PROFESSIONAL NOTICING: A COMPONENT OF THE MATHEMATICS TEACHER'S PROFESSIONAL PRACTICE

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ABSTRACT

In this paper, I characterize the notion of «teaching competency» as being able to use knowledge in a pertinent way to carry out mathematics teaching tasks. One aspect of teaching competency is the teacher's «professional noticing» of students' mathematical thinking. This feature of teacher competency involves the cognitive ability to identify and interpret the salient features of the students' output in order to make informed decisions. This construct is illustrated using a context in which the prospective teachers are learning to interpret the students' answers to linear and nonlinear problems (in order to recognize the students' development of proportional reasoning). In this context, when prospective teachers use increasingly more explicit mathematics elements and features of the development of mathematical understanding to describe and interpret the students' mathematical thinking is considered as evidence of the development of professional awareness.

KEY WORDS

Professional practice; Professional noticing; Teaching expertise; Mathematics teaching; Teachers' tasks in mathematics teaching.

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INTRODUCTION

The mathematics teacher's skill of identifying the relevant features in teaching situations and interpreting them from the learner's perspective in order to make decisions about what course the lesson should take is seen as an important component of the teaching practice (Mason, 2002; Sherin, Jacobs & Philipp, 2010). The teacher's knowledge of mathematics and the didactics of mathematics are central in this skill of mathematics teaching expertise. Indeed, the relationship between the different components of the knowledge mathematics for teaching has led some researchers to try to clarify them in order to understand the relation between knowledge and practice (Ball, Thames & Phelps, 2008). This is ultimately linked to what the teacher needs to know to solve professional problems (mathematics teaching and learning situations).

Here, we complement this perspective by identifying the mathematical knowledge that enhances the teacher's ability to «professionally notice» the students' mathematical thinking. We aim to reflect on the teacher professional practice and knowledge in order characterize the teacher's professional noticing examining the role played by mathematical knowledge in the profes-

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sional activities. We shall look at data from a teacher education program to gain a better understanding of this teaching skill in relation to how prospective teachers interpret the development of proportional reasoning. By examining these situations we can gain a better awareness of how to develop this skill in teacher education programs (Llinares, 2012).

THE MATHEMATICS TEACHER: KNOWLEDGE AND PROFESSIONAL PRACTICE

How to characterize what the mathematics teacher knows and how this is put into practice in the classroom is a topic in mathematics education (Ponte & Chapman, 2006). The teacher's knowledge and using this knowledge are dependent constructs. The mathematics teacher's professional knowledge is characterized by how to apply this knowledge in mathematics teaching contexts. This idea presupposes that the contexts in which one acquires knowledge and where one uses have a didactic relationship (Escudero & Sánchez, 2007a, 2007b).

Another element that goes into shaping the mathematics teacher's professional profile is the complementarity between the knowledge from research (knowledge that can be found in books and scientific journals) and the knowledge acquired through experience. In the long run, the professional's performance as a practitioner depends on the way teacher gathers, selects, integrates and interprets his/her experience. The practice of mathematics teacher's involves a number of professional tasks (Figure 1). One of these professional tasks is the adaptation of mathematical activities to support the students' learning (e.g., Gafanhoto & Canavarro, 2012; Morris, Hiebert & Spitzer, 2009); others aim to guide mathematical discussion in class (e.g., Fortuny & Rodríguez, 2012; Ponte, Quaresma & Branco, 2012); and yet others are to analyse student's mathematical thinking (e.g., Fernández, Llinares & Valls, 2013; Sánchez-Matamoros, Fernández, Valls, García & Llinares, 2012).

The identification of tasks which constitute the mathematics teacher's professional practice is relevant because it allows to relation the teacher's knowledge and «the use of knowledge in context.» In this sense, the focus on «the use of knowledge to resolve professional tasks» is relevant to better understand the practice and professional knowledge of mathematics teacher.

