INSTITUTE OF EDUCATION
UNIVERSITY OF LISBON
VOL. 01 • ISSUE 3 • 2013

SISYPHUS 3

JOURNAL OF EDUCATION VOL. I

SISYPHUS 3

THE PROFESSIONAL PRACTICE AND PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

Edited by
João Pedro da Ponte

INSTITUTE OF EDUCATION
UNIVERSITY OF LISBON
VOL. 01 • ISSUE 3 • 2013

Editor
Jorge Ramos do Ó

Associate editors
João Filipe de Matos, Luís Miguel Carvalho
and Pedro Reis

Invited editor for this issue
João Pedro da Ponte

About this journal
Sisyphus – Journal of Education
ISSN: 2182-8474 (PRINTED VERSION)
ISSN: 2182-9640 (ONLINE VERSION)
Electronic version available, free of charge, at
http://revistas.rcaap.pt/sisyphus

Property
Institute of Education, University of Lisbon
Alameda da Universidade, 1649-013 Lisboa, Portugal
E-mail: sisyphus@ie.ulisboa.pt

Support
This journal is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia within the scope of the Strategic Project – Unidade de Investigação e Desenvolvimento em Educação e Formação (contract PEst-OE/CED/UI4107/2011)

Secretariat
Gabriela Lourenço

Special issue assistant
Joana Mata-Pereira

Proof-reading
AmeriConsulta, Ltd.

Design and Publishing
Edições Tinta-da-china

Printing
Guide, Artes Gráficas

Legal deposit
356 800/13

PVP
12€

SISYPHUS 3
INSTITUTE OF EDUCATION
UNIVERSITY OF LISBON
VOL. 01 • ISSUE 3 • 2013

Sisyphus – Journal of Education aims to be a place for debate on political, social, economic, cultural, historical, curricular and organizational aspects of education. It pursues an extensive research agenda, embracing the opening of new conceptual positions and criteria according to present tendencies or challenges within the global educational arena.

The journal publishes papers displaying original researches – theoretical studies and empiric analysis – and expressing a wide variety of methods, in order to encourage the submission of both innovative and provocative work based on different orientations, including political ones. Consequently, it does not stand by any particular paradigm; on the contrary, it seeks to promote the possibility of multiple approaches. The editors will look for articles in a wide range of academic disciplines, searching for both clear and significant contributions to the understanding of educational processes. They will accept papers submitted by researchers, scholars, administrative employees, teachers, students, and well-informed observers of the educational field and correlative domains. Additionally, the journal will encourage and accept proposals embodying unconventional elements, such as photographic essays and artistic creations.

The Professional Practice and Professional Development of Mathematics Teachers

Introduction by *João Pedro da Ponte* (Editor)

Didactics of mathematics has developed internationally as a scientific field of studies in the late 1960s in the wake of the modern mathematics movement. Particularly important landmarks in this development were the creation of the journal *Educational Studies in Mathematics* by Hans Freudenthal in 1968, and the establishment of the *Journal for Research in Mathematics Education* in 1970, with David Johnson as its first editor.

In Portugal, didactics of mathematics began as a research field between 1980 and 1990 when the first graduates had obtained their PhDs from foreign universities and when master's degree programs were set up at Portuguese universities. The national research journal *Quadrante* was established in 1992. Since then, the mathematics teacher has been one of the subjects to receive the most attention from local and international researchers studying teacher conceptions, knowledge and professional practices, as well as teacher education, development and identity. In the last ten years, the focus on the teacher has clearly been centred on the professional practices, together with institutional conditions and teacher education processes that may promote their transformation in order to further students' learning.

Any research perspective on the mathematics teacher presupposes a perspective on the school, the curriculum, and the role of mathematics as a subject on the curriculum. There is not only one canonical way of viewing the educational role of mathematics. There are many; and each one has its own cultural legitimacy. Therefore, mathematics as a school subject can be viewed in a range of ways: from a winnowing out device that fosters segregation and social stigmatization through school failure, to a tool for the development of creativity, transversal capacities such as communication, reasoning and problem solving. There are examples of both views in many countries including Portugal where, over a few decades, mathematics education has gone from memorizing definitions and procedures to exploratory teaching, emphasizing discovery and student understanding, then back to basics once again in the last few years with an orientation that stresses memorization of terms and rules that are basically meaningless to students.

The important changes that took place in Portugal from 1990 to 2011 with a mathematics education that sought to develop the students' creative and transversal capacities – which is usually termed as exploratory or inquiry-based teaching – was largely based on studies done with and by mathematics teachers in collaborative and/or teacher education settings. The intention was not to improve teaching and learning by inventing solutions in laboratory contexts, but to contribute to student learning by conducting research with groups of teachers in natural settings in order to find solutions that work in such environments, function under usual teaching conditions, and effectively help to solve existing problems.

Any perspective of mathematics teaching subsumes we will have opinions about the teachers themselves. Teachers are at the core of the teaching and learning process and it is tempting to attribute the greatest blame to them for students' poor performance in mathematics. Ironically, it is also tempting to let teachers off the hook, by viewing them as the hapless victims of a top-heavy, inefficient educational system that, very often, blindly follows policies that are based more on subjective preferences than on research-based knowledge.

Yet, it is not easy to have mathematics teachers as the object of one's study. Research carried out in Portugal has sought to avoid the two extreme positions by changing focus: instead of studying the teacher himself, researchers work *with* teachers bearing in mind the conditions they teach under and how they can transform their practice. Research has also sought to take into account a number of different issues related to teaching practice and the teacher's role such as the institutional context, teacher education opportunities, and the surrounding social and political conditions in order to provide the most balanced, wide-ranging view possible of teachers and their professional milieu.

This has been done in tandem with international research, of which Portuguese research initiatives are clearly a part.

The connections between national and international research strands are clearly depicted in this issue of *Sisyphus*. In the first article, Tim Rowland presents the genesis and application of the Knowledge Quartet, one of the most influential frameworks for studying teaching practices and how teachers can develop their knowledge of mathematics teaching. This model includes features that broach knowledge, its transformation and organization for learning purposes, as well as the teachers' ability to make appropriate decisions when confronted with unforeseen classroom situations.

Also examining teaching practice, but from a much more focused point of view, a group of Portuguese researchers, Luís Menezes, António Guerreiro, Maria Helena Martinho, and Rosa Tomás Ferreira discuss the role of questioning in exploratory mathematics teaching. In their paper they explore the different moments in which this type of teaching usually develops in the mathematics classroom. In order to achieve this exploratory stance, they discuss the roles of verification, focusing and inquiry questions during the different stages of a classroom mathematics task.

Salvador Llinares addresses another aspect of mathematics teaching practice – professional noticing, which may be defined as the teacher's skill at identifying and interpreting important aspects of the students' oral and written output. This skill is fundamental if the teacher is to form hypotheses about the students' rationale, undertake new activities and make informed decisions in the classroom.

Three papers in this issue of *Sisyphus* discuss in-service teacher education. First, David Clarke, Hilary Hollingsworth and Radhika Gorur present a model for teacher development in which they closely interlink theory and practice with enaction and reflection as key processes that mediate change in teachers' beliefs, knowledge and practice. They use this model to discuss the contribution of video in facilitating teacher reflection and action.

Working with a group of elementary in-service teachers, Olive Chapman analyses their learning in an inquiry setting in which they are encouraged to take an investigative stance towards their own practice. Her main thesis is that this process may be described as an overarching inquiry cycle in which teachers begin with practice, pose a pedagogical problem, understand a key construct in the problem, hypothesize an inquiry-teaching model, test/apply it, and finally revise/apply the model.

Dario Fiorentini also looks at professional learning, but in a different setting – a mixed collaborative group of teachers and researchers. He argues that this is a formative and powerful environment for participating teachers, especially in terms of developing a research attitude and promoting changes in the way teachers relate to and work with their students. As the author indicates, this is also a powerful way of constructing research knowledge for academics.

Two other papers examine preservice teacher education. In the first, Neusa Branco and João Pedro da Ponte present an algebra course that also stresses the articulation of theory and practice. The framework for this innovative course (which was researched as a teaching experiment) is based on two main premises: the key role of analysing practical situations (represented in different ways) by prospective elementary school teachers and the close connection between content and pedagogy in their development. It also shows the advantages of introducing prospective teachers to the early educational application of algebra.

In another paper, Hélia Oliveira and Márcia Cyrino study prospective mathematics teachers' grasp of inquiry-based teaching. To illustrate their premise they discuss an experiment using multimedia materials. The results show that participants developed an understanding of different dimensions and a heightened awareness of the complexity of inquiry-based teaching.

Finally, Paola Sztajn once again takes up the issue of the relationship between researchers and teachers. At one end of the spectrum we have the prevalent view, among university researchers, that academic knowledge of the craft is superior to that of practitioners. This has often led to another extreme position which argues that, when it comes to the knowledge of these two kinds of professionals, never the twain shall meet. Teachers and researchers work in different institutions, have different practices, belong to different communities and have different kinds of knowledge, and therefore, as some assert, they are incapable of connecting with each other. However, if the knowledge generated in research settings aims to be useful in in-service or preservice education, a solution to this standoff must be found. As several papers in this special issue suggest, collaborative environments and university-school partnerships may be fruitful contexts to explore.

The articles by the international authors from Australia, Brazil, Canada, the United States and Spain in this issue of *Sisyphus*, which is devoted to the professional practice and professional development of mathematics teachers, reflect the research that has been done by renowned scholars. The articles by our Portuguese contributors, on the other hand, emerge from a research project (Project P3M) that I coordinated, whose objective was to study the mathematics teachers' practices and the conditions under which transformation takes place.

