ANALYSIS OF TEACHING AND LEARNING SITUATIONS IN ALGEBRA IN PROSPECTIVE TEACHER EDUCATION

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ABSTRACT

This paper presents a teacher education experiment that was conducted in an algebra course based on an exploratory approach and articulating content and pedagogy. We investigate the contribution of analysing teaching and learning situations, namely student answers and episodes of classroom work, in developing the mathematical and teaching knowledge of prospective primary school teachers. We use a design research methodology to probe the prospective teachers' development after having participated in an experiment in their third year of a primary education degree program. The results show that the prospective teachers' understanding of algebra and grasp of how to use different representations and strategies grew considerably. The results also show that their didactical knowledge regarding tasks, classroom organization, attention to students' reasoning, and teacher's questions grew as well. The variety of tasks proposed to the prospective teachers during the course was of vital importance to this outcome, as was the opportunity to reflect, work with elements of real practice, and participate in whole class discussions.

KEY WORDS

Teacher education; Algebra; Algebraic thinking;
Mathematics; Teaching experiment.

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Analysis of Teaching and Learning Situations in Algebra in Prospective Teacher Education[1]

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INTRODUCTION

The preparation of basic education (1st to 6th grade) prospective teachers must take into account that, when they will start teaching, they will face challenges and demands with regard to algebraic thinking that most of them did not experience as students. Kaput and Blanton (2001) acknowledge that these teachers went to school during a shift in the process of algebra learning and teaching, having had few experiences with generalization and formalization activities that, in their perspective, must form the basis of students' work in school. In this changing context, teachers face many challenges that must be addressed in the education of prospective teachers. Canavarro (2007), for example, talks about the need for teachers to value students' reasoning, to know how to select tasks, and to promote a classroom dynamic that is conducive to the development of their students' algebraic thinking.

In order to be a pre-primary or primary school teacher nowadays in Portugal teacher candidates must have a bachelor's degree in primary education and a master's degree that qualifies them to teach. This research focuses on a teacher education experiment in algebra targeting third year students in

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the bachelor's program in primary education. The quality and effectiveness of teacher education largely determines how future teachers will perform later on. This is why teacher education must include experiences that develop the knowledge and skills required to spark their students' learning.

In the education of a prospective mathematics teacher it is important to articulate mathematical and didactical knowledge (NCTM, 2000; Ponte & Chapman, 2008). This paper discusses a teacher education experiment in algebra that promotes a strong link between content and pedagogy. The goal of our experiment was to analyse and discuss – then come to conclusions – regarding how the presentation of teaching and learning situations affects the mathematical and didactic knowledge of prospective teachers. The experiment followed an exploratory approach, with especial emphasis on analysing teaching in the primary school classroom and the students' responses.

ALGEBRA IN PRIMARY EDUCATION

Over the last decade, several researchers (e.g., Blanton & Kaput, 2005; Carpenter, Franke & Levi, 2003; Carraher & Schliemann, 2007; Kieran, 2004; Lins & Kaput, 2004) and curriculum documents (ME, 2007; NCTM, 2000) have advocated encouraging algebraic thinking starting from the early school years. They claim that doing so, using both mathematical and non-mathematical contexts provides learning experiences in early schooling that will carry over to learning algebra with understanding in the students' future years. Thus, the emphasis is on meanings and understandings (Canavarro, 2007; Kaput, 1999). Kieran (2007) argues that algebra should be regarded «as a way of thinking and reasoning about mathematical situations» (p. 5), and that it cannot be seen as merely a collection of techniques. Approaching algebra in this manner thus leads to a deeper understanding of mathematics and promotes the articulation among different mathematical topics.

In order to promote algebraic thinking, attention must be given to the objects and to the relationships among them with the teacher and her students representing these relationships and engaging in wide-ranging reasoning about them (Ponte, 2006). Generalization involves analysing similarities among given situations and/or analysing regularities, procedures, structures and relationships between situations that form new objects (Kaput, 1999). It assumes a pivotal role in the development of algebraic thinking (Blanton & Kaput, 2011; Kieran, 2007). Situations that broach algebraic thinking as «an activity of generalization of mathematical ideas, using literal symbolic representations, and representing functional relationships» (Blanton & Kaput, 2011, p. 6) have now been included in the primary school classroom.

The way algebra has been addressed over the years has undergone widespread changes. The approach that prevailed for several years gave students initial contact with algebra in school when they were 12 or 13 years old (Lins & Kaput, 2004). More recent perspectives on mathematics education suggest that the work aimed at developing the students' algebraic thinking must start in primary school (Carpenter & Levi, 2000; Carraher & Schliemann, 2007; Lins & Kaput, 2004).

This «early algebra» perspective, advocates the development of algebraic thinking, rather than the teaching and learning of specific concepts of algebra, or the domain of algebraic procedures. Lins and Kaput (2004) highlight the two main features of primary school algebra: (i) generalizations and the expression of such generalizations, and (ii) reasoning with generalizations, including syntactically and semantically guided actions. This perspective does not seek to bump up formal algebra studies from secondary to primary school, but rather to promote the students' mathematics reasoning development by connecting algebra to other primary school mathematics topics. The teacher education experiment that we present in this paper approaches algebra teaching and learning as a leading thread that should run through mathematics teaching in general (NCTM, 2000), using a rationale that articulates mathematical and didactical knowledge.

PROSPECTIVE TEACHER EDUCATION

According to Ponte and Chapman (2008), prospective mathematics teacher education involves three elements: (i) knowledge of mathematics (ii) knowledge of how to teach mathematics, and (iii) a professional identity that supports both the knowledge and the teaching of mathematics. For these authors, the articulation of these aspects will develop the prospective teachers' ability to integrate their knowledge of concepts, representations, and mathematical procedures with their knowledge of students, in line with their level of education and the curricular guidelines.

It should be noted that no matter how much knowledge the teacher (or the prospective teacher) has of mathematics, it «does not ensure that one can teach it in ways that enable students to develop the mathematical power and deep conceptual understanding envisioned in current reform documents» (Mewborn, 2001, pp. 28-29). Sánchez, Llinares, García and Escudero (2000) argue that, in order to teach mathematics, one must know about mathematics, know how one learns mathematical concepts, and know about the process of teaching. Thus, teacher education needs to provide future teachers with opportunities to develop their knowledge and abilities in each mathematical topic and the understanding of the connections between both. It must also provide future teachers with the knowledge of how to effectively convey concepts in the classroom. In other words, prospective teachers «need to know the mathematics they are teaching, as well as how to teach it» (Sullivan, 2011, p. 172). Therefore, it is essential to pinpoint the mathematical contents to be taught, taking into account what teachers need to know to teach it and how the prospective teachers themselves learn (Sánchez *et al.*, 2000).