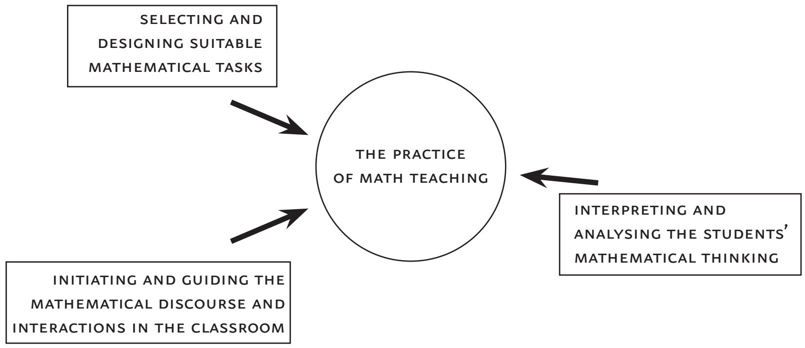


FIGURE I – SYSTEM OF ACTIVITY IN THE MATHEMATICS TEACHER PROFESSIONAL PRACTICE

«NOTICING» OF STUDENTS' MATHEMATICAL THINKING

The idea of «using knowledge to resolve professional tasks» is a core component of the teaching competency. This competence is knowing what, how and when to use specific knowledge to solve the mathematics teaching tasks. The skill «notice professionally» requires that the teacher be able to: identify relevant aspects of the teaching situation; use knowledge to interpret the events, and establish connections between specific aspects of teaching and learning situations and more general principles and ideas about teaching and learning (Jacobs, Lamb & Philipp, 2010; Mason, 2002; Sherin, Jacobs & Philipp, 2010). This way of understanding the construct of «noticing» considers that the teacher's identification of the mathematical elements which are relevant in the problem that the pupils have to solve and in the solution they might produce, allows the teacher to be in a better position to interpret their learning and to take relevant instructional decisions. Specifically, mathematics knowledge for teaching (Ball, Thames & Phelps, 2008; Hill *et al*., 2008) allows the teacher to identify what is relevant and support her interpretation of these facts and evidences deemed relevant. In this sense, the role played by the teacher's mathematics knowledge for teaching in the resolution of professional tasks defines some aspects of his/her teaching competency (An & Wu, 2012; Sánchez-Matamoros *et al*., 2013; Zapatera & Callejo, 2013).

One particular aspect of teacher notice is the ability to be attuned to the students' mathematical thinking. Being able to understand and analyse the students' mathematical reasoning involves the «reconstruction and inference» of the students' understanding from what the student writes, says or does. The teacher's skill of noticing the students' mathematical thinking demands more than just pointing out what is correct or incorrect about their answers. It requires determining in what way the students' answers are or are not meaningful from the mathematics learning standpoint (Hines & McMahon, 2005; Holt, Mojica & Confrey, 2013). In the following examples we illustrate some features of the skill of «professionally noticing» or being aware of the students' mathematical thinking. We will exemplify these features in the context of students' proportional reasoning.

NOTICING THE DEVELOPMENT OF PROPORTIONAL REASONING

During a teacher training course in which the prospective teachers were to develop their ability to «professionally notice» the pupils' mathematical output, the future teachers had to:

i) describe some pupils' solutions to proportional and non-proportional problems, and then;
ii) interpret the pupils' mathematical understanding from the evidence supplied in their answers (i.e., the way the pupils dealt with the problems reflected their mathematical understanding).