Two methodological approaches form the basis of many of the most fruitful studies on mathematics teachers, and we find traces of one or the other in the articles in this issue. One of the approaches is, as we have already discussed, collaborative studies. In these studies, the researcher becomes a member of a team that seeks to deal with a certain professional problem. The team pinpoints possibilities and constraints, and evaluates solutions. The researcher participates fully in the work of the group and shares in its successes, stalemates and failures. Thus, the researcher gains a profound awareness of the nature of the problems being tackled. Collaborative work also creates a collective dynamic, and generates vital energy and professional creativity that allow new educational realities and processes to emerge, enabling all the educational actors to view each other from a completely new angle.

In the other approach, the educational actors assume an inquiry stance with regard to their own teaching. This approach is often dubbed «practitioner research». It brings the logic of exploratory teaching or inquiry-based-teacher education into to the professional realm. However, since this approach is not explicitly required of the teaching professional, and since educational research is most often portrayed as formal and demanding, this perspective is hard to replicate on a large scale. But, if the proper context is provided for such activities and if suitable conditions are created, it can become a very promising framework, both in terms of practitioner development and new research insights. Unlike the other approaches, practitioner researcher studies have the great advantage of yielding immediate results and findings and – in the very least – can benefit the researcher's teaching. However, very often it produces results that impact the practices of the institution itself.

Studies centring on the mathematics teacher undertaken in Portugal are, to a great extent, linked to international research. These studies have had a significant impact, inspiring a number of educational policies regarding mathematics education, especially from 2005 to 2009. They have led to the development of new syllabuses, the production and dissemination of teaching and support materials, large scale national programs for teacher education (in the first and second cycle of basic education), field experimentation, and local support for the introduction of new syllabuses. The fruits of these policies can be seen in the results Portuguese students have achieved in international evaluation programs (such as TIMSS and PISA) and also in the way concepts, practices and the results of mathematics teaching and learning have changed in many schools.

The international and local results discussed in this issue's articles show that the knowledge produced in academic settings may be put to work in in-service and future teacher training programs, in professional development initiatives, and in educational organizations and public policies. Society must ask researchers to make their findings available while forging ties with social actors, so that these findings will be put to efficient use. Researchers, on the other hand, should ask educational actors to reciprocate by finding how the research results, proposals, and materials can be used to improve educational results and processes.

João Pedro da Ponte

The Knowledge Quartet: the genesis and application of a framework for analysing mathematics teaching and deepening teachers' mathematics knowledge

Tim Rowland
tr202@cam.ac.uk | University of East Anglia & University of Cambridge, United Kingdom

ABSTRACT

This paper describes a framework for mathematics lesson observation, the 'Knowledge Quartet', and the research and policy contexts in which it was developed. The framework has application in research and in mathematics teaching development. The research which led to the development of the framework drew on videotapes of mathematics lessons prepared and conducted by elementary pre-service teachers towards the end of their initial training. A grounded theory approach to data analysis led to the emergence of the framework, with four broad dimensions, through which the mathematics-related knowledge of these teachers could be observed in practice. This paper describes how each of these dimensions is characterised, and analyses two lessons, showing how each dimension of the Quartet can be identified in the lesson. The paper concludes by outlining recent developments in the use of the Knowledge Quartet.

KEY WORDS

Mathematics teaching; Teacher knowledge; Teacher education; Knowledge Quartet.

SISYPHUS
JOURNAL OF EDUCATION
VOLUME 1, ISSUE 3,
2013, PP. 15-43

The Knowledge Quartet: The Genesis and Application of a Framework for Analysing Mathematics Teaching and Deepening Teachers' Mathematics Knowledge

Tim Rowland

INTRODUCTION

This paper concerns a framework for the analysis of mathematics teaching – the Knowledge Quartet – which was first developed at the University of Cambridge in the years 2002-4. Since then, the Knowledge Quartet has been applied in several research and teacher education contexts, and the framework has been further refined and developed as a consequence. In order to understand the nature and the status of the Knowledge Quartet, it will be useful to consider first the nature of teacher knowledge in general and mathematics teacher knowledge in particular. The paper then proceeds to a description of the research study which led to the emergence of the Knowledge Quartet, and then discusses some of the ways in which it has been used and developed further.

TEACHER KNOWLEDGE: THE BIG PICTURE

From its historical origins in philosophical deliberation, modern empirical study of teacher knowledge is firmly rooted in the landmark studies of Lee Shulman and his colleagues in the 1980s. In his 1985 presidential address to the American Educational Research Association, Shulman proposed a taxonomy with seven categories that formed a knowledge-base for teaching. Whereas four of these elements (such as knowledge of educational purposes and values) are generic, the other three concern 'discipline knowledge', being specific to the subject matter being taught. They are: subject matter knowledge, pedagogical content knowledge and curricular knowledge.

Shulman's (1986) conceptualisation of subject matter knowledge (SMK) includes Schwab's (1978) distinction between substantive knowledge (the key facts, concepts, principles and explanatory frameworks in a discipline) and syntactic knowledge (the nature of enquiry in the field, and how new knowledge is introduced and accepted in that community). For Shulman, pedagogical content knowledge (PCK) consists of «the ways of representing the subject which make it comprehensible to others (…) [it] also includes an understanding of what makes the learning of specific topics easy or difficult (…)» (Shulman, 1986, p. 9). The identification and de-coupling of this hitherto 'missing link' between knowing something for oneself and being able to enable others to know it is, arguably, Shulman's most enduring contribution to the field. 'PCK' gives educators, whatever their role, a language with which to describe, and to celebrate, what teachers know about and others do not – even those with comparable qualifications in subject matter per se.

The teacher knowledge 'problem' in the UK

International comparative studies (e.g. Mullis, Martin & Foy, 2008), and the related 'league tables', have resulted in a search for scapegoats and demands in a number of countries for improvement of the mathematics knowledge of prospective and serving teachers. Difficulties associated with teachers' mathematical content knowledge are particularly apparent in the elementary sector, where generalist teachers often lack confidence in their own mathematical ability (Brown, McNamara, Jones & Hanley, 1999; Green & Ollerton, 1999). Identifying, developing and deepening teachers' mathematical content knowledge has become a priority for policy makers and mathematics educators around the world.

The rather direct approach to a perceived 'problem' in England was captured by an edict in the first set of government 'standards' for Initial Teacher Training (ITT), first issued in 1997, which required teacher education programmes to «audit trainees' knowledge and understanding of the mathematics contained in the National Curriculum», and where 'gaps' are identified to «make arrangements to ensure that trainees gain that knowledge» (Department for Education and Employment, 1998, p. 48). This process of audit and remediation of subject knowledge within primary ITT became a high profile issue following the introduction of these and subsequent government requirements. Within the teacher education community, few could be found to support the imposition of the 'audit and remediation' culture.

Yet the introduction of this regime provoked a body of research in the UK on prospective elementary teachers' mathematics subject-matter knowledge (e.g., Goulding, Rowland & Barber, 2002). The proceedings of a symposium held in 2003 usefully drew together some of the threads of this research (BSRLM, 2003). One study, with 150 London-based graduate trainee primary teachers (Rowland, Martyn, Barber & Heal, 2000), found that trainees obtaining high (or even middle) scores on a 16-item audit of content knowledge were more likely to be assessed as strong mathematics teachers on school-based placements than those with low scores; whereas those with low audit scores were more likely than other participants to be assessed as weak mathematics teachers.

This was an interesting finding, and a team at the University of Cambridge wanted to find out more about what was 'going on', and took forward this new line of enquiry. If superior content knowledge really does make a difference when teaching elementary mathematics, it ought somehow to be observable in the *practice* of the knowledgeable teacher. Conversely, the teacher with weak content knowledge might be expected to misinform their pupils, or somehow to miss opportunities to teach mathematics 'well'. In a nutshell, the Cambridge team wanted to identify, and to understand better, the ways in which elementary teachers' mathematics content knowledge, or the lack of it, is visible in their teaching.

DEVELOPING THE KNOWLEDGE QUARTET

CONTEXT AND PURPOSE OF THE RESEARCH

Several researchers have argued that mathematical content knowledge needed for teaching is not located in the minds of teachers but rather is realised through the practice of teaching (Hegarty, 2000; Mason & Spence, 1999). From this perspective, knowledge for teaching is constructed in the context of teaching, and can therefore be observed only a *in vivo*' knowledge in this context.

In the UK, the majority of prospective, trainee teachers are graduates who follow a one-year program leading to a Postgraduate Certificate in Education (PGCE) in a university[1] education department. Over half of the PGCE year is spent teaching in schools under the guidance of a school-based mentor, or 'cooperating teacher'. Placement lesson observation is normally followed by a review meeting between the cooperating teacher and the student-teacher. On occasion, a university-based tutor will participate in the observation and the review. Thirty years ago, Tabachnick, Popkewitz and Zeichner (1979) found that «cooperating teacher/student teacher interactions were almost always concerned with (…) procedural and management issues (…) There was little or no evidence of any discussion of substantive issues in these interactions» (p. 19). The situation has not changed, and more recent studies also find that mentor/trainee lesson review meetings typically focus heavily on organisational features of the lesson, with very little attention to the mathematical content of mathematics lessons (Borko & Mayfield, 1995; Strong & Baron, 2004).

The purpose of the research from which the Knowledge Quartet emerged was to develop an empirically-based conceptual framework for lesson review discussions *with a focus on the mathematics* content of the lesson, and the role of the trainee's mathematics subject matter knowledge (SMK) and pedagogical content knowledge (PCK). In order to be a useful tool for those who would use it in the context of *practicum* placements, such a framework would need to capture a number of important ideas and factors about mathematics content knowledge in relation to teaching, within a small number of conceptual categories, with a set of easily-remembered labels for those categories.