Primary school teachers must understand the concepts, representations and algebraic procedures; understand how pupils learn; and be able to use teaching strategies that foster the development of their students' algebraic thinking (Capraro, Rangel-Chavez & Capraro, 2008). It is equally important to articulate content and pedagogy in prospective teacher education courses (Askew, 2008; Davis & Simmt, 2006; Ponte & Chapman, 2008). Davis and Simmt (2006) argue against the separation of content and pedagogy that tends to prevail, emphasizing that mathematics for teaching involves mathematical objects, curricular structures, and an understanding of how the classroom works. This articulation between content and pedagogy aims to develop prospective teachers' knowledge of the students, their learning processes, and how the teacher encourages such learning, as well as how to stimulate their understanding of concepts, representations, procedures, and connections by analysing teaching situations, students' tasks and their strategies and difficulties. Prospective teachers may use this knowledge to foresee what they will face when teaching, and thus be able to identify and integrate suitable resources and set appropriate tasks to develop specific learning goals.

THE TEACHER EDUCATION EXPERIMENT

Teacher education experiments take an exploratory approach toward classroom work. They aim to develop the participants' algebraic thinking, while developing the mathematical knowledge to be taught. They also promote the development of knowledge of algebra-related mathematics teaching and its connection to other mathematics topics. Therefore, teacher education experiments combine content and pedagogy (Ponte & Chapman, 2008) and provide prospective teachers with varied and meaningful learning experiences by putting them into teaching situations that promote sharing, debating and negotiating meanings, i.e., the knowledge and skills that will be essential for their future practice.

ARTICULATION OF CONTENT AND PEDAGOGY

Our teacher education experiment included mathematical and didactic work with regard to teaching and learning of mathematics in primary education, especially in algebra, and its articulation with other mathematical topics. The integration of content and pedagogy aimed to provide prospective teachers with an understanding of mathematics teaching that differs from perceptions they held prior to their experiences in teacher education (Ponte & Chapman, 2008). Thus, teacher education should promote in-depth study of mathematics for prospective teachers, as suggested by the Conference Board of the Mathematical Sciences (CBMS) (2011) but, as this document indicates, it is not sufficient for prospective teachers to study more mathematics than the mathematics they are going to teach. What they must have are educational experiences designed to develop a deep understanding of the mathematics they will teach.

Integrating content and didactic knowledge can produce learning experiences that enhance the prospective educator's teaching of mathematics (Albuquerque *et al.*, 2006; Ponte & Chapman, 2008). In this paper, we present two experiences featuring two tasks: Task 2 (Problems with unknown quantities) and Task 4 (Pictorial sequences). These situations give the participants a chance to: (i) analyse strategies used by students, (ii) observe, explore and connect different representations, (iii) analyse the knowledge shown by students, (iv) identify potential difficulties for students, and (v) discuss working hypotheses with the students that may foster the development of their algebraic thinking.

Open tasks that allow for different solution strategies play an important part in the exploratory approach. However, the variety of tasks is also important, since students should be exposed to a wide range of learning experiences (Ponte, 2005). The prospective teachers we dealt with analysed teaching and learning situations in which students worked on a variety of tasks. During their analysis of the tasks, the student teachers identified key points regarding the use of the task in the classroom as well as the possible contribution of such tasks to student learning. The task usually requires an interpretation of the situation that may involve clarification or reformulation of questions and the appropriate use of representations (Ponte, Quaresma & Branco, 2012). Well-designed tasks also allow students to explore mathematical concepts and ideas, so that «more than a context to apply on already learned concepts, these tasks are useful mainly to promote the development of new concepts and to learn new procedures and mathematical representations» (Ponte *et al.*, 2012, p. 10).

During the teaching experiment, classroom activity involves autonomous work that is supervised and guided by the teacher, and whole class discussions, which involve submitting solutions, debating them, and systematizing the relevant concepts, establishing connections between mathematical ideas and real life situations. The participants' exploration plays a fundamental role and the teacher's activity focuses on promoting and supporting this exploration and managing the different moments of the lesson. The teacher «gathers and analyses information about the strategies and theories being employed by students» (Ruthven, 1989, p. 451), which is essential in promoting a whole class discussion that aims to present and debate ideas and strategies amongst prospective teachers (Ponte, 2005). Thus, the moments of reflection and debate are based on the prospective teachers' activity (Ruthven, 1989), and they assume a pivotal role by reflecting on their own work. On the one hand, this approach aims to promote prospective teachers' learning. On the other, it aims to provide them with the classroom dynamics experiences that they can use in the future, to promote their students' learning, while focusing on the development of their skills and mathematical knowledge.

THE TASKS

This paper presents two of the seven tasks in our teacher education experiment, in which the articulation of content and pedagogy is effected by analysing the students' solutions, describing teachers' practices, and/or discussing classroom situations. Task 2 shows the written answers of 6^{th} grade students and the description of the learning trajectory provided by the teacher (adapted from Reeves, 2000). Analysis of the students' work and understanding of their mathematical thinking is an important part of the educator's practice (NCTM, 1991; Nickerson & Masarik, 2010). Analysing students' answers gives future teachers the opportunity to contextualized learning, gain a better understanding of how students think and hone their ability to make suitable decisions in the classroom (Crespo, 2000; Nickerson & Masarik, 2010). The activity also enabled student teachers to analyse the reasoning of the pupils they observed and the representations they used. The prospective teachers also benefited from discussing the students' written answers and the teacher's approach among themselves.

In Task 4, prospective teachers saw a video of excerpts from a 2^{nd} grade class (described in Silvestre *et al.*, 2010) in which the students were working on a pictorial sequence. The use of video yields significant teacher education opportunities. Llinares and Valls (2009) say that video recordings of classes foster the prospective teachers' ability to analyse and identify key aspects of teaching. The authors also emphasize that classroom videos may reveal and underscore mathematics teaching practices that contrast or collide with the future teachers' own. In this type of experiment, participants can analyse the teacher's practice and line it up with the requirements of the mathematics curriculum (ME, 2007), thus using the observation as a learning opportunity.

Different phases of the class are shown in the video: the introduction of the task, students' autonomous work, and the discussion of the work that includes the formulation of further questions aimed at generalization. The analysis of real life situations offers a productive model of mathematics teaching (Ponte, 2011), and helps prospective teachers to see that it is indeed possible to implement curriculum guidelines in the classroom.

METHODOLOGY

We used a design research methodology to assess how the participants developed from the context we provided, and analysed the teacher education experiment with regard to what worked and how it worked. This method allows one to test and/or improving models of teaching guided by theoretical principles, allowing the verification of how they work, adjust them, improve upon them and re-test them (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003; Collins, Joseph & Bielaczyc, 2004).

The possibility mentioned above was included in this study, which targeted algebra education for future primary school teachers. It involved a planned intervention, implemented by the first author, which took place over a significant period of time and was based on a sequence of teaching episodes, making it possible to analyse the participants' activity (Steffe & Thompson, 2000).

The participants were 20 student teachers attending the 3rd year of the degree program in primary education (referred to in the study as Fx). The study focused on the work and learning achieved by the group as a whole and on the learning experiences of three future teachers with specific educational profiles and different future goals. This enabled us to see to what extent the experiment affected different participants with different characteristics.

