THE KNOWLEDGE QUARTET: THE GENESIS AND APPLICATION OF A FRAMEWORK FOR ANALYSING MATHEMATICS TEACHING AND DEEPENING TEACHERS' MATHEMATICS KNOWLEDGE

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ABSTRACT

This paper describes a framework for mathematics lesson observation, the 'Knowledge Quartet', and the research and policy contexts in which it was developed. The framework has application in research and in mathematics teaching development. The research which led to the development of the framework drew on videotapes of mathematics lessons prepared and conducted by elementary pre-service teachers towards the end of their initial training. A grounded theory approach to data analysis led to the emergence of the framework, with four broad dimensions, through which the mathematics-related knowledge of these teachers could be observed in practice. This paper describes how each of these dimensions is characterised, and analyses two lessons, showing how each dimension of the Quartet can be identified in the lesson. The paper concludes by outlining recent developments in the use of the Knowledge Quartet.

KEY WORDS

Mathematics teaching; Teacher knowledge; Teacher education; Knowledge Quartet.

The Knowledge Quartet: The Genesis and Application of a Framework for Analysing Mathematics Teaching and Deepening Teachers' Mathematics Knowledge

Tim Rowland

INTRODUCTION

This paper concerns a framework for the analysis of mathematics teaching – the Knowledge Quartet – which was first developed at the University of Cambridge in the years 2002-4. Since then, the Knowledge Quartet has been applied in several research and teacher education contexts, and the framework has been further refined and developed as a consequence. In order to understand the nature and the status of the Knowledge Quartet, it will be useful to consider first the nature of teacher knowledge in general and mathematics teacher knowledge in particular. The paper then proceeds to a description of the research study which led to the emergence of the Knowledge Quartet, and then discusses some of the ways in which it has been used and developed further.

TEACHER KNOWLEDGE: THE BIG PICTURE

From its historical origins in philosophical deliberation, modern empirical study of teacher knowledge is firmly rooted in the landmark studies of Lee Shulman and his colleagues in the 1980s. In his 1985 presidential address to the American Educational Research Association, Shulman proposed a taxonomy

with seven categories that formed a knowledge-base for teaching. Whereas four of these elements (such as knowledge of educational purposes and values) are generic, the other three concern 'discipline knowledge', being specific to the subject matter being taught. They are: subject matter knowledge, pedagogical content knowledge and curricular knowledge.

Shulman's (1986) conceptualisation of subject matter knowledge (SMK) includes Schwab's (1978) distinction between substantive knowledge (the key facts, concepts, principles and explanatory frameworks in a discipline) and syntactic knowledge (the nature of enquiry in the field, and how new knowledge is introduced and accepted in that community). For Shulman, pedagogical content knowledge (PCK) consists of «the ways of representing the subject which make it comprehensible to others (…) [it] also includes an understanding of what makes the learning of specific topics easy or difficult (…)» (Shulman, 1986, p. 9). The identification and de-coupling of this hitherto 'missing link' between knowing something for oneself and being able to enable others to know it is, arguably, Shulman's most enduring contribution to the field. 'PCK' gives educators, whatever their role, a language with which to describe, and to celebrate, what teachers know about and others do not – even those with comparable qualifications in subject matter per se.

THE TEACHER KNOWLEDGE 'PROBLEM' IN THE UK

International comparative studies (e.g. Mullis, Martin & Foy, 2008), and the related 'league tables', have resulted in a search for scapegoats and demands in a number of countries for improvement of the mathematics knowledge of prospective and serving teachers. Difficulties associated with teachers' mathematical content knowledge are particularly apparent in the elementary sector, where generalist teachers often lack confidence in their own mathematical ability (Brown, McNamara, Jones & Hanley, 1999; Green & Ollerton, 1999). Identifying, developing and deepening teachers' mathematical content knowledge has become a priority for policy makers and mathematics educators around the world.

The rather direct approach to a perceived 'problem' in England was captured by an edict in the first set of government 'standards' for Initial Teacher Training (ITT), first issued in 1997, which required teacher education programmes to «audit trainees' knowledge and understanding of the mathematics contained in the National Curriculum», and where 'gaps' are identified to

«make arrangements to ensure that trainees gain that knowledge» (Department for Education and Employment, 1998, p. 48). This process of audit and remediation of subject knowledge within primary ITT became a high profile issue following the introduction of these and subsequent government requirements. Within the teacher education community, few could be found to support the imposition of the 'audit and remediation' culture.

Yet the introduction of this regime provoked a body of research in the UK on prospective elementary teachers' mathematics subject-matter knowledge (e.g., Goulding, Rowland & Barber, 2002). The proceedings of a symposium held in 2003 usefully drew together some of the threads of this research (BSRLM, 2003). One study, with 150 London-based graduate trainee primary teachers (Rowland, Martyn, Barber & Heal, 2000), found that trainees obtaining high (or even middle) scores on a 16-item audit of content knowledge were more likely to be assessed as strong mathematics teachers on school-based placements than those with low scores; whereas those with low audit scores were more likely than other participants to be assessed as weak mathematics teachers.

This was an interesting finding, and a team at the University of Cambridge wanted to find out more about what was 'going on', and took forward this new line of enquiry. If superior content knowledge really does make a difference when teaching elementary mathematics, it ought somehow to be observable in the *practice* of the knowledgeable teacher. Conversely, the teacher with weak content knowledge might be expected to misinform their pupils, or somehow to miss opportunities to teach mathematics 'well'. In a nutshell, the Cambridge team wanted to identify, and to understand better, the ways in which elementary teachers' mathematics content knowledge, or the lack of it, is visible in their teaching.

DEVELOPING THE KNOWLEDGE QUARTET

CONTEXT AND PURPOSE OF THE RESEARCH

Several researchers have argued that mathematical content knowledge needed for teaching is not located in the minds of teachers but rather is realised through the practice of teaching (Hegarty, 2000; Mason & Spence, 1999). From this perspective, knowledge for teaching is constructed in the context of teaching, and can therefore be observed only a *in vivo*' knowledge in this context.