The focus of this particular research was therefore to identify ways that teachers' mathematics content knowledge – both SMK and PCK – can be observed to 'play out' in practical teaching. The teacher-participants in this study were novice, trainee elementary school teachers, and the observations were made during their school-based placements. Whilst we believe certain kinds of knowledge to be *desirable* for elementary mathematics teaching, we are convinced of the futility of asserting what a beginning teacher, or a more experienced one for that matter, *ought* to know. Our interest is in what a teacher *does* know and believe, and how opportunities to enhance knowledge

1 It should be noted, however, that the government now actively promotes a range of workplace-based alternatives (such as 'School Direct') to the PGCE. These are effectively located in notions of apprenticeship, and offer little interaction with university-based teacher educators.

can be identified. We have found that the Knowledge Quartet, the framework that arose from this research, provides a means of reflecting on teaching and teacher knowledge, with a view to developing both.

The research reported in this paper was undertaken in collaboration with Cambridge colleagues Peter Huckstep, Anne Thwaites, Fay Turner and Jane Warwick. I frequently, and automatically, use the pronoun 'we' in this text in recognition of their contribution.

METHOD: HOW THE KNOWLEDGE QUARTET CAME ABOUT

The participants in the original study were enrolled on a one-year PGCE course in which each of the 149 trainees specialised either on the Early Years (pupil ages 3–8) or the Primary Years (ages 7–11). Six trainees from each of these groups were chosen for observation during their final school placement. The 12 participants were chosen to reflect a range of outcomes of a subject-knowledge audit administered three months earlier. Two mathematics lessons taught by each of these trainees were observed and videotaped, i.e. 24 lessons in total. The trainees were asked to provide a copy of their planning for the observed lesson. As soon as possible after the lesson, the observer/researcher wrote a succinct account of what had happened in the lesson, so that a reader might immediately be able to contextualise subsequent discussion of any events within it. These 'descriptive synopses' were typically written from memory and field notes, with occasional reference to the videotape if necessary.

From that point, we took a grounded approach to the data for the purpose of generating theory (Glaser & Strauss, 1967). In particular, we identified in the videotaped lessons aspects of trainees' actions in the classroom that seemed to be significant in the limited sense that it could be construed to be informed by a trainee's mathematics subject matter knowledge or their mathematical pedagogical knowledge. We realised later that most of these significant actions related to choices made by the trainee in their planning or more spontaneously. Each was provisionally assigned an 'invented' code. These were grounded in particular moments or episodes in the tapes. This provisional set of codes was rationalised and reduced (e.g. eliminating duplicate codes and marginal events) by negotiation and agreement in the research team.

The 17 codes generated by this inductive process are itemised later in this chapter. The name assigned to each code is intended to be indicative of the type of issue identified by it: for example, the code *adheres to textbook* (AT) was applied when a lesson followed a textbook script with little or no deviation, or when a set of exercises was 'lifted' from a textbook, or other published resource, sometimes with problematic consequences. By way of illustration of the coding process, we give here a brief account of an episode that we labelled with the code *responding to children's ideas* (RCI). It will be seen that the contribution of a child was unexpected. Within the research team, this code name was understood to be potentially ironic, since the observed response of the teacher to a child's insight or suggestion was often to put it to one side rather than to deviate from the planned lesson script, even when the child offered further insight into the topic at hand.

Code RCI*: an illustrative episode*. Jason was teaching elementary fraction concepts to a Year 3 class (pupil age 7–8). Each pupil held a small oblong whiteboard and a dry-wipe pen. Jason asked them to «split» their individual whiteboards into two. Most of the children predictably drew a line through the centre of the oblong, parallel to one of the sides, but one boy, Elliot, drew a diagonal line. Jason praised him for his originality, and then asked the class to split their boards «into four». Again, most children drew two lines parallel to the sides, but Elliot drew the two diagonals. Jason's response was to bring Elliot's solution to the attention of the class, but to leave them to decide whether it was correct. He asked them:

> Jason: What has Elliot done that is different to what Rebecca has done?
> Sophie: Because he's done the lines diagonally.
> Jason: Which one of these two has been split equally? (…) Sam, has Elliot split his board into quarters?
> Sam: Um … yes … no …
> Jason: Your challenge for this lesson is to think about what Elliot's done, and think if Elliot has split this into equal quarters. There you go Elliot.

At that point, Jason returned the whiteboard to Elliot, and the question of whether it had been partitioned into quarters was not mentioned again. What makes this interesting mathematically is the fact that (i) the four parts of Elliot's board are not congruent, but (ii) they have equal areas; and (iii) this is not at all obvious. Furthermore, (iv) an elementary demonstration of (ii) is arguably even less obvious. This seemed to us a situation that posed very direct demands on Jason's SMK and arguably his PCK too. It is not possible to infer whether Jason's «challenge» is motivated by a strategic decision to give the children some thinking time, or because he needs some himself.

Equipped with this set of codes, we revisited each lesson in turn and, after further intensive study of the tapes, elaborated each descriptive synopsis into an analytical account of the lesson. In these accounts, the agreed codes were associated with relevant moments and episodes, with appropriate justification and analysis concerning the role of the trainee's content knowledge in the identified passages, with links to relevant literature.

The identification of these fine categories was a stepping stone with regard to our intention to offer a practical framework for use by ourselves, our colleagues and teacher-mentors, for reviewing mathematics teaching with trainees following lesson observation. A 17-point tick-list (like an annual car safety check) was not quite what was needed. Rather, the intended purpose demanded a more compact, readily-understood scheme which would serve to frame a coherent, content-focused discussion between teacher and observer. The key to the solution of our dilemma was the recognition of an association between elements of subsets of the 17 codes, enabling us to group them (again by negotiation in the team) into four broad, superordinate categories, which we have named (I) foundation (II) transformation (III) connection (IV) contingency. These four units are the dimensions of what we call the 'Knowledge Quartet'.

Each of the four units is composed of a small number of subcategories that we judged, after extended discussions, to be of the same or a similar nature. An extended account of the research pathway described above is given in Rowland (2008). The Knowledge Quartet has now been extensively 'road tested' as a descriptive and analytical tool. As well as being re-applied to analytical accounts of the original data (the 24 lessons), it has been exposed to extensive 'theoretical sampling' (Glaser & Strauss, 1967) in the analysis of other mathematics lessons in England and beyond (see e.g. Weston, Kleve & Rowland, 2013).

As a consequence, three additional codes[2] have been added to the original 17, but in its broad conception, we have found the KQ to be comprehensive as a tool for thinking about the ways that content knowledge comes into play in the classroom. We have found that many moments or episodes within a les-

2 These new codes, derived from applications of the KQ to classrooms within and beyond the UK, are *teacher insight* (Contingency), *responding to the (un)availability of tools and resources* (Contingency) and use of *instructional materials* (Transformation) respectively.

son can be understood in terms of two or more of the four units; for example, a **contingent** response to a pupil's suggestion might helpfully **connect** with ideas considered earlier. Furthermore, the application of content knowledge in the classroom always rests on **foundational** knowledge.

Mathematical Knowledge for Teaching and the Knowledge Quartet

It is useful to keep in mind how the KQ differs from the well-known Mathematical Knowledge for Teaching (MKT) egg-framework due to Deborah Ball and her colleagues at the University of Michigan, USA (Ball, Thames & Phelps, 2008). The Michigan research team refer to MKT as a «practice-based theory of knowledge for teaching» (Ball & Bass, 2003, p. 5). The same description could be applied to the Knowledge Quartet, but while parallels can be drawn between the methods and some of the outcomes, the two theories look very different. In particular, the theory that emerges from the Michigan studies aims to unpick and clarify the formerly somewhat elusive and theoretically-undeveloped notions of SMK and PCK. In the Knowledge Quartet, however, the distinction between different *kinds* of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In this sense, the two theories are complementary, so that each has useful perspectives to offer to the other.

CONCEPTUALISING THE KNOWLEDGE QUARTET

The concise conceptualisation of the Knowledge Quartet which now follows draws on the extensive range of data referred to above. As we observed earlier, the practical application of the Knowledge Quartet depends more on teachers and teacher educators understanding the broad characteristics of each of the four dimensions than on their recall of the contributory codes.

Foundation

§ Contributory codes: awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures.

The first member of the KQ is rooted in the foundation of the teacher's theoretical background and beliefs. It concerns their knowledge, understanding and ready recourse to what was learned at school and at college/university, including initial teacher education in preparation (intentionally or otherwise) for their role in the classroom. It differs from the other three units in the sense that it is about knowledge 'possessed'[3], irrespective of whether it is being put to purposeful use. For example, we could claim to have knowledge about division by zero, or about some probability misconceptions – or indeed to know where we could seek advice on these topics – irrespective of whether we had had to call upon them in our work as teachers. Both empirical and theoretical considerations have led us to the view that the other three units flow from a foundational underpinning.

A key feature of this category is its *propositional* form (Shulman, 1986). It is what teachers learn in their 'personal' education and in their 'training' (pre-service in this instance). We take the view that the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way. By 'fundamental' we have in mind a rational, reasoned approach to decision-making that rests on something other than imitation or habit. The key components of this theoretical background are: knowledge and understanding of mathematics per se; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt.

In summary, this category that we call 'foundation' coincides to a significant degree with what Shulman (1987) calls 'comprehension', being the first stage of his six-point cycle of pedagogical reasoning.

Transformation

§ Contributory codes: teacher demonstration; use of instructional materials; choice of representation; choice of examples.