PROSPECTIVE TEACHER	MATHEMATICS BACKGROUND	SEEKING THE FOLLOWING MASTER'S DEGREE
Alice	9th grade	Pre-school Education
Beatriz	10th grade	Pre-school and Primary Education
Diana	12th grade	Primary and Middle School Education

TABLE I – CHARACTERIZATION OF THE PROSPECTIVE TEACHERS: MATHEMATICS BACKGROUND AND MASTER'S DEGREE SOUGHT

Data was gathered by various means, for a detailed understanding of the situations experienced by the participants (Bogdan & Biklen, 1982) throughout the teacher education experiment. In this paper, we present evidence from three audio and video recorded interviews with the participants (named E1, E2 and E3), documents produced by the participants in the experiment (written solutions of tasks 2 and 4 and portfolios), and participant observation in the classroom, complemented by audio and video recordings. This data collection took place at different times (Figure 1).

T1, T2, T3 E1	INTERNSHIP PERIOD	T4, T5 E2	SCHOOL BREAK	T6, T7 E3

FIGURE 1 – PERIODS OF DATA COLLECTION

Data analysis sought to identify the development of knowledge in mathematics and teaching offered to the three participants within the teacher education experiment. We present and discuss the work carried out in class on both tasks and the data concerning the development of Alice's, Beatriz's and Diana's knowledge in this context. In its essence, the analysis takes on an interpretative nature and tries to highlight the contribution made by analysing teaching and learning situations. The interpretative analysis documents the group's learning and highlights the development that classroom work afforded prospective teachers (Cobb, Zhao & Dean, 2009). We organized the data in order to discover regularities involving the actions and meanings the participants attributed to mathematical knowledge, the students' knowledge and their learning processes, and the knowledge of teaching practice afforded by both tasks.

ANALYSIS OF TEACHING-LEARNING SITUATIONS

In Tasks 2 and 4, the participants had the chance to analyse teaching and learning situations. In Task 2, they examined the written answers of 6th grade students, as well as the description of the work and the teacher's thoughts. In Task 4, question 3 deals with the observation and analysis of a classroom segment containing a task that features a growing pictorial sequence for 2nd graders.

TASK 2

Task 2 offers the analysis of the answers of 6th grade students to the «chicken problem» (Figure 2). On the next pagewe show the answers produced by Matt (Figure 3) and Joanna (Figure 4) (taken from Reeves, 2000, p. 399).

Problem solving by the prospective teachers. Before examining the students' answers, the participants solved the problem themselves, by working in pairs. Many participants acknowledged three unknown values in it, to which three conditions are given (F7, F8 and F19). Then, they formulated a system of three first-degree equations and solved it using the substitution method.

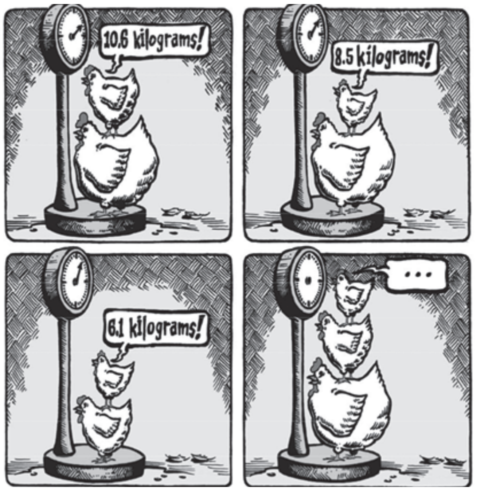


FIGURE 2 – THE CHICKEN PROBLEM,
TASK 2 (REEVES, 2000, P. 399)

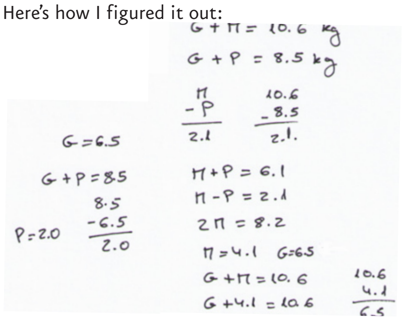
Name: Matt
The mass of the big chicken is 6.5 kgs
The mass of the middle-sized chicken is 4.1 kgs
The mass of the little chicken is 2 kgs

Here's how I figured out. I put the number 1, 2, 3 and 4 around the boxes. Then I added box 2 [6.1] and box 1 [10.6]. I got the sum of 16.7. Then I subtracted box 3 [8.5] from 16.7. I got the sum of 8.2. Then I divided 8.2 by 2. I got 4.1 for the weight of the medium chicken. Then I subtracted 4.1 from box 1 which had one big and one medium chicken. I got 6.5 for the big chicken. Then I subtracted 4.1 from box 2 and got 2 for the small chicken.

FIGURE 3 – MATT'S ANSWER

Name: Joanna
The mass of the big chicken is 6.5
The mass of the middle-sized chicken is 4.1
The mass of the little chicken is 2.0

Here's how I figured it out:



[G presents the mass of the big chicken, M presents the mass of the middle-size chicken, P presents the mass of the little chicken]

FIGURE 4 – JOANNA'S ANSWER

Most of the prospective teachers used the addition method informally, carrying out basic operations to get the value of an unknown. Some participants such as Diana (Figure 5) and Beatriz subtracted pairs of given values in order to establish relationships between the mass of two chickens, and find out which one weighs more than the other.

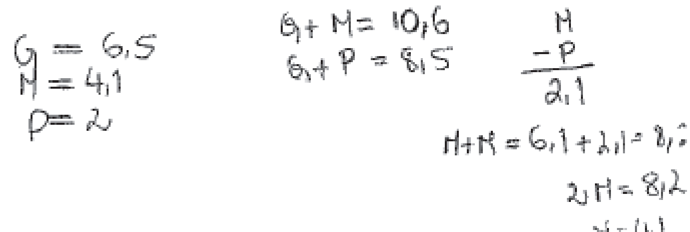


FIGURE 5 – DIANA'S ANSWER

Diana also formulated a system of three first-degree equations but, because it is a system with three equations with three unknown values, she had trouble solving it.

Most of the participants, such as Alice (Figure 6), merged two given values and got twice the weight of a chicken and the weight of the other two chickens. Based on the other equation, she subtracted the weight of the other two chickens to her result. Several participants used letters to represent unknown quantities and write algebraic expressions for the quantities they were looking for.