In the UK, the majority of prospective, trainee teachers are graduates who follow a one-year program leading to a Postgraduate Certificate in Education (PGCE) in a university[1] education department. Over half of the PGCE year is spent teaching in schools under the guidance of a school-based mentor, or 'cooperating teacher'. Placement lesson observation is normally followed by a review meeting between the cooperating teacher and the student-teacher. On occasion, a university-based tutor will participate in the observation and the review. Thirty years ago, Tabachnick, Popkewitz and Zeichner (1979) found that «cooperating teacher/student teacher interactions were almost always concerned with (…) procedural and management issues (…) There was little or no evidence of any discussion of substantive issues in these interactions» (p. 19). The situation has not changed, and more recent studies also find that mentor/trainee lesson review meetings typically focus heavily on organisational features of the lesson, with very little attention to the mathematical content of mathematics lessons (Borko & Mayfield, 1995; Strong & Baron, 2004).

The purpose of the research from which the Knowledge Quartet emerged was to develop an empirically-based conceptual framework for lesson review discussions *with a focus on the mathematics* content of the lesson, and the role of the trainee's mathematics subject matter knowledge (SMK) and pedagogical content knowledge (PCK). In order to be a useful tool for those who would use it in the context of *practicum* placements, such a framework would need to capture a number of important ideas and factors about mathematics content knowledge in relation to teaching, within a small number of conceptual categories, with a set of easily-remembered labels for those categories.

The focus of this particular research was therefore to identify ways that teachers' mathematics content knowledge – both SMK and PCK – can be observed to 'play out' in practical teaching. The teacher-participants in this study were novice, trainee elementary school teachers, and the observations were made during their school-based placements. Whilst we believe certain kinds of knowledge to be *desirable* for elementary mathematics teaching, we are convinced of the futility of asserting what a beginning teacher, or a more experienced one for that matter, *ought* to know. Our interest is in what a teacher *does* know and believe, and how opportunities to enhance knowledge

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1   It should be noted, however, that the government now actively promotes a range of workplace-based alternatives (such as 'School Direct') to the PGCE. These are effectively located in notions of apprenticeship, and offer little interaction with university-based teacher educators.

can be identified. We have found that the Knowledge Quartet, the framework that arose from this research, provides a means of reflecting on teaching and teacher knowledge, with a view to developing both.

The research reported in this paper was undertaken in collaboration with Cambridge colleagues Peter Huckstep, Anne Thwaites, Fay Turner and Jane Warwick. I frequently, and automatically, use the pronoun 'we' in this text in recognition of their contribution.

METHOD: HOW THE KNOWLEDGE QUARTET CAME ABOUT

The participants in the original study were enrolled on a one-year PGCE course in which each of the 149 trainees specialised either on the Early Years (pupil ages 3–8) or the Primary Years (ages 7–11). Six trainees from each of these groups were chosen for observation during their final school placement. The 12 participants were chosen to reflect a range of outcomes of a subject-knowledge audit administered three months earlier. Two mathematics lessons taught by each of these trainees were observed and videotaped, i.e. 24 lessons in total. The trainees were asked to provide a copy of their planning for the observed lesson. As soon as possible after the lesson, the observer/researcher wrote a succinct account of what had happened in the lesson, so that a reader might immediately be able to contextualise subsequent discussion of any events within it. These 'descriptive synopses' were typically written from memory and field notes, with occasional reference to the videotape if necessary.

From that point, we took a grounded approach to the data for the purpose of generating theory (Glaser & Strauss, 1967). In particular, we identified in the videotaped lessons aspects of trainees' actions in the classroom that seemed to be significant in the limited sense that it could be construed to be informed by a trainee's mathematics subject matter knowledge or their mathematical pedagogical knowledge. We realised later that most of these significant actions related to choices made by the trainee in their planning or more spontaneously. Each was provisionally assigned an 'invented' code. These were grounded in particular moments or episodes in the tapes. This provisional set of codes was rationalised and reduced (e.g. eliminating duplicate codes and marginal events) by negotiation and agreement in the research team.

The 17 codes generated by this inductive process are itemised later in this chapter. The name assigned to each code is intended to be indicative of the

type of issue identified by it: for example, the code *adheres to textbook* (AT) was applied when a lesson followed a textbook script with little or no deviation, or when a set of exercises was 'lifted' from a textbook, or other published resource, sometimes with problematic consequences. By way of illustration of the coding process, we give here a brief account of an episode that we labelled with the code *responding to children's ideas* (RCI). It will be seen that the contribution of a child was unexpected. Within the research team, this code name was understood to be potentially ironic, since the observed response of the teacher to a child's insight or suggestion was often to put it to one side rather than to deviate from the planned lesson script, even when the child offered further insight into the topic at hand.

Code RCI: *an illustrative episode*. Jason was teaching elementary fraction concepts to a Year 3 class (pupil age 7–8). Each pupil held a small oblong whiteboard and a dry-wipe pen. Jason asked them to «split» their individual whiteboards into two. Most of the children predictably drew a line through the centre of the oblong, parallel to one of the sides, but one boy, Elliot, drew a diagonal line. Jason praised him for his originality, and then asked the class to split their boards «into four». Again, most children drew two lines parallel to the sides, but Elliot drew the two diagonals. Jason's response was to bring Elliot's solution to the attention of the class, but to leave them to decide whether it was correct. He asked them:

Jason: What has Elliot done that is different to what Rebecca has done?
Sophie: Because he's done the lines diagonally.
Jason: Which one of these two has been split equally? (…) Sam, has Elliot split his board into quarters?
Sam: Um … yes … no …
Jason: Your challenge for this lesson is to think about what Elliot's done, and think if Elliot has split this into equal quarters. There you go Elliot.

At that point, Jason returned the whiteboard to Elliot, and the question of whether it had been partitioned into quarters was not mentioned again. What makes this interesting mathematically is the fact that (i) the four parts of Elliot's board are not congruent, but (ii) they have equal areas; and (iii) this is not at all obvious. Furthermore, (iv) an elementary demonstration of (ii) is arguably even less obvious. This seemed to us a situation that posed very direct demands on Jason's SMK and arguably his PCK too. It is not possible to infer whether Jason's

«challenge» is motivated by a strategic decision to give the children some thinking time, or because he needs some himself.