The remaining three categories, unlike the first, refer to ways and contexts in which knowledge is brought to bear on the preparation and conduct of teaching. They focus on knowledge-in-action as *demonstrated* both in planning to teach and in the act of teaching itself. At the heart of the second member of the KQ, and acknowledged in the particular way that we name it, is Shulman's observation that the knowledge base for teaching is distinguished by «the capacity of a teacher to *transform* the content knowledge he or she possesses

3 The use of this acquisition metaphor for knowing suggests an individualist perspective on Foundation knowledge, but we suggest that this 'fount' of knowledge can also be envisaged and accommodated within more distributed accounts of knowledge resources (see Hodgen, 2011).

into forms that are pedagogically powerful» (1987, p. 15, emphasis added). This characterisation has been echoed in the writing of Ball (1988), for example, who distinguishes between knowing some mathematics 'for yourself' and knowing in order to be able to help someone else learn it. As Shulman indicates, the presentation of ideas to learners entails their re-presentation (our hyphen) in the form of analogies, illustrations, examples, explanations and demonstrations (Shulman, 1986, p. 9).

Our second category, unlike the first, picks out behaviour that is directed towards a pupil (or a group of pupils), and which follows from deliberation and judgement informed by foundation knowledge. This category, as well as the first, is informed by particular kinds of literature, such as the teachers' handbooks of textbook series or in the articles and 'resources' pages of professional journals. Increasingly, in the UK, teachers look to the internet for 'bright ideas', and even for ready-made lesson plans. The trainees' choice and *use of examples* has emerged as a rich vein for reflection and critique. This includes the use of examples to assist concept formation and demonstrate procedures, and the selection of exercise examples for student activity.

Connection

§ Contributory codes: making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness.

The next category binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content – the learning, perhaps, of a concept or procedure. It concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons. Mathematics is notable for its coherence as a body of knowledge and as a field of enquiry. Indeed, a great deal of mathematics is held together by deductive reasoning.

The pursuit of coherence and mathematical connections in mathematics pedagogy has been stimulated recently by the work of Askew, Brown, Rhodes, Johnson and Wiliam (1997): of six case study teachers found to be highly effective, all but one gave evidence of a 'connectionist' orientation. The association between teaching effectiveness and a set of articulated beliefs of this kind lends a different perspective to the work of Ball (1990), who also strenuously argued for the importance of connected knowledge for teaching.

Related to the integrity of mathematical content in the mind of the teacher and his/her management of mathematical discourse in the classroom, our conception of coherence includes the *sequencing* of topics of instruction within and between lessons, including the ordering of tasks and exercises. To a significant extent, these reflect deliberations and choices entailing not only knowledge of structural connections within mathematics itself, but also awareness of the relative cognitive demands of different topics and tasks.

Contingency

§ Contributory codes: responding to students' ideas; deviation from agenda; teacher insight; (un)availability of resources.

Our final category concerns the teacher's response to classroom events that were not anticipated in the planning. In some cases, it is difficult to see how they could have been planned for, although that is a matter for debate. In

DIMENSION	CONTRIBUTORY CODES
Foundation: knowledge and understanding of mathematics per se and of mathematics-specific pedagogy, beliefs concerning the nature of mathematics, the purposes of mathematics education, and the conditions under which students will best learn mathematics	· awareness of purpose · adherence to textbook · concentration on procedures · identifying errors · overt display of subject knowledge · theoretical underpinning of pedagogy · use of mathematical terminology
Transformation: the presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations	· choice of examples · choice of representation · use of instructional materials · teacher demonstration (to explain a procedure)
Connection: the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks	· anticipation of complexity · decisions about sequencing · recognition of conceptual appropriateness · making connections between procedures · making connections between concepts
Contingency: the ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events	· deviation from agenda · responding to students' ideas · (use of opportunities) · teacher insight during instruction · responding to the (un)availability of tools and resources

TABLE I — THE KNOWLEDGE QUARTET: DIMENSIONS AND CONTRIBUTORY CODES commonplace language this dimension of the KQ is about the ability to ‘think on one’s feet’: it is about *contingent action*. Shulman (1987) proposes that most teaching begins from some form of ‘text’ – a textbook, a syllabus, ultimately a sequence of planned, intended actions to be carried out by the teacher and/or the students within a lesson or unit of some kind. Whilst the stimulus – the teacher’s intended actions – can be planned, the students’ responses cannot.

Brown and Wragg (1993) suggested that ‘responding’ moves are the lynchpins of a lesson – important in the sequencing and structuring of a lesson – and observed that such interventions are some of the most difficult tactics for novice teachers to master. The quality of such responses is undoubtedly determined, at least in part, by the knowledge resource available to the teacher, as the earlier illustrative episode with Jason demonstrates. For further details, see Rowland, Thwaites and Jared (2011).

For ease of reference, the account of the KQ above is summarised in Table 1, on the previous page.

In the next two sections, I shall illustrate the application of the KQ in the analysis of mathematics lessons: one primary, one secondary. In both cases, the teachers are pre-service graduate students.

PRIMARY MATHEMATICS TEACHING: THE CASE OF LAURA

The lesson now under scrutiny is one of the original 24 videotaped lessons. The graduate trainee in question, Laura, was teaching a Year 5 (pupil age 9–10) class about written multiplication methods, specifically multiplying a two-digit number by a single digit number.

Laura’s Lesson

Laura reminded the class that they had recently been working on multiplication using the ‘grid’ method. She spoke about the tens and units being «partitioned off». Simon was invited to the whiteboard to demonstrate the method for 9 × 37. He wrote:

×	30	7	
9	270	63	= 333

Laura then said that they were going to learn another way. She proceeded to write the calculation for 9 × 37 on the whiteboard in a conventional but elaborated column format, explaining as she wrote:

$$\begin{array}{rr} & 37 \\ & \times\ 9 \\ \cline{2-2} 30\times 9 & 270 \\ 7\times 9 & 63 \\ \cline{2-2} & 333 \\ & {\scriptstyle 1}\ \ \end{array}$$

Laura performed the sum 270 + 63 by column addition *from the right*, 'carrying' the 1 (from 7 + 6=13) from the tens into the hundreds column. She wrote the headings h, t, u [indicating hundreds, tens, units] above the three columns.

Next, Laura showed the class how to «set out» 49 × 8 in the new format, and then the first question (19 × 4) of the exercises to follow. The class proceeded to work on these exercises, which Laura had displayed on a wall. Laura moved from one child to another to see how they were getting on. She emphasised the importance of lining up the hundreds, tens and units columns carefully, and reminded them to estimate first.

Finally, she called the class together and asked one boy, Sean, to demonstrate the new method with the example 27 × 9. Sean got into difficulty; he was corrected by other pupils and by Laura herself. As the lesson concluded, Laura told the children that they should complete the set of exercises for homework.

We now select from Laura's lesson a number of moments, episodes and issues to show how they might be perceived through the lens of the Knowledge Quartet. It is in this sense that we offer Laura's lesson as a 'case' – it is typical of the way that the KQ can be used to identify for discussion matters that arise from the lesson observation, and to structure reflection on the lesson. Some possibilities for discussion with the trainee, and for subsequent reflection, are flagged below thus: **Discussion point**. We emphasise that the process of *selection* in the commentary which follows has been extreme. Nevertheless, we raise more issues relating to content knowledge than would normally be considered in a post-lesson review meeting.

FOUNDATION

First, Laura's professional knowledge underpins her recognition that there is more than one possible written algorithm for whole number multiplication. We conceptualise this within the domain of fundamental knowledge, being the *foundation* that supports and significantly determines her intentions or actions. Laura's learning objective seems to be taken from the National Numeracy Strategy (NNS) *Framework* (DfEE, 1999) teaching programme for Year 4:

> · Approximate first. Use *informal* pencil and paper methods to support, record or *explain* multiplication. *Develop* and *refine* written methods for TUxU (p. 3/18, emphasis added).

These objectives are clarified by examples later in the *Framework*; these contrast (A) *informal* written methods – the grid, as demonstrated by Simon – with (B) *standard* written methods – the column layout, as demonstrated by Laura in her introduction. In both cases (A and B), an 'approximation' precedes the calculation of a worked example in the *Framework*. Laura seems to have assimilated the NNS guidance and planned her teaching accordingly. It is perhaps not surprising that she does not *question* the necessity of teaching the standard column format to pupils who already have an effective, meaningful algorithm at their disposal. Indeed, many respected educators advocate the adequacy and pedagogical preference of grid-type methods with primary pupils (e.g. Haylock, 2001, pp. 91-94).

> § **Discussion point**: where does Laura stand on this debate, and how did her stance contribute to her approach in this lesson?

Another issue related to Laura's fundamental knowledge is her approach to computational estimation. When she asks the children to estimate 49 × 8, one child proposes 400, saying that 8 × 50 is 400. Laura, however, suggests that she could make this «even more accurate» by taking away two lots of 50. She explains, «Because you know two times five is ten and two times fifty is a hundred, you could take a hundred away». Perhaps she had 10 × 50 in mind herself as an estimate, or perhaps she confused something like subtracting 8 from the child's estimate. She recognises her error and says «Sorry, I was getting confused, getting my head in a spin». The notions of *how* to estimate and *why* it might be desirable to do so are not adequately discussed or explored with the class.

> § **Discussion point**: what did Laura have in mind in this episode, and is there some way she can be more systematic in her approach to computational estimation?

At this stage of her career in teaching, Laura gives the impression that she is passing on her own practices and her own forms of knowledge. Her main resource seems to be her own experience (of using this algorithm), and it seems that she does not yet have a view of mathematics didactics as a *scientific* enterprise.