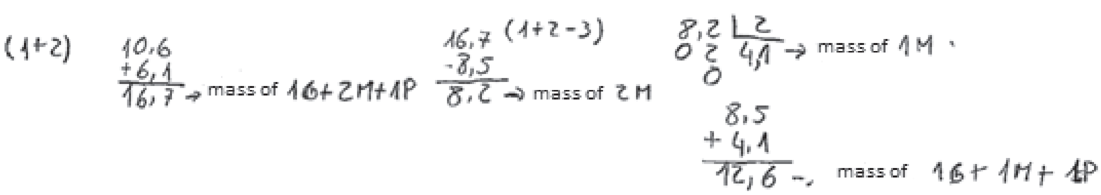


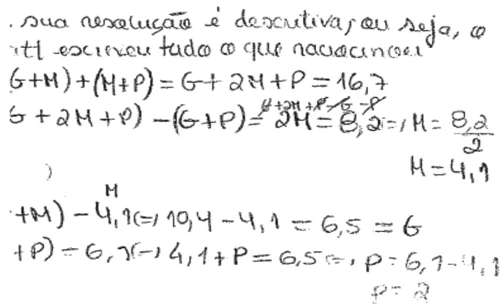
FIGURE 6 – ALICE'S ANSWER

After this autonomous work, the prospective teachers shared and discussed their strategies, presenting solutions based on doing basic operations and establishing relationships that use the algebraic language. This enabled them to analyse systems of equations and the substitution method for solving the problem. This activity was very important to the participants, as twelve of them included it in their portfolios.

Analysis of the students' answers. By its very nature, the analysis of students' answers is a challenge for the participants. Matt's answer (Figure 3) is descriptive, using verbal language. In contrast, Joanna's answer (Figure 4) is based solely on symbolic representations; but it does not explain the computations carried out. In order to analyse Matt's answer, the participants realized that they needed to specify his computations, while Joanna's answer needed to be complemented with verbal representation.

To make sense of Matt's answer, the participants used symbolic representation, showing the basic operations carried out by the student. Some participants used verbal language to indicate what those operations referred to, while others used algebraic language, as in the case of F15 (Figure 7).

His solution is descriptive, ie, Matt wrote his entire reasoning down


. sua resolução é descritiva, ou seja, o
...tt escreveu tudo o que raciocinou

$(G+M)+(M+P)=G+2M+P=16{,}7$

$(G+2M+P)-(G+P)=2M=8{,}2 \Rightarrow M=\frac{8{,}2}{2}$

$M=4{,}1$

$(G+M)-4{,}1 \Leftrightarrow 10{,}4-4{,}1=6{,}5=G$

$(M+P)=6{,}7 \Leftrightarrow 4{,}1+P=6{,}5 \Rightarrow P=6{,}7-4{,}1$

$P=2$

FIGURE 7 – ANSWER OF F15

In the case of Joanna's answer, the participants felt the need to identify the order in which the student carried out the operations. To shed light on her strategy, they used verbal language, just like F1 shows the class:

F1. – Joanna realized that the difference between the medium-sized chicken and the small chicken is 2.1, i.e., she subtracted the total weight of the big chicken with the medium-sized chicken from the total weight of the medium-sized chicken with the small chicken. She discovers that the sum of the medium-sized chicken and the small chicken equals 6.1. So, Joanna understood that if she added 6.1 and 2.1 she would get the weight of two medium-sized chickens. Right? Here it is, two medium-sized chickens [points to Joanna's answer], 8.2. So, if she divides this value by two, she'll get the weight of a medium-sized chicken. It's 4,1.

F4. – She did the compensation.

F1. – Yes. So, if she has a medium-sized chicken and has these expressions [G+M = 0.6 and G+P = 8.5] all she has to do is replace them, and she'll have all the other values.

Researcher – She knows the medium…

F1. – Exactly. After she got the value for the big chicken [6.5], she replaced it and got the value of the small chicken.

Researcher – Where did they differ? Matt was descriptive…

F1. – And Joanna did the computations.

Researcher – And she uses something else. She uses symbols to represent [unknown] quantities.

F4. – It's a way to explain all the computations but with symbols (Class, T2).

In addition to Matt and Joanna's answers, the participants also analysed Leo's answer (Figure 8) to the problem that the teacher presents with different numbers. This student numbered the different boxes from left to right (above, the scale of box 1 is 11 and box 2 is 9.8; below, the scale of box 3 is 6.4 and box 4 has all the chickens with unknown values). He added all the equations, obtaining twice the weight of the three chickens. In this new equation, he subtracted each of the initial equations. Thus, through the addition method, he got each of the unknown values. Given the conditions of the problem, this strategy proved to be very efficient. Some participants indicated that, if they eventually encounter a scenario similar to the chicken problem, they will use this solution strategy, as indicated by F6: «For me it was the simplest, quickest, most effective way to solve it, so much so, that I would use this same strategy for solving further exercises» (Portfolio).

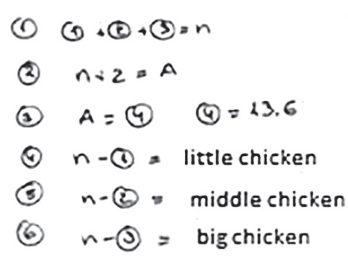


FIGURE 8 – LEO'S ANSWER (ADAPTED FROM REEVES, 2000)

For Alice, the analysis of the students' answers promoted an understanding of their reasoning and the representations they use, and contributed to their learning of different solution strategies. As she said, «After seeing this sheet and how one of the students [Leo] solved one of the problems, in the next sheet (...), I know that I solved a problem taking into account the student's reasoning» (E2, Alice).

For Beatriz, this task contributed to deepening her mathematical knowledge. She mentioned that at first «I could only do it through experimentation» (E2) and she was surprised by the students' answers, having learned different strategies by analysing those answers. For this participant, Leo's strategy was also important:

> They considered the value of the boxes, one of them added [the value of] all the boxes, and found that if he added [the value of] all the boxes, he would obtain two times the chickens' weight (...); and if he divided that value by two, he would obtain the weight of the three chickens (E2, Beatriz).

The opportunity to analyse students' written answers was also meaningful for Diana. She recognized that the teacher has his/her own solution strategies, sometimes formal, and thought that this is not enough when you are a teacher. One must also be able to interpret the students' answers, given that they may have different strategies. Therefore, in her training phase, this prospective teacher was already valuing the knowledge and learning processes that allow students to answer the mathematical problems, and not just the development of conceptual and procedural knowledge.

> I thought it was quite amusing and right for us to analyse how children solved the exercise, because it's good for us to know how to solve the exercises, and is also good for us to understand how children solve them. Sometimes they think in ways we have not thought of, which can be equally correct. And I think it is good for us to not just practice how to do the exercises but also to understand how they did them. Because we will need it in our future teaching. We will need to understand it. We must not just know how to do it in our own way, but we also need to try to decode what they did, because many times it may be right or... Something may be right (E2, Diana).

In other situations featuring problems, the participants continued to explore less formal strategies to solve problems, closer to the primary students' work, and more formal strategies, using symbols and algebraic procedures.

Many participants stressed the importance of analysing the students' answers and checking the possibility of different strategies to solve a given problem, as F3, F8 and F19 suggest. The last two participants were particularly attentive to establishing relationships that are essential in the development of algebraic thinking in basic education. F8 recognized the possibility of solving these problems in different ways, not just t by means of the formal solution that she was acquainted with, i.e., the system of equations:

> I thought it was very interesting (...) also the fact that we analyse different answers and understand the difference between the strategies used by the students (Portfolio, F3).