Equipped with this set of codes, we revisited each lesson in turn and, after further intensive study of the tapes, elaborated each descriptive synopsis into an analytical account of the lesson. In these accounts, the agreed codes were associated with relevant moments and episodes, with appropriate justification and analysis concerning the role of the trainee's content knowledge in the identified passages, with links to relevant literature.

The identification of these fine categories was a stepping stone with regard to our intention to offer a practical framework for use by ourselves, our colleagues and teacher-mentors, for reviewing mathematics teaching with trainees following lesson observation. A 17-point tick-list (like an annual car safety check) was not quite what was needed. Rather, the intended purpose demanded a more compact, readily-understood scheme which would serve to frame a coherent, content-focused discussion between teacher and observer. The key to the solution of our dilemma was the recognition of an association between elements of subsets of the 17 codes, enabling us to group them (again by negotiation in the team) into four broad, superordinate categories, which we have named (I) foundation (II) transformation (III) connection (IV) contingency. These four units are the dimensions of what we call the 'Knowledge Quartet'.

Each of the four units is composed of a small number of subcategories that we judged, after extended discussions, to be of the same or a similar nature. An extended account of the research pathway described above is given in Rowland (2008). The Knowledge Quartet has now been extensively 'road tested' as a descriptive and analytical tool. As well as being re-applied to analytical accounts of the original data (the 24 lessons), it has been exposed to extensive 'theoretical sampling' (Glaser & Strauss, 1967) in the analysis of other mathematics lessons in England and beyond (see e.g. Weston, Kleve & Rowland, 2013).

As a consequence, three additional codes[2] have been added to the original 17, but in its broad conception, we have found the KQ to be comprehensive as a tool for thinking about the ways that content knowledge comes into play in the classroom. We have found that many moments or episodes within a les-

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2   These new codes, derived from applications of the KQ to classrooms within and beyond the UK, are *teacher insight* (Contingency), *responding to the (un)availability of tools and resources* (Contingency) and use of *instructional materials* (Transformation) respectively.

son can be understood in terms of two or more of the four units; for example, a **contingent** response to a pupil's suggestion might helpfully **connect** with ideas considered earlier. Furthermore, the application of content knowledge in the classroom always rests on **foundational** knowledge.

Mathematical Knowledge for Teaching and the Knowledge Quartet

It is useful to keep in mind how the KQ differs from the well-known Mathematical Knowledge for Teaching (MKT) egg-framework due to Deborah Ball and her colleagues at the University of Michigan, USA (Ball, Thames & Phelps, 2008). The Michigan research team refer to MKT as a «practice-based theory of knowledge for teaching» (Ball & Bass, 2003, p. 5). The same description could be applied to the Knowledge Quartet, but while parallels can be drawn between the methods and some of the outcomes, the two theories look very different. In particular, the theory that emerges from the Michigan studies aims to unpick and clarify the formerly somewhat elusive and theoretically-undeveloped notions of SMK and PCK. In the Knowledge Quartet, however, the distinction between different *kinds* of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In this sense, the two theories are complementary, so that each has useful perspectives to offer to the other.

CONCEPTUALISING THE KNOWLEDGE QUARTET

The concise conceptualisation of the Knowledge Quartet which now follows draws on the extensive range of data referred to above. As we observed earlier, the practical application of the Knowledge Quartet depends more on teachers and teacher educators understanding the broad characteristics of each of the four dimensions than on their recall of the contributory codes.

Foundation

§ Contributory codes: awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures.

The first member of the KQ is rooted in the foundation of the teacher's theoretical background and beliefs. It concerns their knowledge, understanding and ready recourse to what was learned at school and at college/university, including initial teacher education in preparation (intentionally or other-

wise) for their role in the classroom. It differs from the other three units in the sense that it is about knowledge 'possessed'[3], irrespective of whether it is being put to purposeful use. For example, we could claim to have knowledge about division by zero, or about some probability misconceptions – or indeed to know where we could seek advice on these topics – irrespective of whether we had had to call upon them in our work as teachers. Both empirical and theoretical considerations have led us to the view that the other three units flow from a foundational underpinning.

A key feature of this category is its *propositional* form (Shulman, 1986). It is what teachers learn in their 'personal' education and in their 'training' (pre-service in this instance). We take the view that the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way. By 'fundamental' we have in mind a rational, reasoned approach to decision-making that rests on something other than imitation or habit. The key components of this theoretical background are: knowledge and understanding of mathematics per se; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt.

In summary, this category that we call 'foundation' coincides to a significant degree with what Shulman (1987) calls 'comprehension', being the first stage of his six-point cycle of pedagogical reasoning.

Transformation

§   Contributory codes: teacher demonstration; use of instructional materials; choice of representation; choice of examples.

The remaining three categories, unlike the first, refer to ways and contexts in which knowledge is brought to bear on the preparation and conduct of teaching. They focus on knowledge-in-action as *demonstrated* both in planning to teach and in the act of teaching itself. At the heart of the second member of the KQ, and acknowledged in the particular way that we name it, is Shulman's observation that the knowledge base for teaching is distinguished by «the capacity of a teacher to *transform* the content knowledge he or she possesses

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3   The use of this acquisition metaphor for knowing suggests an individualist perspective on Foundation knowledge, but we suggest that this 'fount' of knowledge can also be envisaged and accommodated within more distributed accounts of knowledge resources (see Hodgen, 2011).

into forms that are pedagogically powerful» (1987, p. 15, emphasis added). This characterisation has been echoed in the writing of Ball (1988), for example, who distinguishes between knowing some mathematics 'for yourself' and knowing in order to be able to help someone else learn it. As Shulman indicates, the presentation of ideas to learners entails their re-presentation (our hyphen) in the form of analogies, illustrations, examples, explanations and demonstrations (Shulman, 1986, p. 9).

Our second category, unlike the first, picks out behaviour that is directed towards a pupil (or a group of pupils), and which follows from deliberation and judgement informed by foundation knowledge. This category, as well as the first, is informed by particular kinds of literature, such as the teachers' handbooks of textbook series or in the articles and 'resources' pages of professional journals. Increasingly, in the UK, teachers look to the internet for 'bright ideas', and even for ready-made lesson plans. The trainees' choice and *use of examples* has emerged as a rich vein for reflection and critique. This includes the use of examples to assist concept formation and demonstrate procedures, and the selection of exercise examples for student activity.

