representation of the four dialectical phases of DE developed Artigue(2020)

In the first dialectical phase, *the preliminary analyses*, the researcher focuses on the analysis of the mathematical content to be taught, investigating its epistemological, historical, and didactic aspects. Therefore, it is necessary to understand the environment in which teaching will take place, considering the students, their prior knowledge, the sociocultural context, and the characteristics of the educational institution. And so, define the objectives, based on previous analyses, that we aim to achieve with the didactic sequence that will be developed (Almouloud, 2007).

In the second phase of DE, the *a priori analyses*, the researcher will produce a didactic sequence, which corresponds to a set of articulated activities that are planned with the intention of achieving a certain didactic objective. The central objective of this didactic sequence will be to help students overcome obstacles that may arise during the construction of the mathematical knowledge covered (Alves, Dias, and De Lima, 2018).

The third dialectical phase*, experimentation*, consists of the practical implementation of the didactic sequence planned during the previous phase. It is the moment when theoretical results are put into practice, allowing the obtaining of empirical results that complement theoretical analysis. During this phase, the entire planned device is applied in the classroom, providing an opportunity to verify the effectiveness of the proposed pedagogical strategies (Almouloud, 2007).

And finally, the dialectical phase of *posteriori analysis and validation*, a stage in which a detailed investigation takes place on the students' production, observing their behavior during the development of the didactic sequence and using the data collected throughout the experimentation (Artigue,1996). This phase is based on the analysis of the results obtained in the classroom, comparing the predictions made during the a priori analysis with the data observed after implementing the activities.

DOI: https://doi.org/10.29352/mill0218e.39468

The use of systematic criteria and ED assumptions is fundamental to allow this comparison between predicted data and observed data. This approach provides an opportunity for continuous refinement of teaching practices, allowing educators to adjust their pedagogical strategies based on empirical evidence.
However, in this text, we only focus on the first two initial phases of DE, specifically on preliminary and a priori analyses, since the activities have not yet been implemented. This planning phase is essential to ensure that teaching situations are carefully designed and aligned with teaching objectives.

3. RESULTS: A BRIEF PRELIMINARY AND A PRIORI ANALYSIS OF ENVELOPES TEACHING

In the previous sections, a study was conducted on the definitions and concepts related to envelopes, highlighting the importance of understanding concepts from Multivariable Calculus (MVC), such as differentiability, topological notions, and limits. This is due to the mechanisms present in the algorithm for finding the envelope equation, which frequently involve the use of partial derivatives. The relevance of MVC for the study of envelopes and Ordinary Differential Equations (ODEs) is widely recognized, being a prerequisite for advanced courses in various fields of Science, Technology, Engineering, and Mathematics.
In agreement with the perspective defended by JAVARONI (2010), who argues that the study of the epistemic field of Ordinary Differential Equations (ODEs) requires understanding the relationship between the derivative and the rate of change of the function of a family of curves, it is necessary to comprehend how the derivative relates to the function's rate of change and how this affects differential equations. In ODEs, one starts from a relationship between the function and its derivatives to deduce the original function, which contrasts with differential calculus, where one starts from the function to find its derivatives.
In the Preliminary Analysis phase, theoretical surveys and investigations about the teaching context are carried out with the objective of identifying the main difficulties and the teaching strategies that can be adopted.
The studies by HENDRIYANTO et al. (2024) and PRABOWO et al. (2022), as well as the doctoral thesis by ARSLAN (2005), indicate that students who do not master MVC concepts may face difficulties in visualizing and interpreting differential equations and, consequently, envelopes, which limits their ability to apply this knowledge in more advanced mathematical contexts. These studies suggest that obstacles in teaching ODEs can be ontogenetic (gaps in prior knowledge), didactic (teaching methods), and epistemological (incomplete knowledge in specific contexts) in nature. A lack of understanding of prerequisite calculus concepts is a significant factor for these difficulties. Moreover, differentiability itself, a central concept, may be a source of misconceptions for students (KERTIL & KÜPCÜ, 2021) and even for prospective mathematics teachers.
Thus, in addressing these epistemological difficulties in teaching envelopes, the aim is to overcome obstacles by providing a solid foundation in the concepts of differentiability and derivatives, assisting students in relating these concepts to envelopes in a clear and meaningful way. Overcoming these challenges requires not only a strong conceptual foundation but also the adoption of different teaching strategies that promote collaboration and contextualized instruction. The lack of proficiency in differential and integral calculus is frequently cited as a reason why solving Ordinary Differential Equations (ODEs) problems ends up being treated as a mere algorithmic application, without adequate comprehension of the underlying concepts involved. Subsequently, the phase of Conception and A Priori Analysis consists of proposing the development of a didactic sequence based on the previous analyses, accompanied by the anticipation of the possible results of the classroom intervention.
The curves adopted in this proposal were carefully selected to enhance students' understanding of the concept of envelopes. The selection was guided by the intention to create situations in which key elements, such as parameter dependence, the behavior of the family of curves, and the construction of the envelope, could be clearly and meaningfully visualized. This choice is consistent with the second phase of Didactical Engineering and aims to foster the articulation between analytical and geometrical aspects of the concept. In doing so, it supports student learning and helps address the conceptual difficulties frequently encountered in the study of envelopes within the context of Ordinary Differential Equations (ODEs).