One of the topics in the course was proportional reasoning (Fernández & Llinares, 2012). To describe pupils' responses and interpret their mathematical understanding future teachers must be able to identify the mathematical elements of problems that foster proportional reasoning by interpreting the multiplicative relationship between quantities. In other words, the future teacher must «break down» the mathematics that define the problem and recognize the manner in which the mathematical elements that characterize the problem are present or not in the pupil's answer. In the development of proportional reasoning as a component of multiplicative structures, these mathematical elements are (Lamon, 2007; Vergnaud, 1983):

- The difference between linear and non-linear situations

- The scalar ratio (relationships between corresponding elements [$a/b=f(a)/f(b)$; within – internal – ratios, or comparisons within measure space)
- The constancy of the functional ratio ($a / f(a)=k$; between – external – ratios or comparisons between measure spaces)
- The constructive nature of the multiplicative relationship between two magnitudes: $f(ka+pb)=k.f(a)+p.f(b)$

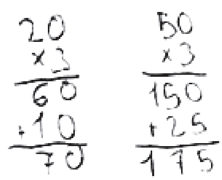
The identification of the relevant mathematical elements in a problem and the interpretation of how they are present in the students' answers allow future teachers to be in better conditions to make relevant instructional decisions and help students develop their proportional reasoning. In this sense, the knowledge of mathematics (Hill *et al*., 2008) allows that the teacher identify and interpret how the students use the mathematical elements when solving proportional problems. For example, recognition of the mathematical elements in the answer given by pupil 1 (figure 2) allows the future teacher to determine whether or not the procedure used is suitable. Moreover, identifying the mathematical elements which give sense to this procedure allows teacher to justify the way this procedure can be generalized (i.e., it is independent of the numbers used).

> «If I multiply the number of metres covered by Sofia by 3, then this also corresponds to the triple of the metres covered by Sara (hence the multiplications 20×3 and 50×3).»
>
> «If 20 corresponds to 50 metres, then half (10) corresponds to half (25, which is half of 50).»
>
> «The total of two quantities of metres covered by Sofia corresponds to the total of the «respective quantities» of metres covered by Sara (therefore, 60 plus10 corresponds to 150 plus25).»
>
> These three points in the process used by the pupil show his recognition of:
>
> $f(20)=50$, so $f(3 \cdot 20)=3 \cdot f(20)=3 \cdot 50$
>
> $f(60+10)= f(60)+f(10)$,
>
> which is the breakdown of the mathematical elements in the problem that define the linear situations
>
> $f(a+b)=f(a)+f(b)$
>
> $f(k.a)=k.f(a)$

On the other hand, the knowledge of mathematics and students, as another component in the mathematics knowledge or teaching (MKT, Hill *et al*., 2008), it is necessary to recognize that using linear in non-linear situations (which happens in pupil 2's answer (Figure 2) is an erroneous approach fairly common (De Bock *et al.*, 2007; Fernández & Llinares, 2012).

PUPIL 1

Sofia and Sara are walking through a field. They began at the same time but Sara is faster. When Sofia has walked 20 metres, Sara has walked 50 metres. When Sofia has walked 70 metres, how many metres will Sara have walked?



PUPIL 2

Juan and Carolina are driving a car around a track. They are driving at the same speed but Juan started later. By the time Juan completed 20 laps, Carolina had completed 60. When Juan has completed 100 laps, how many laps will Carolina have completed?

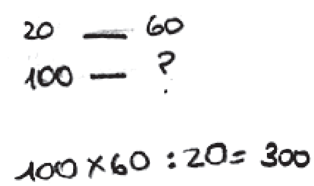


FIGURE 2 – SOME OF THE ANSWERS GIVEN TO FUTURE TEACHERS TO INTERPRET THE STUDENTS' MATHEMATICAL LEARNING

Examining the student's answers in fig. 2, a future teacher gave the following explanation:

> [in relation to pupil 1]. The student realized that in the first time-period, Sofia had covered 20 metres and in triple that time she had covered 60 metres. If he added half of what was covered in one time-period to those 60 metres (20=> 10) he would obtain the total metres we are told Sofia covered. Therefore, you would infer that the distance was covered in 3 and a half time-periods and you would have calculated the distance covered by Sara in that same time frame. (Emphasis added).
> [in relation to pupil 2]. In this problem the pupil thinks that the relationship of laps Juan and Carolina do, with respect to one another, is proportional. He does not realize that one started later than the other and that they are going at the same speed; consequently, they <u>have to</u> complete the same number of laps in a certain time, once they have begun (Emphasis added).