Connection
§ Contributory codes: making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness.

The next category binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content – the learning, perhaps, of a concept or procedure. It concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons. Mathematics is notable for its coherence as a body of knowledge and as a field of enquiry. Indeed, a great deal of mathematics is held together by deductive reasoning.

The pursuit of coherence and mathematical connections in mathematics pedagogy has been stimulated recently by the work of Askew, Brown, Rhodes, Johnson and Wiliam (1997): of six case study teachers found to be highly effective, all but one gave evidence of a 'connectionist' orientation. The association between teaching effectiveness and a set of articulated beliefs of this kind lends a different perspective to the work of Ball (1990), who also strenuously argued for the importance of connected knowledge for teaching.

Related to the integrity of mathematical content in the mind of the teacher and his/her management of mathematical discourse in the classroom, our conception of coherence includes the *sequencing* of topics of instruction within and between lessons, including the ordering of tasks and exercises. To a significant extent, these reflect deliberations and choices entailing not only knowledge of structural connections within mathematics itself, but also awareness of the relative cognitive demands of different topics and tasks.

Contingency

§ Contributory codes: responding to students' ideas; deviation from agenda; teacher insight; (un)availability of resources.

Our final category concerns the teacher's response to classroom events that were not anticipated in the planning. In some cases, it is difficult to see how they could have been planned for, although that is a matter for debate. In

DIMENSION	CONTRIBUTORY CODES
Foundation: knowledge and understanding of mathematics per se and of mathematics-specific pedagogy, beliefs concerning the nature of mathematics, the purposes of mathematics education, and the conditions under which students will best learn mathematics	· awareness of purpose · adherence to textbook · concentration on procedures · identifying errors · overt display of subject knowledge · theoretical underpinning of pedagogy · use of mathematical terminology
Transformation: the presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations	· choice of examples · choice of representation · use of instructional materials · teacher demonstration (to explain a procedure)
Connection: the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks	· anticipation of complexity · decisions about sequencing · recognition of conceptual appropriateness · making connections between procedures · making connections between concepts
Contingency: the ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events	· deviation from agenda · responding to students' ideas · (use of opportunities) · teacher insight during instruction · responding to the (un)availability of tools and resources

TABLE I — THE KNOWLEDGE QUARTET: DIMENSIONS AND CONTRIBUTORY CODES

commonplace language this dimension of the KQ is about the ability to 'think on one's feet': it is about *contingent action*. Shulman (1987) proposes that most teaching begins from some form of 'text' – a textbook, a syllabus, ultimately a sequence of planned, intended actions to be carried out by the teacher and/or the students within a lesson or unit of some kind. Whilst the stimulus – the teacher's intended actions – can be planned, the students' responses cannot.

Brown and Wragg (1993) suggested that 'responding' moves are the lynch-pins of a lesson – important in the sequencing and structuring of a lesson – and observed that such interventions are some of the most difficult tactics for novice teachers to master. The quality of such responses is undoubtedly determined, at least in part, by the knowledge resource available to the teacher, as the earlier illustrative episode with Jason demonstrates. For further details, see Rowland, Thwaites and Jared (2011).

For ease of reference, the account of the KQ above is summarised in Table 1, on the previous page.

In the next two sections, I shall illustrate the application of the KQ in the analysis of mathematics lessons: one primary, one secondary. In both cases, the teachers are pre-service graduate students.

PRIMARY MATHEMATICS TEACHING: THE CASE OF LAURA

The lesson now under scrutiny is one of the original 24 videotaped lessons. The graduate trainee in question, Laura, was teaching a Year 5 (pupil age 9–10) class about written multiplication methods, specifically multiplying a two-digit number by a single digit number.

Laura's Lesson

Laura reminded the class that they had recently been working on multiplication using the 'grid' method. She spoke about the tens and units being «partitioned off». Simon was invited to the whiteboard to demonstrate the method for 9×37. He wrote:

×	30	7	
9	270	63	= 333

Laura then said that they were going to learn another way. She proceeded to write the calculation for 9 × 37 on the whiteboard in a conventional but elaborated column format, explaining as she wrote:

$$
\begin{array}{rr}
 & 37 \\
 & \times\ 9 \\
\hline
30 \times 9 & 270 \\
7 \times 9 & 63 \\
\hline
 & 333 \\
 & \scriptstyle 1
\end{array}
$$

Laura performed the sum 270 + 63 by column addition *from the right*, 'carrying' the 1 (from 7 + 6=13) from the tens into the hundreds column. She wrote the headings h, t, u [indicating hundreds, tens, units] above the three columns.

Next, Laura showed the class how to «set out» 49 × 8 in the new format, and then the first question (19 × 4) of the exercises to follow. The class proceeded to work on these exercises, which Laura had displayed on a wall. Laura moved from one child to another to see how they were getting on. She emphasised the importance of lining up the hundreds, tens and units columns carefully, and reminded them to estimate first.

Finally, she called the class together and asked one boy, Sean, to demonstrate the new method with the example 27 × 9. Sean got into difficulty; he was corrected by other pupils and by Laura herself. As the lesson concluded, Laura told the children that they should complete the set of exercises for homework.

We now select from Laura's lesson a number of moments, episodes and issues to show how they might be perceived through the lens of the Knowledge Quartet. It is in this sense that we offer Laura's lesson as a 'case' – it is typical of the way that the KQ can be used to identify for discussion matters that arise from the lesson observation, and to structure reflection on the lesson. Some possibilities for discussion with the trainee, and for subsequent reflection, are flagged below thus: **Discussion point**. We emphasise that the process of *selection* in the commentary which follows has been extreme. Nevertheless, we raise more issues relating to content knowledge than would normally be considered in a post-lesson review meeting.

FOUNDATION

First, Laura's professional knowledge underpins her recognition that there is more than one possible written algorithm for whole number multiplication. We conceptualise this within the domain of fundamental knowledge, being the

foundation that supports and significantly determines her intentions or actions. Laura's learning objective seems to be taken from the National Numeracy Strategy (NNS) *Framework* (DfEE, 1999) teaching programme for Year 4:

· Approximate first. Use *informal* pencil and paper methods to support, record or *explain* multiplication. *Develop* and *refine* written methods for TUxU (p. 3/18, emphasis added).