> This [task] developed my ability [to] interpret and compare the students' answers. Besides that, I understood that the aim of these exercises is not to develop solution techniques but to lead the students' to understand the relationships (Portfolio, F19).

> Through this task, I realized the variety of solutions that may exist for the same exercise, because for me, until now, the only way to solve this kind of problem was through the system of equations (Portfolio, F8).

Alice, especially, was surprised by the different ways 6th graders dealt with the situation and found the answer to the problem, as well as their reasoning ability. She relates this to the difficulties she had in solving the task herself:

> I thought this was very difficult. I thought it was funny that 6th graders had a thousand ways to solve this. I was thinking... They really have ability to solve the problem and explain it in different ways, which I also found amusing (E2, Alice).

For this participant, it was very important to look at the different strategies used to solve the same problem. It enabled this future teacher to gain a deeper understanding of the diversity of representations and rationales that students may use, and especially how the relationships may be expressed in a formal or informal way. Beatriz also underlined the value seeing that students can use different strategies and that there may be «several solutions to the same problem» (E2).

Analysis of the teaching approach. This task presents a description of the work done to promote the development of students' reasoning and their ability to use organized solution strategies. Part of that description is presented here (Figure 9, on the next page).

The teacher presented the problem of the chickens in Figure 1, so that the students would solve it using intuitive approaches, before initiating the formal study of algebra.
The problem was solved at home and the answers were given in the classroom. (...)
The students [like Matt and Joanna] presented their solutions to the rest of the class and explained their reasoning. (...)

Reflections

Reeves makes the following observations:

§ Some students can learn how to solve problems like this one from listening to one another and to their parents. Not all solutions will be efficient. The problem itself, as a precursor to systems of equation, is within the reach of sixth graders.
§ If students are to teach problem-solving strategies to others, they should be asked to emphasize strategies rather than computations. They should be encouraged to explain their approaches without using numbers.
§ Students will not automatically learn to use variables even after hearing their classmates use them as shortcuts. The use of variables will have to be encouraged by the teacher if the outcomes is a goal of an algebraic-thinking strand. (2000, p. 401)

FIGURE 9 – PARTIAL DESCRIPTION OF THE WORK DONE IN CLASS, TASK 2

With the description of the teacher's strategy and reflection, the participants found that this type of work in the context of problem solving and sharing strategies is meaningful to the students. They realized the importance of students presenting their answers orally in class, as indicated by F19, since it «requires them to explain their reasoning, so that their classmates may adopt an easier way to solve this kind of problems» (T2-1.3).

They also noted that in choosing the chicken problem, the teacher provided the students with an experience that involved the use of letters to represent unknown quantities. The nature of the problem facilitated the use of algebraic symbolism, as F1 says: «Variables appear naturally with a task like this» (Class, T2). Debating students' answers allows those who use algebraic symbols to share this kind of representation with their classmates, as F1 points out: «Since they explain their reasoning, they're the ones who'll explain the existence of that variable to a fellow student» (Class, T2). F1 is referring to the variables in a broad perspective, focusing on established relationships and not specifying that in this case there is a set of conditions that must be met, that is, each letter is an unknown.

The participants verified the importance of the whole-class debate on different strategies for student learning, and they remarked that they valued opportunities such as these in teaching:

> It's important for the teacher to provide the class with problem-solving moments like the chicken problem, so that students may develop (informal) mathematical reasoning, and also to give them the opportunity, through presenting different solutions, to choose a strategy that is different from theirs, for example (Portfolio, F8).

> It is important that teachers encourage a «debate» about the solutions to the problems at the end of the task and that teachers stress this kind of problems, because the students, by discussing the results and solutions with their classmates, have the opportunity to know and learn new strategies (Portfolio, F15).

F6 highlighted that this task provided her with an experience similar to the ones she will have in the classroom – examining the students' answers and «understanding their computations, and how they reached that answer, and so on» (Portfolio). She considered it an important task, because it contributed to «the preparation for my future [professional] activity, because not all children think like me and follow the same path to reach the same outcome» (Portfolio, F6).

Alice highlighted the fact that the teacher promoted the exchange of strategies among students, so that the class saw different ways to solve the problem and students with more difficulties were able to understand it, which may enable the development of algebraic thinking. She discussed this idea in her portfolio:

> I also emphasize the pedagogical attitude and strategies used by this teacher, making her students compare and discuss their reasoning and strategies among themselves. Thus, students with more difficulties in their algebraic thinking may observe more elaborate strategies and reasoning, which may be helpful to the evolution of their own thinking (...) (Portfolio, Alice).

Alice highlighted the way the teacher offered hints and conducted teaching and learning situations by stressing reasoning over results. She stressed the importance of this in honing the students' ability to establish relationships between different quantities in order to determine the unknown values. She also indicated that the teacher must possess enough knowledge to answer the students' questions, and be able to «answer and explain in many ways, to see how the student understands best» (E2, Alice).

This task led Beatriz to reflect on the importance of teacher preparation for this kind of task:

> I would have to solve such an exercise, before proposing it to the class. Maybe I would have to know other ways of solving it, so that when the students presented several solutions, I would already know about them. But sometimes we are not able to know [all the solutions], and when we listen to theirs, we know whether they are right, even if we have not thought of it ourselves (E2, Beatriz).

Beatriz indicated that the teacher must try to solve the same problem in many ways to be prepared for any questions the students may have. She must understand and be acquainted with their solutions, even when she did not come up with the strategy they followed.

Diana, however, emphasized another important aspect of the teacher's practice: the need to adapt the tasks to the students, for example, by adjusting the values: «I think it's an amusing task to work with them. We can adjust the numbers to their age or education, and we can work with them» (E3, Diana). She also realised that based on the examples analysed, with this task middle school students would be working on problems with unknown quantities, i.e., with unknowns appearing in a natural way. As she commented, «They would be working with unknowns without even realizing it» (E3, Diana).

TASK 4

In question 3 of Task 4, the participants watched a video of a 2nd grade class and analysed how students handled sequences (Figure 10).

Look at the sequence of blocks.

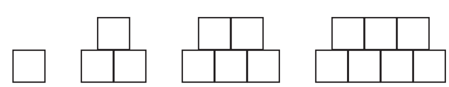


Fig. 1 Fig. 2 Fig. 3 Fig. 4

a) Continue the sequence and draw figures 5 and 6.
b) How many pieces were used to construct each of the figures? Write your answer in the following table.
c) Without using drawings, are you able to figure out how many blocks figure 20 of the sequence has? Explain how you figured it out.

FIGURE 10 – 2ND GRADE TASK, T4-3

Sequence analysis by prospective teachers. Before this task, the participants completed other tasks involving sequences and patterns. Before question 3 from

Task 4, there were other questions regarding pictorial sequences. In analysing this pictorial sequence, they immediately identified relationships between parts of each term and their order, indicating an overall term for the numerical sequence, concerning the number of squares in each pictorial term.