3ºExample: It is the family of straight lines defined by the equation

$$f(x, y, \alpha) = y - \alpha x - {}^{1}/_{\alpha}$$

Here, x and , y are the independent and dependent variables, respectively, and α is the parameter that defines each member of the family of curves.
The core of the envelope algorithm involves differentiating the family's equation with respect to its parameter and setting the result to zero. This step identifies the condition under which the tangent lines of the family coincide with the envelope.
Differentiate $f(x, y, \alpha)$ with respect to α:

$$\frac{\partial f}{\partial \alpha} = \frac{\partial(y - \alpha x - {}^{1}/_{\alpha})}{\partial \alpha} = 0$$

DOI: https://doi.org/10.29352/mill0218e.39468

Applying the rules of differentiation:

$$-x - \left(-\frac{1}{\alpha^2}\right) = 0 \Rightarrow -x + \frac{1}{\alpha^2} = 0 \quad .$$

Express the Parameter in Terms of Variables: From the differentiated equation, isolate the parameter α in terms of x (or y if applicable). This expression will be crucial for eliminating the parameter from the original family equation.

From $-x + \frac{1}{\alpha^2} = 0 \quad \Rightarrow \quad \frac{1}{\alpha^2} = x \quad \Rightarrow \alpha^2 = \frac{1}{x} \Rightarrow \alpha = \pm \frac{1}{\sqrt{x}}$

The final step involves substituting the expression for α (obtained in the previous step) back into the original equation of the family of curves. This eliminates the parameter, resulting in an equation that describes the envelope.
The original equation is:

$$y - \alpha x - {}^{1}/_{\alpha} = 0 \Longrightarrow y = \alpha x + {}^{1}/_{\alpha}.$$

Substitute $\alpha = \pm \frac{1}{\sqrt{x}}$ into this equation:

$$y = \pm \frac{1}{\sqrt{x}} x + \frac{1}{\pm \frac{1}{\sqrt{x}}}$$

Simplify the terms:

$$y = \pm \frac{x}{\sqrt{x}} \pm \sqrt{x} \Rightarrow y = \pm 2\sqrt{x}$$

To express this in a more standard form, can square both sides:

$$y^2 = (\pm 2\sqrt{x})^2 = 4x$$

Therefore, the equation $y^2 = 4x$ represents the envelope of the family of curves $y = \alpha x + {}^{1}/_{\alpha}$. This derivation demonstrates how the envelope algorithm systematically leads to the singular solution that is tangent to every member of the given family of curves.

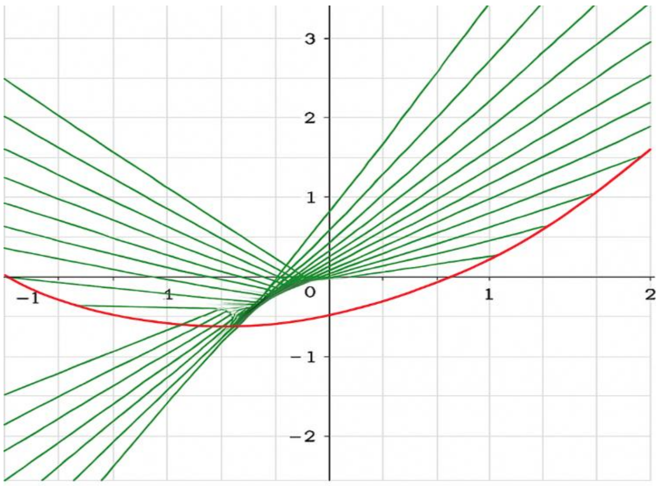

Figure 5 - Representation of the envelope of the curve family the $y = \alpha x + {}^{1}/_{\alpha}$

In Figure 5, a dynamic visualization of the family of curves $y = \alpha x + {}^{1}/_{\alpha}$ is displayed in green, and its envelope $y^2 = 4x$ in red.