TRANSFORMATION

Laura's own ability to perform column multiplication is secure, but her pedagogical challenge is to *transform* what she knows for herself into a form that can be accessed and appropriated by the children. Laura's choice of demonstration examples in her introduction to column multiplication merits some consideration and comment. Her first example is 37'9; she then goes on to work through 49'8 and 19'4. Now, the NNS emphasises the importance of *mental* methods, where possible, and also the importance of choosing the most suitable strategy for any particular calculation. 49'8 and 19'4 can all be more efficiently performed by rounding up, multiplication and compensation e.g. 49'8 = (50'8)-8. Perhaps Laura had this in mind in her abortive effort to make the estimate of 400 «even more accurate».

Her choice of exercises – the practice examples – also invites some comment. The sequence is: 19 × 4, 27 × 9, 42 × 4, 23 × 6, 37 × 5, 54 × 4, 63 × 7, 93 × 6, with 99 × 9, 88 × 3, 76 × 8, 62 × 43, 55 × 92, 42 × 15 as 'extension' exercises (although no child actually attempts these in the lesson). Our earlier remark about the suitability of the column algorithm relative to alternative mental strategies applies to several of these, 99 × 9 being a notable example.

But suppose for the moment that it is understood and accepted by the pupils that they are to put aside consideration of such alternative strategies – that these exercises are there merely as a vehicle for them to gain fluency with the algorithm. In that case, the *sequence* of exercises might be expected to be designed to present the pupils with increasing challenge as they progress though them.

§ **Discussion point**: on what grounds did Laura choose and sequence these particular examples and exercises? What considerations might contribute to the choice?

CONNECTION

Perhaps the most important connection to be established in this lesson is that between the grid method and the column algorithm. Laura seems to have this connection in mind as she introduces the main activity. She reminds them that they have used the grid method, and says that she will show them a «different way to work it out». She says that the answer would be the same whichever way they did it «because it's the same sum». However, Laura does not clarify the connections between the two methods: that the same processes and principles – partition, distributivity and addition – are present in both. No reference to the grid method is made in her second demonstration example, 49 × 8. Her presentation of this example now homes in on procedural aspects – the need to «partition the number down», «adding a zero» to 8 × 4, getting the columns lined up, adding the partial products from the right. The fact that the connection is tenuous for at least one pupil is apparent in the plenary. Sean actually volunteers to calculate 27 × 9 on the whiteboard. He writes 27 and x9 in the first two rows as expected, but then writes 20 × 7 and 2 × 9 to the left in the rows below.

> § **Discussion point**: Laura is clearly trying to make a connection between the grid method and the column method. What reasons did she have in mind for doing so? To what extent did she think she was successful?

CONTINGENCY

Sean's faulty attempt (mentioned above) to calculate 27 × 9 on the whiteboard appears to have surprised Laura – it seems that she fully expected him to apply the algorithm faultlessly, and that his actual response really was unanticipated. In the event, there are several 'bugs' in his application of the procedure. The partition of 27 into 20 and 2 is faulty, and the multiplicand is first 9, then 7. This would seem to be a case where Sean might be encouraged to reconsider what he has written by asking him some well-chosen questions. One such question might be to ask how he would do it by the grid method. Or simply why he wrote those particular numbers where he did. Laura asks the class «Is that the way to do it? Would everyone do it that way?». Leroy demonstrates the algorithm correctly, but there is no diagnosis of where Sean went wrong, or why.

> § **Discussion point**: what might be the reason for Sean's error? In what ways could this have been addressed in the lesson, or subsequently?

final remark concerning laura's lesson

It is all too easy for an observer to criticise a novice teacher for what they omitted or committed in the high-stakes environment of a school placement, and we would emphasise that the KQ is intended as a tool to *support* teacher development, with a sharp and structured focus on the impact of their SMK and PCK on teaching. Indications of how this might work are explicit in our analysis of Laura's lesson. We have emphasised that our analysis has been *selective*: we raised some issues for attention, but there were others which, not least out of space considerations, we chose not to mention.

SECONDARY MATHEMATICS TEACHING: THE CASE OF HEIDI

revised method

The lesson to be described and analysed in this section took place some years after the original project described earlier in this paper. The objective in this phase of our research programme was systematic testing of the KQ as an analytical framework in the context of secondary education. As before, lessons were video-recorded, and trainees were asked to provide a copy of their lesson plan for reference in later analysis. At this point, the data collection was extended to include a post-lesson interview, as follows. As soon as possible after the lesson, the research team met to undertake preliminary analysis of the videotaped lesson, and to identify some key episodes in it with reference to the KQ framework. Then, again with minimum delay, one team member met with the trainee to view some episodes[4] from the lesson and to discuss them in the spirit of stimulated-recall (Calderhead, 1981). These interview-discussions addressed some of the issues that had come to light in the earlier KQ-structured preliminary analysis of the lesson. An audio recording was made of this discussion, to be transcribed later. In some cases, the observation, preliminary analysis and stimulated-recall interview all took place on the same day.

4 A DVD of the full lesson was given to the trainee soon afterwards, as a token of our appreciation, but their reflections on viewing this DVD in their own time are not part of our data.

The lesson to be considered now was taught by Heidi, who had come to the course direct from undergraduate study in mathematics at a well-regarded UK university. Her practicum placement secondary school was state-funded, with some 1400 pupils across the attainment range. In keeping with almost all secondary schools in England, pupils were 'setted' by attainment in mathematics, with 10 or 11 sets in most years.

heidi's lesson

Heidi's class was one of two parallel 'top' mathematics sets in Year 8 (pupil age 12–13), and these pupils would be expected to be successful both now and in the high-stakes public examinations in the years ahead. 17 boys and 13 girls were seated at tables facing an interactive white board (IWB)[5] located at the front of the room. The objectives stated in Heidi's lesson plan were as follows: «Go over questions from their most recent test, and then introduce direct proportion».

Heidi returned the test papers from a previous lesson to the students, and proceeded to review selected test questions with the whole class, asking the pupils about their solution methods. The first question to be revisited was on percentages, and the next two on simultaneous linear equations. Several pupils offered solution methods, and these were noted on the IWB. Heidi responded to requests for review of two more questions, and nearly 30 minutes of the 45-minute lesson had elapsed before Heidi moved on to the topic of direct proportion.

She began by displaying images of three similar cuboids on the IWB: she explained that the cuboids were boxes, produced in the same factory, and that the dimensions were in the same proportions. The linear scale factor between the first and second cuboids was 2 [Heidi wrote x2], and the third was three times the linear dimensions of the second [x3]. Heidi identified one rectangular face, and asked what would happen to the area of this face as the dimensions increased. They calculated the areas, and three pupils made various conjectures about the relationship between them. The third of these said «I think it is that number [the linear scale factor] squared».

Heidi then introduced two straightforward direct proportion word problems. One, for example, began «6 tubes of toothpaste have a mass of 900g.

5 Interactive whiteboards, with associated projection technology, are now more-or-less universal in secondary classrooms in England.

What is the mass of 10 tubes?». Different solutions were offered and discussed. Heidi then gave them six exercise questions of a similar kind (e.g., «In 5 hours a man earns £30. How much does he earn in 6 hours?»).

THE KNOWLEDGE QUARTET: HEIDI'S LESSON

We now offer our interpretation of some ways in which we observed or inferred *foundation*, *transformation*, *connection* and *contingency* (but not in that order) in Heidi's lesson. It will become apparent that many moments or episodes within a lesson can be understood in terms of two or more of the four dimensions. We also draw upon her lesson plan and upon her contributions to the post-lesson, stimulated-recall discussion with Anne, one of the research team. This discussion had homed in on two fragments of the lesson: the first was Heidi's review of a test question on simultaneous equations; the second was the introduction of the proportion topic using the IWB-images of the three cuboids.

TRANSFORMATION

Heidi had little or no influence regarding the choice of examples (a key component of this KQ dimension) in her test review, since the test had been set by a colleague. However, the stimulated-recall interview gave an opportunity and a motive for her to reflect on the test items. There had been two questions (7 and 8) on simultaneous equations, and the related pairs of equations were

$$\text{Q7: } 2x + 3y = 16,\ 2x + 5y = 20 \qquad \text{Q8: } 3b - 2c = 30,\ 2b + 5c = 1$$

In response to an interview question, Heidi thought the sequencing appropriate. In particular (regarding Q7) she said «They could do it the way it was», seeming to refer to the fact that one variable (x) could be eliminated by subtraction, without the need for scaling either equation. In fact, the pupils' response to Heidi's invitation to offer solution methods suggested that this opportunity was not recognised or not welcomed. The first volunteer, Max, had proposed multiplying the first equation by 10, and the second by 6, suggesting a desire on his part to eliminate y, not x. (Heidi's response to this will be considered under *Contingency*). Heidi was able to explain this in her answer to Anne's question, «What if the y-coefficients were the same?» Heidi's first response was «That would be less difficult, because they tended to want to get rid of the y. I don't know why».

In fact, in this lesson segment, when eliminating one variable by adding or subtracting two equations, Heidi reminds the class several times about a 'rule', namely: if the signs are the same, then subtract; if they are different then add. Heidi suggests, later in the interview, that the pupils tend to want to make the y-coefficients equal, as Max did, because their signs are *explicit* in both equations. This can be seen in both Q7 and Q8 where the coefficient of the first variable is positive in both equations, and the sign left implicit, whereas + or – is explicit in the coefficient of the second variable. This insight of Heidi's is typical of the way that focused reflection on the disciplinary *content* of mathematics teaching, structured by the KQ, has been found to provoke valuable insights on how to improve it (Turner & Rowland, 2010). Heidi's observation is that restricting the x-coefficients to positive values (and emphasising the 'rule') has somehow imposed unintended limitations on student solution methods, with a preference for eliminating y even when «they could do it the way it was» by eliminating x.