Analysis of students' strategies. Viewing the classroom excerpts enabled prospective teachers to identify the students' real difficulties with pictorial sequences, the different representations and strategies they used, and how the teacher acted in the various situations:

> [It allowed us] to identify different approaches suggested by the students and be aware of some of their difficulties. The analysis of such strategies prepares us, prospective teachers, for the variety of answers that we can get from them (Portfolio, F3).

In the analysis and discussion of the students' work, the participants were able to identify the relationships they established, which allowed them to set an algebraic generalization using natural language:

> [Regarding the 20th term]
> Researcher – What did [the student] discover?
> F19. – That it was always an odd number.
> Researcher – It was always an odd number, correct. And what did he do?
> F11. – He did 20 twice.
> Researcher –He did 20 twice, exactly.
> F11. – Then, he realized that it had to be an odd number: either 39 or 41.
> Researcher – Exactly. What's the teacher doing?
> F15. – She keeps asking.
> [searching for the generalization. In the video, the student indicates he discovered the secret: «it's twice minus 1»]
> Researcher – What has the [student] just done?
> Beatriz: He discovered the general term.
> F15. – He discovered the secret. (Class, T4)

The prospective teachers highlighted the fact that students made a generalization using natural language, even if they did not know that it was the general term of the numerical sequence. Alice's group wrote the generalization down in symbolic language, while some students expressed it in natural language (Figure 11).

• $2n - 1$ → • $n+n-1$	Some students gathered the row of squares on top with the one below, but since the upper row is always one square short, they always take one square from the sum total.

FIGURE II – ANSWER FROM ALICE'S GROUP, T4-3.2

For Alice, it was important to observe the classroom, the students' work, and the strategies they used to determine distant terms and how they managed to express a generalization. She was surprised that 2nd graders could generalize. She recalled a strategy of a student who used his knowledge of numbers to find a general rule in order to find a term of the numerical sequence. The analysis of this teaching and learning situation gave Alice a better understanding of what students are capable of and their ability to establish generalizations from these contexts. She said that students «can relate concepts, as was the case with Bruno, whose answer was twice minus 1» (E2, Alice).

Beatriz pointed out several features of the students' work on pictorial sequences. In finding close terms and in using the table, she stressed a recursive analysis a child had performed and how another had established a direct relationship between the total number of blocks in a term and its order:

> After looking at the table filled with the figure number and the number of matching blocks [up to 6th term], a child found that each time we changed pictures, the pieces increased 2 by 2; another child added the number of the requested figure with the previous figure [based on table data] (Portfolio, Beatriz).

In finding distant terms, she emphasizes two other things as well. One has to do with what was already identified in the analysis of the table, in which the students relate the composition of the pictorial term to the order, and to the order of the previous term. This is a contextualized generalization, because the students always present an example to express it:

> He replaced the blocks with numbers. In figure two, the 1 is on top and the 2 is below. In figure three, the 2 that was below in the previous figure went on top of the next figure, and the 3 went below (...) As to figure 20, the child thought that the 19 that was below in the previous figure shifts up in the next figure and the 20 is now below this 19. If we add the 19 to 20, we obtain the number of blocks of figure 20, which equals 39 (Portfolio, Beatriz).

The other situation that she identified dealt with the formulation of a general term of the sequence in natural language by a student, which he calls «the secret». The student presents a direct rule to find any term of the sequence, relating the number of blocks to the order.

> This child suggested that the secret «is not double, but twice minus 1». The teacher asked the child to explain the «secret». The child used the order in figure 12 and said: «12 + 12 = 24, but it can't be, because 24 is even, so, it's 23, because it's always twice minus 1» (Portfolio, quotation marks in original, Beatriz).

Based on her mathematical knowledge, Beatriz interpreted the students' answer and gave it an algebraic meaning. The teacher education experience and the reflection upon this work allowed her to develop knowledge of how students work with sequences, particularly their strategies, and how they express generalizations.

The student presented a direct rule to find any term of the sequence, establishing an algebraic generalization. However, a second grader does not use symbolic algebraic language to express that rule, he uses natural language instead, as Beatriz acknowledged. Thus, the prospective teacher understood that the work with pictorial sequences acts as a precursor to the students' development of algebraic thinking:

> I found it very important and interesting that we learned from a classroom excerpt with second grade children. We were able to observe how children think, and although they have not yet been taught algebra, they have already unconsciously internalized it. That is why one of the children talked about the «secret» [refers to a general term] (Portfolio, Beatriz).

This situation also helped Diana envision the work she would be doing in primary school. She suggested that, for primary school students, «it's easier to work with pictorial sequences than with numerical sequences» (E2). She stated that at this level, students do not use symbolic algebraic language; nevertheless, it makes sense to work on these issues to find close and distant terms. She recalls the strategy of some students, to which she associates a general term:

> Diana – With or without algebraic language, I think that they may get it, for example, if we ask them the term, the closest term, I think that they get

it. That was the case yesterday [referring to the video]. (…) They knew that there was always 1 less on top, so when it was 50 they took 1 and put 49, then they put 50 below. Then, they can do it. They are doing it without realizing it, by doing $n - 1 + n$.

Researcher – $n - 1 + n$, yes it was one of the situations.

Diana – They are doing it without realizing it, but they didn't write it. Nobody from primary school or middle school wrote it, I think. (E2)

The generalization that the students establish is algebraic, as Diana saw. She says that it is important for students to analyse close and distant terms to «see the relationship between the order and the terms. I think that's it… They compare the relationship between the order and the terms and, if they understand the first cases, they will also understand more [distant ones]» (E2, Diana). According to her, the search for terms of a more distant order establishes this relationship more than an indication of terms of a very near order, because «for example, here in the fourth, they would probably draw it and say, without understanding the relationship, what was more and what was less» (E2, Diana). She realised that, for near orders, students may choose a strategy of representing and counting and still not establish the generalization. The first strategy is ineffective in finding distant terms and, therefore, the generalization becomes easier.

Analysis of the teacher's practice. Based on the video of the lesson, the participants identified significant aspects regarding the dynamics and organization of the class when it came to their work on pictorial sequences. Some participants stated that the experience was important to their teacher training, because it enabled them «to know the classroom environment and to analyse the methodologies used by the teacher (…)» (Portfolio, F3).

The experience led the participants to recognize how important the teacher's role is in managing and leading the classroom in ways that provide opportunities for their students to develop their algebraic thinking. The participants saw that the teacher asked students to find distant terms so that they could use both strategies discussed in the classroom, to add to the order, the order of the previous term (based on the pictorial representation) used by most students, and the subtraction of double the order plus one, followed by one student. At the end of the class, the grade 2 teacher suggests that they all use the latter strategy in finding some distant terms.

Alice's group highlighted various important aspects of the teacher's practice, namely, task presentation, the organization of the students' work and the use of resource materials (Figure 12).