These objectives are clarified by examples later in the *Framework*; these contrast (A) *informal* written methods – the grid, as demonstrated by Simon – with (B) *standard* written methods – the column layout, as demonstrated by Laura in her introduction. In both cases (A and B), an 'approximation' precedes the calculation of a worked example in the *Framework*. Laura seems to have assimilated the NNS guidance and planned her teaching accordingly. It is perhaps not surprising that she does not *question* the necessity of teaching the standard column format to pupils who already have an effective, meaningful algorithm at their disposal. Indeed, many respected educators advocate the adequacy and pedagogical preference of grid-type methods with primary pupils (e.g. Haylock, 2001, pp. 91-94).

§ **Discussion point**: where does Laura stand on this debate, and how did her stance contribute to her approach in this lesson?

Another issue related to Laura's fundamental knowledge is her approach to computational estimation. When she asks the children to estimate 49×8, one child proposes 400, saying that 8×50 is 400. Laura, however, suggests that she could make this «even more accurate» by taking away two lots of 50. She explains, «Because you know two times five is ten and two times fifty is a hundred, you could take a hundred away». Perhaps she had 10×50 in mind herself as an estimate, or perhaps she confused something like subtracting 8 from the child's estimate. She recognises her error and says «Sorry, I was getting confused, getting my head in a spin». The notions of *how* to estimate and *why* it might be desirable to do so are not adequately discussed or explored with the class.

§ **Discussion point**: what did Laura have in mind in this episode, and is there some way she can be more systematic in her approach to computational estimation?

At this stage of her career in teaching, Laura gives the impression that she is passing on her own practices and her own forms of knowledge. Her main resource seems to be her own experience (of using this algorithm), and it seems that she does not yet have a view of mathematics didactics as a *scientific* enterprise.

Laura's own ability to perform column multiplication is secure, but her pedagogical challenge is to *transform* what she knows for herself into a form that can be accessed and appropriated by the children. Laura's choice of demonstration examples in her introduction to column multiplication merits some consideration and comment. Her first example is 37 ' 9; she then goes on to work through 49 ' 8 and 19 ' 4. Now, the NNS emphasises the importance of *mental* methods, where possible, and also the importance of choosing the most suitable strategy for any particular calculation. 49 ' 8 and 19 ' 4 can all be more efficiently performed by rounding up, multiplication and compensation e.g. 49 ' 8 = (50 ' 8)-8. Perhaps Laura had this in mind in her abortive effort to make the estimate of 400 «even more accurate».

Her choice of exercises – the practice examples – also invites some comment. The sequence is: 19 × 4, 27 × 9, 42 × 4, 23 × 6, 37 × 5, 54 × 4, 63 × 7, 93 × 6, with 99 × 9, 88 × 3, 76 × 8, 62 × 43, 55 × 92, 42 × 15 as 'extension' exercises (although no child actually attempts these in the lesson). Our earlier remark about the suitability of the column algorithm relative to alternative mental strategies applies to several of these, 99 × 9 being a notable example.

But suppose for the moment that it is understood and accepted by the pupils that they are to put aside consideration of such alternative strategies – that these exercises are there merely as a vehicle for them to gain fluency with the algorithm. In that case, the *sequence* of exercises might be expected to be designed to present the pupils with increasing challenge as they progress though them.

§ **Discussion point**: on what grounds did Laura choose and sequence these particular examples and exercises? What considerations might contribute to the choice?

Perhaps the most important connection to be established in this lesson is that between the grid method and the column algorithm. Laura seems to have

this connection in mind as she introduces the main activity. She reminds them that they have used the grid method, and says that she will show them a «different way to work it out». She says that the answer would be the same whichever way they did it «because it's the same sum». However, Laura does not clarify the connections between the two methods: that the same processes and principles – partition, distributivity and addition – are present in both. No reference to the grid method is made in her second demonstration example, 49 × 8. Her presentation of this example now homes in on procedural aspects – the need to «partition the number down», «adding a zero» to 8 × 4, getting the columns lined up, adding the partial products from the right. The fact that the connection is tenuous for at least one pupil is apparent in the plenary. Sean actually volunteers to calculate 27 × 9 on the whiteboard. He writes 27 and x9 in the first two rows as expected, but then writes 20 × 7 and 2 × 9 to the left in the rows below.

§ **Discussion point**: Laura is clearly trying to make a connection between the grid method and the column method. What reasons did she have in mind for doing so? To what extent did she think she was successful?

CONTINGENCY

Sean's faulty attempt (mentioned above) to calculate 27 × 9 on the whiteboard appears to have surprised Laura – it seems that she fully expected him to apply the algorithm faultlessly, and that his actual response really was unanticipated. In the event, there are several 'bugs' in his application of the procedure. The partition of 27 into 20 and 2 is faulty, and the multiplicand is first 9, then 7. This would seem to be a case where Sean might be encouraged to reconsider what he has written by asking him some well-chosen questions. One such question might be to ask how he would do it by the grid method. Or simply why he wrote those particular numbers where he did. Laura asks the class «Is that the way to do it? Would everyone do it that way?». Leroy demonstrates the algorithm correctly, but there is no diagnosis of where Sean went wrong, or why.

§ **Discussion point**: what might be the reason for Sean's error? In what ways could this have been addressed in the lesson, or subsequently?

FINAL REMARK CONCERNING LAURA'S LESSON

It is all too easy for an observer to criticise a novice teacher for what they omitted or committed in the high-stakes environment of a school placement, and we would emphasise that the KQ is intended as a tool to *support* teacher development, with a sharp and structured focus on the impact of their SMK and PCK on teaching. Indications of how this might work are explicit in our analysis of Laura's lesson. We have emphasised that our analysis has been *selective*: we raised some issues for attention, but there were others which, not least out of space considerations, we chose not to mention.