Turning now to Heidi's introduction of the direct proportion topic, in our preliminary lesson analysis we misinterpreted Heidi's use of the three cuboids. Her lesson plan included: «Discussion point: What happens to the area of the rectangular face as the dimensions increase? What happens to the volumes of the cuboids as the dimensions increase?». We took this to mean that she intended to investigate the relationship between linear scale factor (between similar figures) and the scale factors for area and volume. Although she had been drawn into this topic, this had not been her intention, as the subsequent word problems indicated. In the event, there was discussion in the lesson of the area of one rectangular face of the cuboid, and how its area increases as the cuboids grow larger: there was not time to consider the volumes. When probed about her choice of context for the introduction of the direct proportion topic, Heidi said that she had chosen the cuboids because it was «a nice visual» which contrasted with the «wordy» presentation of the other problems. In the interview, when asked whether she agreed that she could have done the work on area comparison with rectangles, she replied «You're absolutely right, rectangles would be enough (…) but I did like my box factory». Here we see an example of trainees' propensity to choose representations in mathematics teaching on the basis of their superficial attractiveness at the expense of their mathematical relevance (Turner, 2008). In this instance, the preference for these 'visuals' took Heidi into mathematical territory for which she was not mathematically prepared (see *Contingency*).

CONTINGENCY

Analysis of this dimension of the KQ in Heidi's lesson intertwines with the component of *foundation* concerned with teachers' beliefs about mathematics and mathematics teaching. Here, we begin by taking up the story of Max's suggestion to solve Q7 by multiplying the first equation by 10, and the second by 6. In the interview, Anne asked Heidi why she had «run with» Max's suggestion. Heidi replied «Because it would work. You're trying to find the lowest common denominator, but it would work. Like adding fractions, it would work with any common multiple. I didn't want him to think he was wrong». This kind of openness to pupils' suggestions and ability to anticipate where they would lead was very characteristic of Heidi's teaching, and several examples of it can be found in our data.

In the class discussion which followed, Heidi's introduction of the three cuboids, the pupils calculated (in cm^2) the areas of the rectangles with sides (respectively) 2×3, 4×6, 12×18 (all cm) viz. 6, 24, 216 (in cm^2). Heidi had annotated x2, x3, as I noted earlier. One pupil suggested that the relationships between the areas were «timesed by 4 and timesed by 6». Heidi made it clear that she was not checking these calculations numerically («I'm going to take your word for that»), recorded this second proposal on the IWB (writing x4 and x6) and said «So two times what this has been timesed by [*pointing to the linear scale factors*]. Good observations». This seemed to be the end of the matter, until a third pupil, Lay Tun, said «I think it is that number squared». Heidi paused, then changed the second factor (from x6) to x9, and emphasised the squares.

Now, this length/area relationship in similar figures was not what Heidi had set out to teach, and it became clear at the interview that Heidi (unlike Lay Tun) did not know in advance about «that number squared». In the interview, the discussion proceeded:

Anne: Then you go on to areas. They give a range of options. Now, you take all these responses and give value to all of them. But this was different, in that two of these responses were not correct.

Heidi: I want to take everyone's ideas on board. When you do put something on the board they correct each other rather than me being the authority. In that case, I had a bit of a brain freeze. I hadn't worked out how many times 24 goes into 216, but they're used to me putting up everything.

We see here, paradoxically, a situation in this secondary teaching data in which some subject-matter in the school curriculum lies outside the scope of the content knowledge of the trainee at that moment in time. This should come as no great surprise. For all their university education in mathematics, and their knowledge of topics such as analysis, abstract algebra and statistics, there remain facts from the secondary curriculum that they will have had no good reason to revisit since they left school. What is significant, however, exemplified by Heidi but more-or-less absent in our observations of primary mathematics classrooms, is a teacher with the confidence to negotiate and make sense of mathematical situations such as this (the length/area relationship) 'on the fly', as they arise.

FOUNDATION

This lesson does raise a few issues about Heidi's content knowledge that might be brought to her attention, and some of them were raised in the interview. Briefly, these include: her use of mathematical terminology, which is either very careful and correct (e.g. 'coefficient'), or quite the opposite (e.g. 'timesing'); her lack of fluency and efficiency in mental calculation, such that she did not question the suggestion that $6 \times 24 = 216$ herself in the cuboids situation: on occasion it appeared that she was puzzled by some of the pupils' mental calculations; thirdly, she was not aware of the length/area/volume scale-factor relationships referred to earlier.

But, after many hours spent scrutinising the recording of this lesson, and that of the post-lesson interview, our lasting impression relates to the beliefs component of the *Foundation* dimension, in particular, Heidi's beliefs about her role in the classroom in bringing pupils' ideas and solution strategies into the light, even – as we remarked earlier – when she believed that 'her way' would, in some sense, be better. As she told Anne, «I want to take everyone's ideas on board. When you do put something on the board they correct each other rather than me being the authority». Her perception of this aspect of her role as teacher and the possibility of the pupils themselves contributing to pupil learning is resonant of various constructivist and fallibilist manifestos. Heidi constantly assists this 'letting go' by acknowledging pupils' suggestions, and making them available for scrutiny by writing them on the board. Occasionally she finds herself in deep water as a consequence, but she never seems to doubt her [mathematical] ability to stay afloat.

CONNECTION

We coded a few events in this lesson under *connection*. For example, Heidi's introduction to direct proportionality with the cuboids seemed quite unrelated to the word problems which followed. In any case, the rather diverse objectives for the lesson were likely to make it somewhat 'bitty', and we omit further analysis of this KQ dimension from the present narrative.

SUPPORTING RESEARCH AND TEACHING DEVELOPMENT

The KQ has found two intersecting user groups since its emergence a decade ago. In this section, we outline resources developed to support these user groups.

TEACHER EDUCATION AND TEACHING DEVELOPMENT

As we remarked earlier, one of the goals of our original 2002 research was to develop an empirically-based conceptual framework for mathematics lesson review discussions with a focus on the *mathematics content* of the lesson and the role of the trainee's mathematics subject matter knowledge (SMK) and pedagogical content knowledge (PCK). In addition to the kind of 'knowledgeable-other' analysis and formative feedback exemplified in the cases of Laura and Heidi in this paper, it has also been used to support teachers wanting to develop their teaching by means of reflective evaluation on their own classroom practice (Corcoran, 2011; Turner, 2012). Specifically, the KQ is a tool which enables teachers to focus reflection on the mathematical content of their teaching.

However, both teacher educators and teachers must first learn about the tool, and how to put it to good use. A book (Rowland, Turner, Thwaites & Huckstep, 2009) was written to address the needs of this user-group, especially in relation to primary mathematics. It describes the research-based origins of the KQ, with detailed accounts of the four dimensions, and separate chapters on key codes such as Choice of Examples. The narrative of the book is woven around accounts of over 30 episodes from actual mathematics lessons. We return to this use of the KQ towards the end of this paper.

In some respects, the needs of researchers using the KQ as a theoretical framework for lesson analysis are the same as those of teacher educators, but they are different in others. In particular, a broad-brush approach to the four KQ dimensions often suffices in the teacher education context, and may even be preferable to detailed reference to constituent codes. For example, identifying Contingent moments and actual or possible responses to them need not entail analysis of the particular triggers of such unexpected events. On the other hand, reflections or projections on Transformation usually involve reference to examples and representations. Our writing about the KQ (e.g. Rowland, Huckstep & Thwaites, 2005) initially focused on explaining the essence of each of the four dimensions rather than identifying definitions of each of the underlying codes. However, a detailed KQ- analysis of a record (ideally video) of instruction necessarily involves labelling events at the level of individual KQ-codes prior to synthesis at dimension level (Foundation, Transformation, etc.). This, in turn, raises reliability issues: the coder needs a deep understanding of what is intended by each code, going beyond any idiosyncratic connotations associated with its name. Addressing this issue, a Cambridge colleague of ours wrote as follows:

> Essentially, the Knowledge Quartet provides a repertoire of ideal types that provide a heuristic to guide attention to, and analysis of, mathematical knowledge-in-use within teaching. However, whereas the basic codes of the taxonomy are clearly grounded in *prototypical teaching actions*, their grouping to form a more discursive set of superordinate categories – Foundation, Transformation, Connection and Contingency – appears to risk introducing too great an interpretative flexibility *unless these categories remain firmly anchored in grounded exemplars of the subordinate codes* (Ruthven, 2011, p. 85, emphasis added).

In 2010, a Norwegian doctoral student wrote to us as follows: «I need a more detailed description on the contributory codes to be able to use them in my work. Do you have a coding manual that I can look at?». This enquiry, Ruthven's comment, and our growing sense of the risk of «interpretive flexibility» led us to initiate a project to develop an online coding manual, with the needs of researchers particularly in mind.

The aim of the project was to assist researchers interested in analysing classroom teaching using the Knowledge Quartet by providing a comprehensive collection of «grounded exemplars» of the 20 contributory codes from primary and secondary classrooms. An international team of 15 researchers was assembled. All team members were familiar with the KQ and had used it in their own research as a framework with which to observe, code, comment on and/or evaluate primary and secondary mathematics teaching across various countries, curricula, and approaches to teaching. The team included representatives from the UK, Norway, Ireland, Italy, Cyprus, Turkey and the United States.

In Autumn 2011, team members individually scrutinised their data and identified prototypical classroom-exemplars of some of the KQ codes. To begin with, a written account of each selected classroom *scenario* was drafted. Often this included excerpts of transcripts and/or photographs from the lesson. Then a *commentary* was written, which analysed the excerpt, explaining why it is representative of the particular code, and why it is a strong example. Each team member submitted scenarios and commentary for at least three codes from his/her data to offer as especially strong, paradigmatic exemplars. In March 2012, 12 team members gathered in Cambridge, and worked together for two days. Groups of three team members evaluated and revised each scenario and commentary. The scenarios and commentaries were then revised on the basis of the conference feedback. Further details of the participants and methodology are given in Weston, Kleve and Rowland (2013).