- Reading the statement and explaining the tasks
- Student questions
- Teacher's orientation of the work, ie, in pairs and in case there are different opinions or explanations register them
- Providing materials for the students to solve the work sheet (checkered sheet and pieces – squares)

FIGURE 12 – THE TEACHER'S PRACTICE (ALICE'S GROUP)

Moreover, the group underlined the teacher's role in the classroom dynamics, particularly during the presentation and discussion of student strategies (Figure 13).

To explore the solutions and explanations that students had, the teacher was always asking the students questions about their solutions.

FIGURE 13 – THE TEACHER'S ROLE (ALICE'S GROUP)

Alice examined what the teacher in the video said to promote the student involvement: «The teacher asked many questions and never gave the answer, and she let the students figure out the answer themselves» (E2). Moreover, she indicated that teachers must know the students so that they can «adapt the knowledge to the students inside the classroom. You can try one way or another, and use several strategies in order to meet the students' needs» (E2). For this participant, the teacher should promote a classroom dynamic aimed at the students' learning progress, without directly providing the answer.

Analysis of the classroom enabled Beatriz to see that students must have the opportunity to explore the assignments and that the teacher must be prepared to assist them if they have any questions or difficulties as the teacher in the video did. Thus, with regard to working with sequences, she suggested:

> First, either (...) say nothing and let them explore and succeed, but if they fail, warn them and (...) tell them, for example, to try and look at the order

> and the number of elements of various orders, see what each of them has in common, how often it increases, for instance (E2, Beatriz).

Diana noted in the teacher's practice that she offered blocks to students to represent the pictorial terms. She discusses the possibility of using manipulative materials in primary school to make the situations more concrete:

> *For* example, to divide the class into groups and to give them materials for them to work with. Making bead strings with different colours, making a sequence. Then, for example, one thing that the program now encompasses, is to present and explain to the class how they made the sequence. What could be made after that… The teacher may ask questions to the class to see if the kids understood what the regularity was. Which is the tenth piece… Or the tenth bead, for example. I think they must do practical work that they can manipulate (E3, Diana).

Diana suggested that, for the situation she presented, group work is more appropriate, followed by a whole class discussion. This is what happened in the taped excerpt, in which the students shared what they did and the teacher asked the children listening if they understood the regularity that was presented. For primary school students, she suggests completion of a practical assignment that engages them, where they can manipulate objects and discuss ideas with their classmates, justifying their answers.

DISCUSSION

The prospective teachers' mathematical knowledge grew considerably, particularly their ability to establish relationships and use different representations and strategies to solve tasks. The experience also enabled them to learn more formal strategies, geared toward establishing relationships and abandoning trial and error strategies, as Beatriz did. It also taught them about strategies that were closer to the students' knowledge, so that they could solve situations other than by using only systems of equations, as F8 did. The study shows that the analysing students' answers to the task 2 problem with unknown quantities, helped the participants to understand different representations and solution strategies and may have acted as the precursor to the use of letters to designate the unknown.

The experience also made them aware of how they themselves learned and how they could use different representations and strategies to suit the students' grade level. In task 2, Diana highlighted the different solutions that the problem evoked, and Beatriz became cognizant of how important the teacher's knowledge was in dealing with the different solutions. Beatriz also emphasized the importance of knowing different strategies and the skill required by the teacher to assess the students' answers and orient classwork so as to encourage discussion and sharing of ideas. Task 4 contributed to the participants' understanding of the various ways in which students can analyse a pictorial sequence, and how they express generalization. During the teacher education experiment, the participants became more aware of the teacher's role in promoting student participation and encouraging them to share their reasoning.

The prospective teachers homed in on specifics of the teacher's practice, such as the selection of tasks, how they used different strategies to complete the tasks, and the importance of understanding the students and their learning processes. Analysing teaching and learning situations involving grade 6 students gave the participants a glimpse of the material and methods they could design for their future students, as well as the representations and reasoning that they could use. Diana stressed the importance of fostering moments of autonomous work and group discussions, and Alice noted that the time for sharing and discussing strategies contributes to students' learning, especially in the case of students who are struggling with math. As Crespo (2000), Capraro *et al.* (2008) and Nickerson and Masarik (2010) suggested, the participants in this study also showed development in their understanding of students' reasoning and the ability to see the teaching of algebra in terms of the tasks and ways of working in the classroom.

The results of the participants' work in these two tasks show that the analysis of students' solutions and of teacher's practice is a meaningful addition to the education of prospective teachers. In line with the theoretical principles that guided the teacher education experiment and the results in each task, the analysis of classroom situations was conducted throughout the teacher education experiment in other tasks on these and other topics, such as the study of functions.

The study also bears out the importance of using video excerpts for mathematics teaching development. The viewing of the grade 2 class with pictorial sequences enabled the participants to jointly analyse the students' and the teacher's work, the students' strategies, and how they communicated them. Without the video it would have been hard to broach the context during the prospective teachers' course.

Beatriz valued this learning opportunity for the chance it gave her to analyse the different ways students look at a pictorial sequence. All the participants were surprised that the students succeeded in generalising the pictorial sequence, thereby finding distant terms. They acknowledged that the students do not use symbolic algebraic language, but rather express such generalizations in natural language. The study shows, as Llinares and Valls (2009) mention, that video analysis identifies key aspects of teaching and learning, particularly with regard to the students' ability to generalize; it also sheds light on the way they express such generalizations. Since analysing what goes on in the classroom is eminently served by videos, it would be logical to use them even more in prospective teacher education.

CONCLUSION

The experiment gave future teachers a general view of how algebraic thinking could be promoted in primary school. It also led the participants to appreciate the teacher's role as a promoter of learning through suitably designed tasks, one who fosters communication, and as a classroom manager and supervisor. Experiments such as these also promote the analysis of the students' role and work, their degree of understanding, their strategies, and the difficulties they demonstrate, the representations they use and the connections they establish.

By analysing teaching and learning situations in primary school, future teachers are also learning mathematics themselves. They see representations and strategies that are different from their own, and this enhances their own knowledge. The experience also enabled Alice, Beatriz and Diana to understand the teaching of algebra, and allowed them to envision situations that were conducive to algebraic thinking, which is an important aspect of the teacher's role (Canavarro, 2007). This aspect is of particular significance, because not all participants had experiences with generalization and formalization when they were in primary school (Kaput & Blanton, 2001).

The discussion of the work done in the classroom with sequences and relationships gave the prospective teachers a vehicle for reflecting on key aspects of teaching practice, classroom dynamics, and how to follow the students' train of thought and get them to adjust their reasoning. Analysing teaching and learning through video put a spotlight on teaching practice itself, in particular the way the task is introduced, what materials are provided, and how student communication is promoted by asking questions.

Our study indeed suggests that by analysing and reflecting on teaching and learning situations help future teachers learn how to teach algebra, as Doerr (2004) suggests. More specifically, it helps them learn to engage their future students in situations that promote their algebraic thinking, taking into account aspects of the teacher's practice close to those analysed during the teacher education experiment and if it is in line with the exploratory approach.