SECONDARY MATHEMATICS TEACHING: THE CASE OF HEIDI

REVISED METHOD

The lesson to be described and analysed in this section took place some years after the original project described earlier in this paper. The objective in this phase of our research programme was systematic testing of the KQ as an analytical framework in the context of secondary education. As before, lessons were video-recorded, and trainees were asked to provide a copy of their lesson plan for reference in later analysis. At this point, the data collection was extended to include a post-lesson interview, as follows. As soon as possible after the lesson, the research team met to undertake preliminary analysis of the videotaped lesson, and to identify some key episodes in it with reference to the KQ framework. Then, again with minimum delay, one team member met with the trainee to view some episodes[4] from the lesson and to discuss them in the spirit of stimulated-recall (Calderhead, 1981). These interview-discussions addressed some of the issues that had come to light in the earlier KQ-structured preliminary analysis of the lesson. An audio recording was made of this discussion, to be transcribed later. In some cases, the observation, preliminary analysis and stimulated-recall interview all took place on the same day.

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4   A DVD of the full lesson was given to the trainee soon afterwards, as a token of our appreciation, but their reflections on viewing this DVD in their own time are not part of our data.

The lesson to be considered now was taught by Heidi, who had come to the course direct from undergraduate study in mathematics at a well-regarded UK university. Her practicum placement secondary school was state-funded, with some 1400 pupils across the attainment range. In keeping with almost all secondary schools in England, pupils were 'setted' by attainment in mathematics, with 10 or 11 sets in most years.

heidi's lesson

Heidi's class was one of two parallel 'top' mathematics sets in Year 8 (pupil age 12–13), and these pupils would be expected to be successful both now and in the high-stakes public examinations in the years ahead. 17 boys and 13 girls were seated at tables facing an interactive white board (IWB)[5] located at the front of the room. The objectives stated in Heidi's lesson plan were as follows: «Go over questions from their most recent test, and then introduce direct proportion».

Heidi returned the test papers from a previous lesson to the students, and proceeded to review selected test questions with the whole class, asking the pupils about their solution methods. The first question to be revisited was on percentages, and the next two on simultaneous linear equations. Several pupils offered solution methods, and these were noted on the IWB. Heidi responded to requests for review of two more questions, and nearly 30 minutes of the 45-minute lesson had elapsed before Heidi moved on to the topic of direct proportion.

She began by displaying images of three similar cuboids on the IWB: she explained that the cuboids were boxes, produced in the same factory, and that the dimensions were in the same proportions. The linear scale factor between the first and second cuboids was 2 [Heidi wrote x2], and the third was three times the linear dimensions of the second [x3]. Heidi identified one rectangular face, and asked what would happen to the area of this face as the dimensions increased. They calculated the areas, and three pupils made various conjectures about the relationship between them. The third of these said «I think it is that number [the linear scale factor] squared».

Heidi then introduced two straightforward direct proportion word problems. One, for example, began «6 tubes of toothpaste have a mass of 900g.

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5   Interactive whiteboards, with associated projection technology, are now more-or-less universal in secondary classrooms in England.

What is the mass of 10 tubes?». Different solutions were offered and discussed. Heidi then gave them six exercise questions of a similar kind (e.g., «In 5 hours a man earns £30. How much does he earn in 6 hours?»).

THE KNOWLEDGE QUARTET: HEIDI'S LESSON

We now offer our interpretation of some ways in which we observed or inferred *foundation*, *transformation*, *connection* and *contingency* (but not in that order) in Heidi's lesson. It will become apparent that many moments or episodes within a lesson can be understood in terms of two or more of the four dimensions. We also draw upon her lesson plan and upon her contributions to the post-lesson, stimulated-recall discussion with Anne, one of the research team. This discussion had homed in on two fragments of the lesson: the first was Heidi's review of a test question on simultaneous equations; the second was the introduction of the proportion topic using the IWB-images of the three cuboids.

TRANSFORMATION

Heidi had little or no influence regarding the choice of examples (a key component of this KQ dimension) in her test review, since the test had been set by a colleague. However, the stimulated-recall interview gave an opportunity and a motive for her to reflect on the test items. There had been two questions (7 and 8) on simultaneous equations, and the related pairs of equations were

$$\text{Q7: } 2x + 3y = 16, \ 2x + 5y = 20 \qquad \text{Q8: } 3b - 2c = 30, \ 2b + 5c = 1$$

In response to an interview question, Heidi thought the sequencing appropriate. In particular (regarding Q7) she said «They could do it the way it was», seeming to refer to the fact that one variable (x) could be eliminated by subtraction, without the need for scaling either equation. In fact, the pupils' response to Heidi's invitation to offer solution methods suggested that this opportunity was not recognised or not welcomed. The first volunteer, Max, had proposed multiplying the first equation by 10, and the second by 6, suggesting a desire on his part to eliminate y, not x. (Heidi's response to this will be considered under *Contingency*). Heidi was able to explain this in her answer to Anne's question, «What if the y-coefficients were the same?» Heidi's first response was «That would be less difficult, because they tended to want to get rid of the y. I don't know why».

In fact, in this lesson segment, when eliminating one variable by adding or subtracting two equations, Heidi reminds the class several times about a 'rule', namely: if the signs are the same, then subtract; if they are different then add. Heidi suggests, later in the interview, that the pupils tend to want to make the y-coefficients equal, as Max did, because their signs are *explicit* in both equations. This can be seen in both Q7 and Q8 where the coefficient of the first variable is positive in both equations, and the sign left implicit, whereas $+$ or $-$ is explicit in the coefficient of the second variable. This insight of Heidi's is typical of the way that focused reflection on the disciplinary *content* of mathematics teaching, structured by the KQ, has been found to provoke valuable insights on how to improve it (Turner & Rowland, 2010). Heidi's observation is that restricting the x-coefficients to positive values (and emphasising the 'rule') has somehow imposed unintended limitations on student solution methods, with a preference for eliminating y even when «they could do it the way it was» by eliminating x.

Turning now to Heidi's introduction of the direct proportion topic, in our preliminary lesson analysis we misinterpreted Heidi's use of the three cuboids. Her lesson plan included: «Discussion point: What happens to the area of the rectangular face as the dimensions increase? What happens to the volumes of the cuboids as the dimensions increase?». We took this to mean that she intended to investigate the relationship between linear scale factor (between similar figures) and the scale factors for area and volume. Although she had been drawn into this topic, this had not been her intention, as the subsequent word problems indicated. In the event, there was discussion in the lesson of the area of one rectangular face of the cuboid, and how its area increases as the cuboids grow larger: there was not time to consider the volumes. When probed about her choice of context for the introduction of the direct proportion topic, Heidi said that she had chosen the cuboids because it was «a nice visual» which contrasted with the «wordy» presentation of the other problems. In the interview, when asked whether she agreed that she could have done the work on area comparison with rectangles, she replied «You're absolutely right, rectangles would be enough (…) but I did like my box factory». Here we see an example of trainees' propensity to choose representations in mathematics teaching on the basis of their superficial attractiveness at the expense of their mathematical relevance (Turner, 2008). In this instance, the preference for these 'visuals' took Heidi into mathematical territory for which she was not mathematically prepared (see *Contingency*).