These scenarios and commentaries now combine to form a «KQ coding manual» for researchers to use. This is a collection of primary and secondary classroom vignettes, with episodes and commentaries provided for each code. The collection of codes and commentaries is now freely available online at *www.knowledgequartet.org*. At the time of writing, the website is 'live' but subject to further development. We encourage researchers and teacher educator to use and share this website in the cause of improved clarity about what each of the KQ codes 'looks like' in a classroom setting.

CONCLUSION

Mathematics teaching is a highly complex activity; this complexity ought to be acknowledged when teaching is analysed and discussed, and due attention is given to discipline-specific aspects of pedagogical decision and actions beyond generic aspects of the management of learning. Strong, clear conceptual frameworks assist in the management of this complexity. By attending to events enacted and observed in actual classrooms, with a specific focus on the subject-matter under consideration, the KQ offers practitioners and researchers such a conceptual framework, particularly suited to understanding the contribution of teacher knowledge to mathematics teaching.

For practitioners and teacher educators, the KQ is a tool for identifying opportunities and possibilities for teaching development through the enhancement of teacher knowledge, as indicated, for example, in the book Rowland *et al.* (2009). Especially in the case of pre-service teacher education, it is beneficial to limit the post-observation review meeting to one or two lesson fragments, and also to only one or two dimensions of the KQ, in order to focus the analysis and avoid overloading the trainee-teacher with action points.

In this paper, I have emphasised the progression from observation of teaching to its description and analysis, but I have been less explicit about the evaluation of teaching. In the spirit of reflective practice, the most important evaluation must be that of the teacher him/herself. However, this self-evaluation is usefully provoked and assisted by a colleague or mentor. Earlier in this paper, in the account of the Case of Laura, I have exemplified this provocation through the identification, using the KQ, of tightly-focused discussion points to be raised in a post-observation review. We have suggested that these points be framed in a relatively neutral way, such as «Could you tell me why you … ?» or «What were you thinking when … ?». It would be naïve, however, to suggest that the mentor, or teacher educator, makes no evaluation of what they observe. Indeed, the observer's evaluation is likely to be a key factor in the identification and prioritisation of the discussion points. In post-observation review, it is expected that the 'more knowledgeable other' will indicate what the novice did well, what they did not do and might have, and what they might have done differently. The KQ is a framework to organise such evaluative comments, and to identify ways of learning from them.

The KQ has been successfully applied across different phases of schooling, and in diverse cultures, but we mention, in conclusion, a development that we had not originally anticipated. Having attended presentations about the KQ in cross-disciplinary settings, some teacher education colleagues working in subjects other than mathematics – such as language arts, science and modern foreign languages education – have seen potential in the KQ for their own lesson observations and review meetings. They sometimes ask whether they could adapt and adopt the KQ for their own purposes. This raises the issue: can a framework for knowledge-in-teaching developed in one subject discipline be legitimately adopted in another? My reply usually begins as follows: what might the conceptualisations of the dimensions of the KQ, beginning with Foundation, look like in this other discipline? An answer to this question could set the scene for empirical testing of the KQ in another subject area.

REFERENCES

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy.* London: Kings College.

Ball, D. L. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics, 8*(1), 40-48.

Ball, D. L. (1990). The Mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90*(4), 449-466.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, Alberta, Canada: Canadian Mathematics Education Study Group.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Ball, H., & Mayfield, V. (1995). The roles of the cooperating teacher and university supervisor in learning to teach. *Teaching and Teacher Education, 11*, 501-518.

Borko, H., & Mayfield, V. (1995). The roles of the cooperating teacher and university supervisor in learning to teach. *Teaching and Teacher Education*, 11, 501-518.

Brown, G., & Wragg, E. C. (1993). *Questioning.* London: Routledge.

Brown, T., McNamara, O., Jones, L., & Hanley, U. (1999). Primary student teachers' understanding of mathematics and its teaching. *British Education Research Journal, 25*(3), 299-322.

BSRLM (2003). *Proceedings of the British Society for Research into Learning Mathematics 23*(2). Retrieved from http://www.bsrlm.org.uk/IPs/ip23-2/index.html.

Calderhead, J. (1981). Stimulated recall: a method for research on teaching. *British Journal of Educational Psychology, 51*(2), 211-217.

Corcoran, D. (2011). Learning from lesson study: Power distribution in a community of practice. In L. Hart, A. Alston & A. Murata (Eds.), *Lesson-study Research and Practice in Mathematics Education: Learning Together* (pp. 251-268). New York: Springer.

DfEE (1998). *Teaching: High Status, High Standards: Circular 4/98*. London: HMSO.

DfEE (1999). *The National Numeracy Strategy: framework for teaching mathematics from Reception to Year 6*. Sudbury: DfEE Publications.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. New York: Aldine de Gruyter.

Goulding, M., Rowland, T., & Barber, P. (2002). Does it matter? Primary teacher trainees' subject knowledge in mathematics. *British Educational Research Journal, 28*(5), 689-704.

Green, S., & Ollerton, M. (1999). Mathematics anxiety amongst primary QTS students. *Proceedings of the Conference of the British Society for Research in Learning Mathematics held at St Martin's College, Lancaster, 19*(2), 43-48.

Haylock, D. (2001). *Mathematics Explained for Primary Teachers*. London: Paul Chapman.

Hegarty, S. (2000). Teaching as a knowledge-based activity. *Oxford Review of Education, 26*(3-4), 451-465.

Hodgen, J. (2011). Knowing and identity: a situated theory of mathematical knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 27-42). London and New York: Springer.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics, 38*, 135-161.

Mullis, I. V. S., Martin, M. O., & Foy, P. (2008). *TIMSS 2007 International mathematics report: Findings from IEA's trends in international mathematics and science study at the fourth and eighth grades*. Chestnut Hill, MA: TIMSS and PIRLS International Study Center, Boston College.

Rowland, T. (2008). Researching teachers' mathematics disciplinary knowledge. In P. Sullivan & T. Wood (Eds.), *International handbook of mathematics teacher education: Vol.1. Knowledge and beliefs in mathematics teaching and teaching development* (pp. 273-298). Rotterdam, The Netherlands: Sense Publishers.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*(3), 255-281.

Rowland, T., Martyn, S., Barber, N. P., & Heal, C. (2000). Primary teacher trainees' mathematics subject knowledge and classroom performance. *Research in Mathematics Education, 2*, 3-18.

Rowland, T., Thwaites, A., & Jared, L. (2011). Triggers of contingency in mathematics teaching. In B. Ubuz (Ed.), *Proceedings of the 35th Conference* *of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 73-80). Ankara, Turkey: PME.

Rowland, T., Turner, F., Thwaites, A., & Huckstep, P. (2009). *Developing Primary Mathematics Teaching: Reflecting on practice with the Knowledge Quartet*. London: Sage.

Ruthven, K. (2011). Conceptualising mathematical knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching* (pp. 83-96). London and New York: Springer.

Schwab, J. J. (1978). Education and the structure of the disciplines. In I. Westbury & N. J. Wilkof (Eds.), *Science, curriculum and liberal education* (pp. 229-272). Chicago: University of Chicago Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Strong, M., & Baron, W. (2004). An analysis of mentoring conversations with beginning teachers: Suggestions and responses. *Teaching and Teacher Education, 20*, 47-57.

Tabachnick, B. R., Popkewitz, T. S., & Zeichner, K. M. (1979). Teacher education and the professional perspectives of student teachers. *Interchange on Educational Policy, 10*(4), 12-29.

Turner, F. (2008). Growth in teacher knowledge: Individual reflection and community participation. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda (Eds.), *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education* (Vol 4, pp. 353-360). Morelia, Mexico: University of Saint Nicholas of Hidalgo.

Turner, F. (2012). Using the Knowledge Quartet to develop mathematics content knowledge: the role of reflection on professional development. *Research in Mathematics Education, 14*(3), 253-271.

Turner, F., & Rowland, T. (2011). The Knowledge Quartet as an organising framework for developing and deepening teachers' mathematics knowledge. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 195-212). London and New York: Springer.

Weston, T. L., Kleve, B., & Rowland, T. (2013). Developing an online coding manual for the Knowledge Quartet: An international project. *Proceedings of the British Society for Research into Learning Mathematics, 32*(3), 179-184.

ESSAY ON THE ROLE OF TEACHERS' QUESTIONING IN INQUIRY-BASED MATHEMATICS TEACHING

Luís Menezes
menezes@esev.ipv.pt | Instituto Politécnico de Viseu, Portugal

António Guerreiro
aguerrei@ualg.pt | Universidade do Algarve, Portugal

Maria Helena Martinho
mhm@ie.uminho.pt | Universidade do Minho, Portugal

Rosa Antónia Tomás Ferreira
rferreir@fc.up.pt | Universidade do Porto, Portugal

ABSTRACT

This paper is an essay on the role of the mathematics teacher's questioning in inquiry-based teaching. Questions are important communication tools that are used by the teacher for various purposes and underpin different visions of what it means to teach mathematics. Inquiry-based mathematics teaching has achieved relevance as a powerful alternative to direct teaching, which is inefficient in complying with current demands of mathematics learning. The paper constitutes a reflection on teachers´ questioning within an inquiry-based approach to teaching mathematics, based on available research and illustrated by classroom episodes of three basic education teachers. Our reflection has led us to advocate the central role of the teacher's questions in inquiry-based mathematics teaching, having two main goals: (i) verification of knowledge, a questioning goal that is common to the direct teaching approach; and (ii) development of knowledge, a questioning goal that is specific to inquiry-based teaching. These two goals are attained using three types of questions which may be present in all phases of an inquiry-based lesson, albeit with different weights according to the lesson phases and the teacher's own goals.