Through the visualization provided by the GeoGebra software, it is understood that the envelope of the family of curves $\alpha x + {}^{1}/_{\alpha}$is a parabola that is sufficiently larger to touch at least one point, all curves generated by this family.

DOI: https://doi.org/10.29352/mill0218e.39468

4ºExample: Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having the same area. The area of an ellipse is

$$S = \pi ab.$$

Then the equation of the family of curves can be written as

$$\frac{x^2}{a^2}+\frac{y^2}{\left(\frac{S}{\pi a}\right)^2}=1 \text{ or } \frac{x^2}{a^2}+\frac{\pi^2 y^2 a^2}{S^2}=1\ (eq.09)$$

where the semi-axis a is a parameter. By differentiating a equation 09 with respect to a:

$$-\frac{2x^2}{a^3}+\frac{2\pi^2 y^2 a}{S^2}=0 \Rightarrow \frac{x^2}{a^3}=\frac{\pi^2 y^2 a}{S^2} \Rightarrow x^2S^2=\pi^2y^2a^4 \Rightarrow S|x|=\pi|y|a^2.$$

$$a^2=\frac{S|x|}{\pi|y|}.$$

Replacing, a^2 in the expression $\frac{x^2}{a^2}+\frac{\pi^2 y^2 a^2}{S^2}=1$:

$$\frac{x^2}{\frac{S|x|}{\pi|y|}}+\frac{\pi^2y^2\frac{S|x|}{\pi|y|}}{S^2}=1 \Rightarrow \frac{\pi|x||y|}{S}+\frac{\pi|x||y|}{S}=1 \Rightarrow 2\frac{\pi|x||y|}{S}=1 \Rightarrow 2\pi|x||y|=S$$

$$|y|=\frac{S}{2\pi|x|},$$

this equation consists of 4 hyperbolic branches and describes the envelope of the family of ellipses.

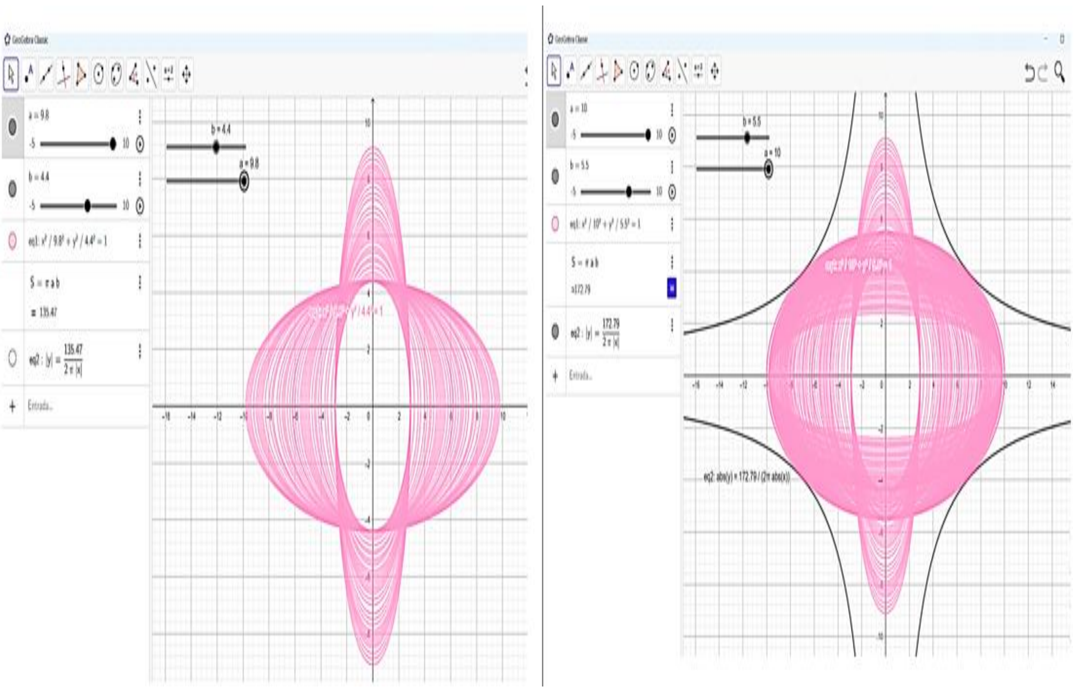

Figure 6 - The envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, having the same area(left), and on the right, the description of the envelope of the family of these ellipses.