This future teacher's discussion shows that he deems relevant the relationship between the operations carried out by the students and the different relationships between the quantities. Describing pupil 1's answer, he mentions how the student has identified the multiplicative relationship between the quantities and has translated this relationship into the operations he carries out («in the first time-period Sofia had covered 20 metres and *in triple that time* she had covered 60 metres [triple the distance]»). Moreover, the fact that he has pinpointed the multiplicative relationship between the quantities of the two magnitudes (the distance covered by both Sofia and Sara) can be seen when he justifies the students' operations in the sense that adding 10 metres covered by Sara corresponds to adding 25 metres covered by Sofia. This manner of describing pupil 1's answer shows that the future teacher was able to recognize the way in which the mathematical elements of the proportional and non-proportional situations were present in the pupils' answers. The identification of these mathematical elements is the first step in the teacher's ability to correctly interpret the students' level of proportional reasoning.

When the future teacher then describes pupil 2's answer, he recognizes the discrepancy between the relation between the quantities in the problem and the operations the student is carrying out. He notes that pupil 2 is carrying out operations that do not suit the structure of the problem (the relationship between quantities). In other words, this future teacher can differentiate between the proportional situation (the problem of Sofia and Sara) and the non-proportional situation (the situation with Juan and Carolina), and he can recognize when they are - or are not – being picked up by the students.

This way of «noticing» the students' answers allows the future teacher to glean evidence from the students' answers and interpret them in light of the mathematical understanding they reflect. In this case, pupil 1's answers reflect his knowledge of proportional relations $f(k.x)=k.f(x)$ and $f(a+b) = f(a) + f(b)$. Whilst at the same time, the teacher is aware that student 2 does not recognize the additive relationships between quantities and therefore does not discriminate between proportional and non-proportional situations (saying «and he does not realize that one has started later than the other and that they are going at the same speed») making the student apply inappropriate proportional procedures. These examples illustrate how the future teacher can «professionally notice» or be attuned to the students' answers, which in turn allow the teacher to interpret student learning styles. We can see why being attuned/aware is such an important component of math teaching. They are examples of how the knowledge is being used to successfully carry out a professional task.

In this sense, *identify* relevant aspects of the students' output in order to *interpret* their mathematical understanding are teacher's cognitive activities that set professional noticing as a component of his/her teaching competence. That is to say, identifying and interpreting are cognitive actions the future teacher has to undertake in which he/she is using his/her mathematics knowledge for teaching. In this example, the prospective teacher is demonstrating that he/she is aware of the differences between proportional and non-proportional situations and how these differences impact the pupils' solutions; he/she is also aware of the student's misuse of linearity in non-linear situations.

In short, professional noticing or being attuned is a component of the mathematics teacher's professional practice and can be characterized by the teacher's

- possessing mathematical knowledge that facilitates identifying what is relevant from the perspective of learning mathematics in a teaching context, and
- using it to interpret the evidence according to the goals desired.

In other words, to become attuned or aware, the teacher not only needs to have an interpretive viewpoint toward math teaching and learning, but theoretical knowledge as well. Having the theoretical background that allows one to interpret or «professionally notice» is what justifies the use of the word »professional.» In this context, the teacher must assess to what extent her knowledge is relevant to the professional task at hand. In order for knowledge to become «relevant» to a professional task, the teacher must be aware of how his/her own knowledge dovetails with the task to be carried out (Mason, 2002).