Analysis of this dimension of the KQ in Heidi's lesson intertwines with the component of *foundation* concerned with teachers' beliefs about mathematics and mathematics teaching. Here, we begin by taking up the story of Max's suggestion to solve Q7 by multiplying the first equation by 10, and the second by 6. In the interview, Anne asked Heidi why she had «run with» Max's suggestion. Heidi replied «Because it would work. You're trying to find the lowest common denominator, but it would work. Like adding fractions, it would work with any common multiple. I didn't want him to think he was wrong». This kind of openness to pupils' suggestions and ability to anticipate where they would lead was very characteristic of Heidi's teaching, and several examples of it can be found in our data.

In the class discussion which followed, Heidi's introduction of the three cuboids, the pupils calculated (in cm^2) the areas of the rectangles with sides (respectively) 2×3, 4×6, 12×18 (all cm) viz. 6, 24, 216 (in cm^2). Heidi had annotated x2, x3, as I noted earlier. One pupil suggested that the relationships between the areas were «timesed by 4 and timesed by 6». Heidi made it clear that she was not checking these calculations numerically («I'm going to take your word for that»), recorded this second proposal on the IWB (writing x4 and x6) and said «So two times what this has been timesed by [*pointing to the linear scale factors*]. Good observations». This seemed to be the end of the matter, until a third pupil, Lay Tun, said «I think it is that number squared». Heidi paused, then changed the second factor (from x6 to x9), and emphasised the squares.

Now, this length/area relationship in similar figures was not what Heidi had set out to teach, and it became clear at the interview that Heidi (unlike Lay Tun) did not know in advance about «that number squared». In the interview, the discussion proceeded:

Anne: Then you go on to areas. They give a range of options. Now, you take all these responses and give value to all of them. But this was different, in that two of these responses were not correct.

Heidi: I want to take everyone's ideas on board. When you do put something on the board they correct each other rather than me being the authority. In that case, I had a bit of a brain freeze. I hadn't worked out how many times 24 goes into 216, but they're used to me putting up everything.

We see here, paradoxically, a situation in this secondary teaching data in which some subject-matter in the school curriculum lies outside the scope of the content knowledge of the trainee at that moment in time. This should come as no great surprise. For all their university education in mathematics, and their knowledge of topics such as analysis, abstract algebra and statistics, there remain facts from the secondary curriculum that they will have had no good reason to revisit since they left school. What is significant, however, exemplified by Heidi but more-or-less absent in our observations of primary mathematics classrooms, is a teacher with the confidence to negotiate and make sense of mathematical situations such as this (the length/area relationship) 'on the fly', as they arise.

FOUNDATION

This lesson does raise a few issues about Heidi's content knowledge that might be brought to her attention, and some of them were raised in the interview. Briefly, these include: her use of mathematical terminology, which is either very careful and correct (e.g. 'coefficient'), or quite the opposite (e.g. 'timesing'); her lack of fluency and efficiency in mental calculation, such that she did not question the suggestion that $6 \times 24 = 216$ herself in the cuboids situation: on occasion it appeared that she was puzzled by some of the pupils' mental calculations; thirdly, she was not aware of the length/area/volume scale-factor relationships referred to earlier.

But, after many hours spent scrutinising the recording of this lesson, and that of the post-lesson interview, our lasting impression relates to the beliefs component of the *Foundation* dimension, in particular, Heidi's beliefs about her role in the classroom in bringing pupils' ideas and solution strategies into the light, even – as we remarked earlier – when she believed that 'her way' would, in some sense, be better. As she told Anne, «I want to take everyone's ideas on board. When you do put something on the board they correct each other rather than me being the authority». Her perception of this aspect of her role as teacher and the possibility of the pupils themselves contributing to pupil learning is resonant of various constructivist and fallibilist manifestos. Heidi constantly assists this 'letting go' by acknowledging pupils' suggestions, and making them available for scrutiny by writing them on the board. Occasionally she finds herself in deep water as a consequence, but she never seems to doubt her [mathematical] ability to stay afloat.

We coded a few events in this lesson under *connection*. For example, Heidi's introduction to direct proportionality with the cuboids seemed quite unrelated to the word problems which followed. In any case, the rather diverse objectives for the lesson were likely to make it somewhat 'bitty', and we omit further analysis of this KQ dimension from the present narrative.

SUPPORTING RESEARCH AND TEACHING DEVELOPMENT

The KQ has found two intersecting user groups since its emergence a decade ago. In this section, we outline resources developed to support these user groups.

TEACHER EDUCATION AND TEACHING DEVELOPMENT

As we remarked earlier, one of the goals of our original 2002 research was to develop an empirically-based conceptual framework for mathematics lesson review discussions with a focus on the *mathematics content* of the lesson and the role of the trainee's mathematics subject matter knowledge (SMK) and pedagogical content knowledge (PCK). In addition to the kind of 'knowledgeable-other' analysis and formative feedback exemplified in the cases of Laura and Heidi in this paper, it has also been used to support teachers wanting to develop their teaching by means of reflective evaluation on their own classroom practice (Corcoran, 2011; Turner, 2012). Specifically, the KQ is a tool which enables teachers to focus reflection on the mathematical content of their teaching.

However, both teacher educators and teachers must first learn about the tool, and how to put it to good use. A book (Rowland, Turner, Thwaites & Huckstep, 2009) was written to address the needs of this user-group, especially in relation to primary mathematics. It describes the research-based origins of the KQ, with detailed accounts of the four dimensions, and separate chapters on key codes such as Choice of Examples. The narrative of the book is woven around accounts of over 30 episodes from actual mathematics lessons. We return to this use of the KQ towards the end of this paper.