The participants were aware of the importance of a classroom environment that is focused on generalization, sharing, and discussing students' reasoning, features that Kaput and Blanton (2001) and Blanton and Kaput (2011) recommend for teaching practice. Overall, the prospective teachers were surprised by the students' ability to generalize and by the diversity of representations and strategies they used.

Thus, being able to analyse a wide range of teaching and learning situations and the class-wide discussion of the experience help to develop the prospective teachers' algebraic thinking, in particular their ability to use and interpret different representations and strategies. It also hones the participants' skill at algebra teaching, making them aware of the challenges they face in selecting tasks, conducting the class, and focusing on reasoning and its representations. In all, this study showed how teacher education based upon a variety of tasks, aimed at exploring concepts, generalizing and analysing student output and teacher practice, combined with moments of reflection and group discussion, enhances the prospective teachers' understanding of algebra and of how they can foster it in their students.

REFERENCES

Albuquerque, C., Veloso, E., Rocha, I., Santos, L., Serrazina, L., & Nápoles, S. (2006). *A Matemática na formação inicial de professores*. Lisboa: APM & SPCE.

Askew, M. (2008). Mathematical discipline knowledge requirements for prospective primary mathematics teachers, and the structure and teaching approaches of programs designed to develop that knowledge. In P. Sullivan & T. Wood (Eds.), *International handbook of mathematics teacher education* (Vol. 1., pp. 13-35). Rotterdam: Sense.

Blanton, M., & Kaput, J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education, 36*(5), 412-446.

Blanton, M., & Kaput, J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 5-23). Berlin: Springer.

Bogdan, R., & Biklen, S. K. (1982). *Qualitative research for education: An introduction to theory and methods*. Boston, MA: Allyn & Bacon.

Canavarro, A. P. (2007). O pensamento algébrico na aprendizagem da Matemática nos primeiros anos. *Quadrante, 16*(2), 81-118.

Capraro, M. M., Rangel-Chavez, A., & Capraro, R. M. (2008). Effective preparation for teaching of algebra at the primary level. Paper presented at *11th International Conference on Mathematics Education (ICME-11)*, Monterrey, México. Retrieved from http://tsg.icme11.org/document/get/382.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Carpenter, T., & Levi, L. (2000). *Developing conceptions of algebraic reasoning in the primary grades. Research report*. Madison, WI: Wisconsin Center for Education Research, University of Wisconsin, School of Education. Retrieved from http://www.eric.ed.gov/ERICDocs/data/ericdocs2sql/content_storage_01/0000019b/80/1a/8d/19.pdf.

Carraher, D., & Schliemann, A. (2007). Early algebra and algebraic reasoning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669-705). Charlotte, NC: NCTM & IAP.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Cobb, P., Zhao, Q., & Dean, C. (2009). Conducting design experiment to support teachers' learning: A reflection from the field. *Journal of the Learning Sciences, 18*(2), 165-199.

Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. *The Journal of the Learning Sciences, 13*(1). 15-42.

CBMS – Conference Board of the Mathematical Sciences. (2011). *The mathematical education of teachers II*. Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Crespo, S. (2000). Seeing more than right and wrong answers: prospective teachers' interpretation of students' mathematical work. *Journal of Mathematics Teacher Education, 3*, 155-181.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61*, 293-319.

Doerr, H. M. (2004). Teacher's knowledge and teaching of algebra. K. Stacey, H. Chick & M. Kendal (Eds.) *The future of the teaching and learning of algebra: The 12th ICMI Study* (pp. 267-289) New York, NY: Kluwer.

Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema & T. Romberg (Orgs.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Erlbaum.

Kaput, J., & Blanton, M. (2001). Algebrafying the elementary mathematics experience, Part I: Transforming Task Structures. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The future of the teaching and learning of algebra* (Vol. 1, pp. 344-351). Melbourne, Australia: The University of Melbourne.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator, 8*(1), 139-151.

Kieran, C. (2007). Developing algebraic reasoning: The role of sequenced tasks and teacher questions from the primary to the early secondary school levels. *Quadrante, 16*(1), 5-26.

Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick & M. Kendal (Eds.), *The future of the teaching and learning of algebra The 12th ICMI Study* (pp. 47-70). Dordrecht: Kluwer.

Llinares, S., & Valls, J. (2009). The building of pre-service primary teachers' knowledge of mathematics teaching: Interaction and online video case studies. *Instructional Science, 37*, 247-271.

Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Education Research Journal, 3*, 28-36.

ME – Ministério da Educação. (2007). *Programa de matemática do ensino básico.* Lisboa: ME.

NCTM – National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: NCTM.

NCTM – National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Nickerson, S., & Masarik, D. (2010). Assessing teachers' developing interpretive power: Analysing students thinking. *Mathematics Teacher Education and Development, 12*(1), 19-29.

Ponte, J. P. (2005). Gestão curricular em Matemática. In GTI (Org.), *O professor e o desenvolvimento curricular* (pp. 11-34). Lisboa: APM.

Ponte, J. P. (2006). Números e álgebra no currículo escolar. In I. Vale, T. Pimentel, A. Barbosa, L. Fonseca, L. Santos & P. Canavarro (Orgs.), *Números e Álgebra na aprendizagem da Matemática e na formação de professores* (pp. 5-27). Lisboa: SEM-SPCE.

Ponte, J. P. (2011). Using video episodes to reflect on the role of the teacher in mathematical discussions. In O. Zaslavsky & P. Sullivan (Eds.), *Constructing knowledge for teaching secondary mathematics: Tasks to enhance prospective and practicing teacher learning* (pp. 249-261). New York, NY: Springer.

Ponte, J. P., & Chapman, O. (2008). Preservice mathematics teachers' knowledge and development. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 225-263). New York, NY: Routledge.

Ponte, J. P., Quaresma, M., & Branco, N. (2012). Práticas profissionais dos professores de Matemática. *Avances en Investigación en Educación Matemática, 1*, 65-86.

Reeves, C. A. (2000). The chicken problem. *Mathematics Teaching in the Middle School, 5*(6), 398-402.

Ruthven, K. (1989). An exploratory approach to advanced mathematics. *Educational Studies in Mathematics, 20*, 449-467.

Sánchez, V., Llinares, S., García, M., & Escudero, I. (2000). La formación de profesores de primaria desde la didáctica de las matemáticas. *Números: Revista de Didáctica de las Matemáticas, 43-44*, 143-146.

Silvestre, A., Faria, A., Sousa, H., Cristo, I., Santos, I., Molarinho, M. J., & Veladas, M. (2010). Sequências pictóricas: estratégias de generalização dos alunos de 2.º, 3.º e 5.º anos. In GTI (Org.), O *professor e o programa de matemática* (pp. 89-119). Lisboa: APM.

Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Sullivan, P. (2011). Identifying and describing the knowledge needed by teachers of mathematics. *Journal of Mathematics Teacher Education*, *14*, 171-173.