Thus, through examples, other epistemological obstacles, which may be developed as highlighted by Javaroni (2010), within the perspective of Didactic Engineering in teaching EDOs, such as the difficulty in understanding a basic concept: the differentiation of concepts of general solution, singular solution, and their specificities.

Accordingly, Bisognin E. and Bisognin V.(2011) complement this view by highlighting that students with difficulties in relating the analytical and graphical representations of the derivative are often unable to demonstrate understanding of the concept of

DOI: https://doi.org/10.29352/mill0218e.39468

derivative and its various interpretations. And this difficulty in understanding the differentiability of functions and the relationship between analytical and graphical representations of the derivative has direct implications for teaching envelopes.
Furthermore, Macuácua, Catarino & Soares (2024, p. 7) state that "the use of computer simulations aims not only to motivate students to participate in learning but also to make it easier to visualize more complex phenomena".
Thus, the use of computer simulations in the development of didactic sequences planned with Didactic Engineering, as carried out in Geogebra, proves to be an effective strategy for motivating and involving students in learning abstract concepts. This visual approach promotes understanding of complex specifics.

CONCLUSION

The investigation conducted in this study has elucidated the epistemological complexity inherent in teaching the concept of envelope curves within the framework of Ordinary Differential Equations (ODEs). The analyses, grounded in the principles of Didactic Engineering (DE), revealed that this concept encompasses a mathematical structure requiring prior knowledge of differentiability, partial derivatives, and the behavior of parameterized families of curves—elements that may constitute substantial learning obstacles.
The identified difficulties extend beyond the abstract nature of the content, arising also from the predominantly algebraic approach prevalent in traditional teaching materials, which impedes the development of a deeper understanding and the integration of multiple representations. In response, the methodological approach proposed herein aimed to provide alternatives that support overcoming such challenges by leveraging visual resources, constructing interactive simulations, and emphasizing didactic strategies that synthesize analytical and geometric perspectives on the envelope concept.
The study further highlighted the critical need to address epistemological, ontogenetic, and didactic obstacles that hinder the acquisition of advanced mathematical concepts within the domain of ODEs. In conclusion, this teaching proposal, framed by Didactic Engineering and supported by digital technologies, presents a pertinent methodological alternative for constructing learning environments conducive to students' conceptual development, especially concerning the understanding of envelopes and their relationship with differential equations. Future research may focus on extending these resources to more complex curve families and integrating empirical assessments to evaluate the effectiveness of such educational interventions on student learning outcomes.

ACKNOWLEDGEMENTS

Work financed by National Funds through the National Council for Scientific and Technological Development - CNPQ, the Ceará Foundation to Support Scientific and Technological Development - FUNCAP and Foundation for Science and Technology, I.P., within the scope of the projects UIDB/00194/2020 (https://doi.org/10.54499/UIDB/00194/2020) and UIDP/00194/2020 (https://doi.org/10.54499/ UIDP/00194/2020) (CIDTFF)

AUTHORS' CONTRIBUTION

Conceptualization, A.P., F.A. and H.C.; data curation, A.P., and H.C.; formal analysis, A.P., F.A. and H.C.; funding acquisition, A.P. and F.A.; investigation, A.P. and F.A.; methodology, F.A.; project administration, A.P., F.A. and H.C.; supervision, F.A. and H.C.; writing-original draft, A.P., F.A. and H.C.; writing-review and editing, H.C.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

REFERENCES

Alves, F.R.V., Catarino, P. M. M. C., & Borges, A. M. F. V. R. (2020). Uma proposta de investigação histórica, epistemológica e matemática sobre a noção de sequência recorrente com o amparo de uma Engenharia (Didática) de Formação*. Unión - Revista Iberoamericana de Educación Matemática, 16*(60), 76-103. https://union.fespm.es/index.php/UNION/article/view/141

Almouloud, S. A. (2007). *Fundamentos da didática da matemática*. Editora da Universidade Federal de Parana. https://abrir.link/ghiQU