We use the term «professionally» aware because this skill may not be innate in the math teacher. Research into the development of this awareness in teachers has shown how complex it is (An & Wu, 2012; Fernández, Llinares & Valls, 2011; Fernández, Llinares & Valls, 2013; Prediger, 2010; Prieto & Valls, 2010; Sánchez-Matamoros *et al.*, 2012; Spitzer, Phelps, Beyers, Johnson & Sieminski, 2010). For example, faced with the same task as described earlier, (Figure 2) another future teacher remarked:

> [In relation to pupil 1]. The student tried to solve the problem without using proportions. From 20 he tries to arrive at 70 using multiplications and addition.

> He knew he had to go from 20 to 70. He did this by multiplying the first number by 3 and adding 10. Applying the same operation to 50 he arrived at 175.
> [In relation to pupil 2]. He applied the proportions method, although he did not write 20:100=60: x, but what he did do was directly write down the formula 100 × 60/20.

This future teacher describes the operations that appear in the student's answer but he is not capable of giving them meaning in relation to the two structures of the situations (one situation is equivalent to the type «SOFIA'S metres=20/50 x SARA'S metres» and the other is equivalent to the type: «Carolina's laps=Juan's laps + 40»).

With regard to problem 1, this future teacher seems to recognize that the student is translating the operations (multiply by three and add half) between the two magnitudes. In describing student 1's solution, we can see that he recognizes the correct relationship between the operations carried out and the relationship between the quantities. However, when discussing problem 2, this future teacher only describes the operations carried out by the pupil, without establishing any relationship with the structure of the quantities involved. His description, which centres on the operations, not the relationship between the quantities, reveals that the future teacher is not capable of recognising that problem 2 is a situation with an additive structure. Therefore, what he calls «the proportions method» is not applicable. The future teacher describes the solving of the problem in terms of the operations carried out, but he does not relate these operations to the structure of the problem. He therefore does not recognize that the solution was incorrect for this type of problem. In this case the future teacher was not able to «break down» the relevant mathematical elements in this situation with the aim of targeting what was relevant in order to pinpoint the student's difficulties.

Answers of this type have been obtained from teacher education programs that are designed to enhance the teachers' ability to «professionally notice» or become attuned. Yet, at the same time, they reveal how difficult it is for some future teachers at this early stage to go further than just describing the operations used to solve a problem. It is indeed hard to make accurate inferences about the pupils' mathematical thinking by offering more than just a superficial description.

In teacher education programs, prospective teachers are usually able to offer different interpretations of the pupils' answers and manage to pinpoint the mathematical elements used in their answers. On the other hand, interpreting the mathematical reasoning in the students' answers is far more complex and future teachers should use as a reference from the mathematics education research. For example, as a reference, future teachers could use the information collated on the levels of development of proportional reasoning (figure 3).

Level 0. Non-proportional reasoning	Level I - Illogical	· Incapable of recognizing multiplicative relations. · Uses numbers and does procedures without sense · Applies proportional strategies to non-proportional situations
	Level A – Addition	· Uses addition relations between numbers. Applies addition strategies in a systematic way (in proportional and non-proportional problems)
Level 1. Informal reasoning in proportional situations		· Uses drawings or manipulatives to give sense to situations · Carries out qualitative comparisons
Level 2. Quantitative reasoning		· Uses constructive strategies · Identifies and uses functional ratios when the ratios are whole numbers
	Level T – Transition	· Begins to constructively use multiplicative relations between the quantities
Level 3. Proportional reasoning	Level R - Reasoning	· Identifies and uses functional ratios when the ratios are NOT whole numbers · Identifies and uses scalar ratios when the reasons are NOT whole numbers · Understands the constancy of scalar ratios · Understands that functional ratio is constant

FIGURE 3 – LEVELS OF DEVELOPMENT OF PROPORTIONAL REASONING WHICH CAN BE USED AS A REFERENCE TO INTERPRET PUPILS' MATHEMATICAL UNDERSTANDING, AS GLEANED FROM THEIR ANSWERS TO PROBLEMS.