In some respects, the needs of researchers using the KQ as a theoretical framework for lesson analysis are the same as those of teacher educators, but they are different in others. In particular, a broad-brush approach to the four KQ dimensions often suffices in the teacher education context, and may even be preferable to detailed reference to constituent codes. For example, identifying Contingent moments and actual or possible responses to them need not entail analysis of the particular triggers of such unexpected events. On the other hand, reflections or projections on Transformation usually involve reference to examples and representations. Our writing about the KQ (e.g. Rowland, Huckstep & Thwaites, 2005) initially focused on explaining the essence of each of the four dimensions rather than identifying definitions of each of the underlying codes. However, a detailed KQ- analysis of a record (ideally video) of instruction necessarily involves labelling events at the level of individual KQ-codes prior to synthesis at dimension level (Foundation, Transformation, etc.). This, in turn, raises reliability issues: the coder needs a deep understanding of what is intended by each code, going beyond any idiosyncratic connotations associated with its name. Addressing this issue, a Cambridge colleague of ours wrote as follows:

> Essentially, the Knowledge Quartet provides a repertoire of ideal types that provide a heuristic to guide attention to, and analysis of, mathematical knowledge-in-use within teaching. However, whereas the basic codes of the taxonomy are clearly grounded in *prototypical teaching actions*, their grouping to form a more discursive set of superordinate categories – Foundation, Transformation, Connection and Contingency – appears to risk introducing too great an interpretative flexibility *unless these categories remain firmly anchored in grounded exemplars of the subordinate codes* (Ruthven, 2011, p. 85, emphasis added).

In 2010, a Norwegian doctoral student wrote to us as follows: «I need a more detailed description on the contributory codes to be able to use them in my work. Do you have a coding manual that I can look at?». This enquiry, Ruthven's comment, and our growing sense of the risk of «interpretive flexibility» led us to initiate a project to develop an online coding manual, with the needs of researchers particularly in mind.

The aim of the project was to assist researchers interested in analysing classroom teaching using the Knowledge Quartet by providing a comprehensive col-

lection of «grounded exemplars» of the 20 contributory codes from primary and secondary classrooms. An international team of 15 researchers was assembled. All team members were familiar with the KQ and had used it in their own research as a framework with which to observe, code, comment on and/or evaluate primary and secondary mathematics teaching across various countries, curricula, and approaches to teaching. The team included representatives from the UK, Norway, Ireland, Italy, Cyprus, Turkey and the United States.

In Autumn 2011, team members individually scrutinised their data and identified prototypical classroom-exemplars of some of the KQ codes. To begin with, a written account of each selected classroom *scenario* was drafted. Often this included excerpts of transcripts and/or photographs from the lesson. Then a *commentary* was written, which analysed the excerpt, explaining why it is representative of the particular code, and why it is a strong example. Each team member submitted scenarios and commentary for at least three codes from his/her data to offer as especially strong, paradigmatic exemplars. In March 2012, 12 team members gathered in Cambridge, and worked together for two days. Groups of three team members evaluated and revised each scenario and commentary. The scenarios and commentaries were then revised on the basis of the conference feedback. Further details of the participants and methodology are given in Weston, Kleve and Rowland (2013).

These scenarios and commentaries now combine to form a «KQ coding manual» for researchers to use. This is a collection of primary and secondary classroom vignettes, with episodes and commentaries provided for each code. The collection of codes and commentaries is now freely available online at *www.knowledgequartet.org*. At the time of writing, the website is 'live' but subject to further development. We encourage researchers and teacher educator to use and share this website in the cause of improved clarity about what each of the KQ codes 'looks like' in a classroom setting.

CONCLUSION

Mathematics teaching is a highly complex activity; this complexity ought to be acknowledged when teaching is analysed and discussed, and due attention is given to discipline-specific aspects of pedagogical decision and actions beyond generic aspects of the management of learning. Strong, clear conceptual frameworks assist in the management of this complexity. By attending to

events enacted and observed in actual classrooms, with a specific focus on the subject-matter under consideration, the KQ offers practitioners and researchers such a conceptual framework, particularly suited to understanding the contribution of teacher knowledge to mathematics teaching.

For practitioners and teacher educators, the KQ is a tool for identifying opportunities and possibilities for teaching development through the enhancement of teacher knowledge, as indicated, for example, in the book Rowland *et al.* (2009). Especially in the case of pre-service teacher education, it is beneficial to limit the post-observation review meeting to one or two lesson fragments, and also to only one or two dimensions of the KQ, in order to focus the analysis and avoid overloading the trainee-teacher with action points.

In this paper, I have emphasised the progression from observation of teaching to its description and analysis, but I have been less explicit about the evaluation of teaching. In the spirit of reflective practice, the most important evaluation must be that of the teacher him/herself. However, this self-evaluation is usefully provoked and assisted by a colleague or mentor. Earlier in this paper, in the account of the Case of Laura, I have exemplified this provocation through the identification, using the KQ, of tightly-focused discussion points to be raised in a post-observation review. We have suggested that these points be framed in a relatively neutral way, such as «Could you tell me why you … ?» or «What were you thinking when … ?». It would be naïve, however, to suggest that the mentor, or teacher educator, makes no evaluation of what they observe. Indeed, the observer's evaluation is likely to be a key factor in the identification and prioritisation of the discussion points. In post-observation review, it is expected that the 'more knowledgeable other' will indicate what the novice did well, what they did not do and might have, and what they might have done differently. The KQ is a framework to organise such evaluative comments, and to identify ways of learning from them.

The KQ has been successfully applied across different phases of schooling, and in diverse cultures, but we mention, in conclusion, a development that we had not originally anticipated. Having attended presentations about the KQ in cross-disciplinary settings, some teacher education colleagues working in subjects other than mathematics – such as language arts, science and modern foreign languages education – have seen potential in the KQ for their own lesson observations and review meetings. They sometimes ask whether they could adapt and adopt the KQ for their own purposes. This raises the issue: can a framework for knowledge-in-teaching developed in one subject discipline be legitimately adopted in another? My reply usually begins as follows: what

might the conceptualisations of the dimensions of the KQ, beginning with Foundation, look like in this other discipline? An answer to this question could set the scene for empirical testing of the KQ in another subject area.

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