Artigue, M. (2020). Méthodologies de recherche en didactique des mathématiques: Où en sommes-nous? *Educação Matemática Pesquisa, 22*(2), 025-064. https://doi.org/10.23925/1983-3156.2020v22i3p025-064

DOI: https://doi.org/10.29352/mill0218e.39468

Arslan, S. (2005). *L'approche Qualitative Des Équations Differentielles em Classe de Terminale S: Est-elle viable? Quels sont les enjeux et les conséquences?* [Tese de Doutorado, Université Joseph-Fourier- Grenoble I]. https://theses.hal.science/tel-00009594/

Alves, F. R. V. (2014). Família de Curvas Planas e sua Envoltória: Visualização com o software Geogebra. *Conexões-Ciência e Tecnologia, 8*(3). https://doi.org/10.21439/conexoes.v8i3.634.

Alves, F. R. V. (2016). Engenharia didática (análises preliminares e análise a priori): o caso das equações diferenciais de segunda ordem. *Revista de Ensino de Ciências e Tecnologia em Revista–ENCITEC, 6*(2), 1-16. http://orcid.org/0000-0003-3710-1561

Alves, F. R. V., Dias, M. A., & Lima, M. V. M. de. (2018). On the Teaching of Generalized Integrals (GI): a Didactical Engineering Contribution. *Jornal Internacional De Estudos Em Educação Matemática, 11*(2), 130–144. https://doi.org/10.17921/2176-5634.2018v11n2p130-144

Bisognin, E., & Bisognin, V. (2011). Análise do desempenho dos alunos em formação continuada sobre a interpretação gráfica das derivadas de uma função. *Educação Matemática Pesquisa Revista do Programa de Estudos Pós-Graduados em Educação Matemática*, *13*(3), 509–526. https://revistas.pucsp.br/index.php/emp/article/view/7056.

Boyce, W., Diprima, R., & Meade, D. (2020). *Equações diferenciais elementares e problemas de valores de contorno - 11 edição*. Livros Técnicos e Científicos Editora Ltda. https://abrir.link/RhUhX

Figueiredo, D. G. D. & Neves, A. F. (2018). *Equações diferenciais aplicadas*. Instituto de Matemática Pura e Aplicada.

Hendriyanto, A., Suryadi, D., Juandi, D., Dahlan, J. A., Hidayat, R., Wardat, Y., Sahara, S., & Muhaimin, L. H. (2024). The didactic phenomenon: Deciphering students' learning obstacles in set theory. *Journal on Mathematics Education*, *15*(2), 517–544. https://doi.org/10.22342/jme.v15i2.pp517-544

Javaroni, S. L. (2010). O Processo de Visualização no curso de Introdução às Equações Diferenciais Ordinárias. *Revista de Ensino de Engenharia*, *28*(1). https://doi.org/10.5335/ree.v28i1.155

Kertil, M., & Küpcü, A. R. (2021). Prospective mathematics teachers' ways of reasoning on differentiability and corner points in a real-life context. *International Journal of Mathematical Education in Science and Technology,* 52(9), 1361–1384. https://eric.ed.gov/?id=EJ1305317

Macuácua, R. E., Catarino, P., & Soares, A. (2024). Uso de simulações computacionais como estratégia didática de ensino e de aprendizagem do tópico "efeito fotoelétrico": estudo de caso. *Millenium - Journal of Education, Technologies, and Health, 2*(25), e35404. https://doi.org/10.29352/mill0225.35404

Paiva, A. C. P., Alves, F. R. V., de Barros Campos, H. M., & Cidrão, G. G. (2024). Um Estudo de Envoltórias: Investigando o lugar geométrico dos pontos característicos com o *software* Geogebra. *REPPE-Revista de Produtos Educacionais e Pesquisas em Ensino*, *8*(2), 2215-2227. https://periodicos.uenp.edu.br/index.php/reppe/article/view/1596

Prabowo, A., Wulandari, S., & Nuraeni, E. (2022). Learning Obstacles in the Making of Lesson Plans by Prospective Mathematics Teacher Students. *Jurnal Pendidikan Matematika dan Sains*, *6*(2), 103-114. http://dx.doi.org/10.1155/2022/2896860

Vilches, Maurício. (2009). Envoltórias de Curvas Planas. *Cadernos do IME-Série Matemática, 21*(3), 1-32. 2009. https://doi.org/10.12957/cadmat.2009.11852