The examples presented in this section illustrate that the teacher needs to have mathematical knowledge of the topic (domain-specific mathematical knowledge) and knowledge of mathematics and student in order to identify and interpret the students' mathematical thinking. Thus, one can see how important it is in this context to recognize, for example, the characteristics of proportional and non-proportional situations, the role of different contexts and the relationship between numbers when considering whole and non-whole number ratios.

KNOWLEDGE OF MATHEMATICS AND PROFESSIONAL NOTICING

Professional noticing as a component of teacher teacher's professional competence allows to mathematics teacher «notice» the mathematics teaching situations differently from another person that not is mathematics teacher. Although in recent years this skill has been conceptualized from different perspectives, the common approach involves highlighting the way in which teachers interpret mathematics teaching situations.

Mason (2002), in discussing this particular teaching skill, says that the teacher should be aware of how he/she interprets teaching and learning situations, by taking a structured view of what is relevant to his/her students' learning objectives. According to Mason (2002), one way of noticing in a «structured manner» is to be aware of *how* you are «noticing». The more explicitly future teachers use the mathematical elements of the situation to analyse teaching and learning situations, the more actively they are using specialized knowledge of mathematics (Llinares & Valls, 2009, 2010; Sánchez-Matamoros, Fernández, Llinares & Valls, 2013; Zapatera & Callejo, 2013). The difference in the level of explicitness with which future teachers use relevant mathematical elements to analyse the pupils' work determines to what extent they can develop this skill. Some of the different levels of development have been discussed in previous examples.

However, research results (Fernández *et al*., 2011; Sánchez-Matamoros *et al*., 2012) indicate that although future teachers may have adequate background preparation in mathematics, some find it hard to describe the students' solutions using relevant mathematical elements and identifying the features of the students' mathematical understanding. This demonstrates how important it is for future teachers to develop an explicit «awareness» of the mathematical elements involved in solving problems and their role in determining the students' reasoning.

In the previous examples the mathematical elements regarding the proportional situations would be $f(k.x)=k.f(x)$, $f(a+b)=f(a)+f(b)$ and in the non-proportional, $f(x)=ax+b$, with $b \neq 0$ and $a=1$. Another mathematical element to bear in mind is the type of scalar or functional ratio between the quantities, as well as the numerical relationship between the scalar ratios (relationships between quantities of the same magnitude, 20/70 in problem 1 or the ratio 20/100 in problem 2) and the functional ratios (relationships between quantities of a different magnitude, 20/50 in problem 1 or the ratio 20/60 in problem 2), which can be whole numbers or not and therefore facilitate the pupil's recognition of multiplicative and additive relationships. The future teacher needs to realize that the type of relationship between the quantities (whole numbers or not) introduces different levels of difficulty for students and therefore influences the development of proportional reasoning.

This is an example of the mathematical knowledge for teaching the teacher should use to professionally notice teaching and learning relating to ratio and proportion. It is an example of «knowledge in use» and a characteristic of the teacher's skill at becoming attuned to the students' mathematical thinking. It shows the meaningful use of mathematical knowledge, especially about the different meanings of mathematical objects. Being able to analyse the students' mathematical thinking allows the teacher to build his/her mathematical knowledge for teaching (MKT). Thus, issues of mathematical knowledge that teachers need to teach are linked to the knowledge of mathematics that teachers need to understand the students' mathematical thinking.

THE DEVELOPMENT OF TEACHERS' PROFESSIONAL NOTICING

Some research supports the hypothesis that the teacher's ability to «professional noticing» can be developed (Holt *et al.*, 2013; Llinares, 2012; Schack, Fisher, Thomas, Eisenhardt, Tassel & Yoder, 2013). On the other hand, the learning trajectories of currently practicing and future teachers is now being conceived as a process of enculturation that involves consider the nature and extent of the teacher's professional knowledge and how the teacher uses the knowledge in teaching practice. The challenge for teacher education programs is to coordinate the integrate nature of the knowledge (for example the relationship between the knowledge of mathematics and knowledge of the students' learning) (Hill *et al.*, 2008) and how teacher identify and interpret relevant elements of mathematics teaching (Fernández, Llinares & Valls, 2012; Penalva, Rey & Llinares, 2013; Roig, Llinares & Penalva, 2011).

However, our experience with pre-service teacher education has shown that analysing the students' work in order to infer levels of mathematical understanding is a difficult task. Currently, research centred on the development of this skill in initial teacher education has enabled us to begin to provide information about as we can characterize this development (Fernández *et al.*, 2013; Holt *et al.*, 2013; Schack *et al.*, 2013). Figure 4 shows some of the results obtained with regard to developing the teacher's ability to «professionally notice» when it comes to the subject of proportionality.

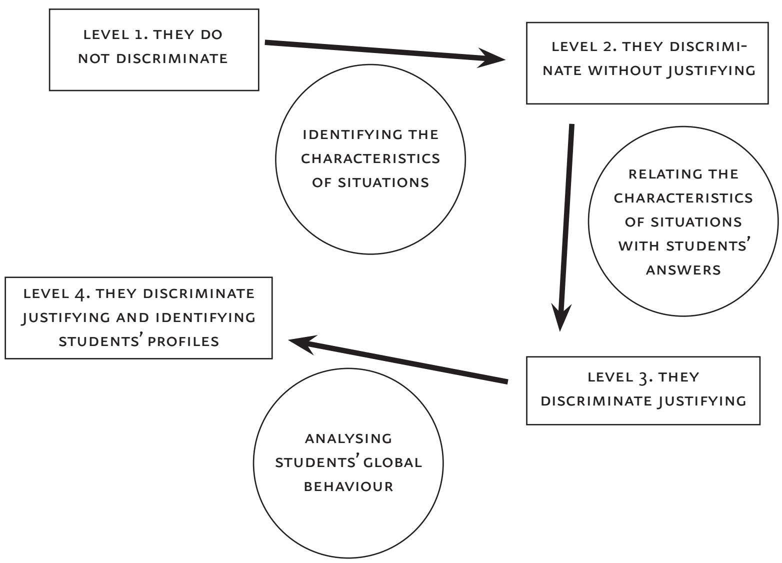


FIGURE 4 – LEVELS OF DEVELOPMENT OF THE TEACHER'S «PROFESSIONAL NOTICE» THE STUDENTS' MATHEMATICAL THINKING IN THE CONTEXT OF PROPORTIONALITY (FERNÁNDEZ, LLINARES & VALLS, 2013, P. 459).

Since it is difficult to develop this skill during teacher training, teacher educators have had to create opportunities – learning environments – for future and practising teachers to acquire new knowledge and skills and enhance their ability to learn through teaching (Llinares, 2012).

CONCLUSION

The emphasis placed recently on a teacher's professional notice or become aware of her students' thought processes is posited on the belief that this skill has a relevant impact on the teaching of mathematics. Some research has proven that when teachers bring enhanced awareness to their teaching, actual teaching practice was enhanced. But what it means to be aware or to «notice professionally» has to be clear to the prospective teachers and also new teaching methods should be introduced in teacher education programmes to improve this skill.

This work has discussed the role played by mathematics knowledge in articulating cognitive actions such as identifying and interpreting the manner in which the student is solving a problem. Hopefully, in the extent in which we as teacher educators link mathematical knowledge with the findings from the findings of mathematical education research on how students learn in different domains, we will be better positioned in order to help prospective teachers to develop this skill.

Research on the development of this teaching skill has allowed us to identify certain characteristics that, in turn, have enabled us to begin to define learning trajectories to describe how this skill evolves (Fernández *et al.*, 2013; Sánchez *et al.*, 2013). However, additional research is needed on the factors that constrain and/or promote this development, while better theoretical models must be developed that enable us to understand it.

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