KEY WORDS

Mathematics communication; Questioning; Teacher;
Inquiry-based mathematics teaching.

SISYPHUS
JOURNAL OF EDUCATION
VOLUME 1, ISSUE 3,
2013, PP. 44-75

Essay on the Role of Teachers' Questioning in Inquiry-Based Mathematics Teaching

Luís Menezes | António Guerreiro

Maria Helena Martinho | Rosa Antónia Tomás Ferreira

INTRODUCTION

Questioning has long been a tradition in teacher discourse and has played an important role in structuring teaching activities (Gall, 1970; Menezes, 1996, 2004; Nicol, 1999; Tomás Ferreira, 2005). In the direct teaching approach, most of the teacher's questions are aimed at testing students' knowledge, and are usually posed after an initial presentation of content made by the teacher (Ainley, 1988; Mason, 2010). Inquiry-based mathematics teaching is characterized by a strongly interactive discourse and by new roles played by both teachers and students. Such an approach to teaching mathematics has gradually gained relevance (Canavarro, Oliveira & Menezes, 2012; ME, 2007; Ponte, 2005). However, there has not yet been sufficient examination of this approach to teaching mathematics, when it comes to teacher discourse and the teacher's use of questions. Thus, as part of the project entitled P3M – *Professional Practices of Mathematics Teachers* – which studies, among other issues, the communication practices of mathematics teachers, we aim to discuss the role of teacher questions in inquiry-based mathematics teaching.

In this essay, we start by laying the grounds on communication and inquiry-based mathematics teaching. We then advocate for the specificity and centrality of teacher questioning in an inquiry-based mathematics classroom, perceiving it as a discursive tool for learning.

COMMUNICATION AND INQUIRY-BASED MATHEMATICS TEACHING

Communication is a structuring element of human activity. In particular, it is a structuring element of the act of teaching. Since we do not live in isolation but rather in interaction with others, our activity is characterized by a strong communicative element. In other words, much of what we do is, or involves, communicating. Given that communication is closely related to our daily actions, the decisions that we make at every moment, which lead us to choose one path over another, are motivated by our vision of what surrounds us, in particular by our conception of communication (Brendefur & Frykholm, 2000).

Mathematics teaching is effected through a set of actions carried out by the teacher, supported and justified by the teacher's knowledge of mathematics, of students and their forms of learning, of curricula, and of instructional practice (Ponte, 2012). Such knowledge transversally embodies the idea of communication as a generative and as a disseminative element (Sierpinska, 1998). Communication is embedded in the generation and representation of mathematical knowledge. At the same time, communication plays a central role in the teaching and learning of mathematics.

In our paper, we discuss some of the main conceptions of mathematical communication in the classroom. Then we focus on the inquiry-based approach to teaching mathematics, which embodies one of those conceptions.

MATHEMATICAL COMMUNICATION IN THE CLASSROOM

When analysing the mathematical communication that occurs in the classroom – globally viewed as the communication that focuses on mathematical ideas and uses mathematical processes and representations – we can identify two main conceptions of mathematical communication. One sees communication as *transmission* of information, knowledge and ideas, a process that is anchored in knowledge and in the various forms of disseminating it. The other conception views communication as *social interaction*, in which the subjects interact with each other, searching shared meanings, and collectively constructing knowledge and forms of circulating it (e.g., Bauersfeld, 1994; Godino & Llinares, 2000; Sierpinska, 1998). The existence of communicative relationships amongst those who communicate (which occur in a certain context and involve the use of a shared code) is assumed by these two visions of communication, but they are distinguished essentially by the intentions of those who communicate. Thus, mathematics communication in the classroom assumes the existence of knowledge, culturally shared codes and relations among the actors (i.e., among the students and between these and the teacher). It is, thus, an essentially communicative process that can be either a transmission/circulation instrument for mathematical knowledge, employing its own language, or a basis for the social construction of mathematical knowledge amongst different actors in the classroom. Sierspinka (1998) clarifies this divergence: «From the interactionist perspective, transmission of knowledge is not an issue because knowledge is not in the head of the teacher. It is something that emerges from shared discursive practices that develop within the cultures of the classroom» (p. 57).

When communication is seen as a transmission (as a tool), its goal of communication is to persuade the other. Thus, based on a relationship of authority, the sender intends for the receiver to react as predicted, in accordance with the message sent. It is important that the message be preserved as much as possible, avoiding noise, in order to ensure that the receiver gets the message with the greatest possible accuracy in terms of the sender's intentions (Bitti & Zani, 1997). Under such a vision of communication, the interlocutors act neutrally toward what is being communicated, and the act is labelled «message decoding» instead of «interpretation». This perspective of communication entails the existence of a mathematical knowledge, previously coded by the teacher, transmissible to the students, in a culturally recognizable language, through constant noise reduction, regardless of how many students are in the classroom.

However, when communication is seen as social interaction (as foundation), it is a social process in which the subjects interact with each other, exchanging information, influencing one another, but looking to build shared meanings. This is a process of successive approximations, in which both parts supply additional information which helps to construct meaning through interpretation. In this perspective, mathematical knowledge emerges from collective processes of communication and interaction among the subjects and the classroom culture, including the teacher's interactions with the students about mathematics (Sierpinska, 1998). The social interactions amongst the students and between them and the teacher are fundamental in the mathematics teaching and learning process, namely in the interpretation and negotiation of social and mathematical meanings (Bauersfeld, 1994). The students' mathematical knowledge is influenced by the nature of the communicative actions happening in the classroom and is, therefore, socially constructed and conditioned by the teacher's and the students' ability to understand, reflect, negotiate meanings, and establish mathematical connections.

These two perspectives of communication match general orientations for teaching practices. In daily classroom life, we can find evidence of each of these perspectives, but also intermediate forms of communication (Brendefur & Frykholm, 2000).

INQUIRY-BASED MATHEMATICS TEACHING

There are several differences between inquiry-based and direct teaching. These two approaches to teaching mathematics are distinguished essentially by the mathematical communication generated in the classroom, the teacher's and students' roles in classroom activities, the status of mathematical knowledge, and the tasks that are posed to students and developed by the class as a collective group (Ponte, 2005). In inquiry-based teaching, «the emphasis is moved from the 'teaching' activity to the more complex activity of 'teaching and learning'» (Ponte, 2005, p. 13). The teacher's role is no longer merely to transmit mathematical knowledge to attentive and silent students. Above all, the teacher is expected to engage the students in rich mathematical activities based on challenging mathematical tasks, working autonomously (usually in small groups) and also collectively (with the whole class), emphasizing discussion and negotiation of meanings (Bishop & Goffree, 1986; Ponte, 2005).

An inquiry-based mathematics lesson is usually organized around three or four phases, according to whether or not the last phase is unfolded. Stein, Engle, Smith, and Hughes (2008) propose a three-phase model (the *launch* phase, the *explore* phase, and the *discuss and summarize* phase), while Canavarro *et al*. (2012) advocate four phases, emphasizing the systematization of mathematical learning as a phase of particular importance. In each of these phases, the teacher carries out a set of actions directly aimed at promoting mathematical learning and a set of actions targeting classroom management. The actions aimed at fostering mathematical learning have a greater impact on the classroom discourse and mathematical communication. Next we describe each of those four phases.

In the first phase of the lesson – *Introduction of the task* – the teacher looks to ensure that the students appropriate the task (by acquainting them with the context and interpreting the task, namely its goals, while avoiding reducing the task's cognitive demand) and to promote task engagement. As far as management is concerned, the teacher organizes the students' work (individually, in pairs, in small groups), creating an adequate environment for students to complete the task (for example, supplying materials that support students' work).

In the second phase – *Development of the task* – students work autonomously, usually in small groups. In this phase, the teacher must guarantee that the students complete the task by posing questions, offering hints, suggesting forms of representations, and asking for clarification and justification. The teacher must also maintain the cognitive demand of the task and stimulate students' autonomy by fostering mathematical reasoning and avoiding validating answers. In terms of management, the teacher promotes pair and group work, regulating students' interactions and asking them to keep a record of all their work to support the collective discussion.

The *Discussion of the task* is a rather important phase in an inquiry-based lesson that goes beyond the presentation of solutions. It is a rich moment concerning mathematical communication and the search for common grounds, whose ultimate aim is the construction of knowledge. Canavarro *et al.* (2012) emphasize, in this phase, the teacher's intention of promoting the mathematical quality of students' presentations and regulating their interactions through questioning, asking for explanations and the underlying rationale behind the strategies and reasoning presented. As far as classroom management is concerned, it is crucial that the teacher maintains an environment conducive to the discussion of ideas by managing students' participation and encouraging the sharing of mathematical ideas, regardless of whether they are incomplete, confusing or even wrong. The aim is thus to deconstruct incorrect knowledge and construct mathematical knowledge in a precise language that is recognizable to the students.

The main mathematical ideas that are discussed and shared in the previous phase are expected to be recalled, systematized, and recorded during the last phase – *Systematization of mathematical learning*. In this phase, the teacher, with the students' collaboration, institutionalizes ideas or procedures and establishes connections with the students' own knowledge. This is done by means of actions such as identifying representations and pointing at connections to previously learned concepts. In terms of classroom management, the teacher must focus students' attention on the systematizing activity and ensure that the ideas emerging from that activity are recorded in written form. It is important to note that this phase of systematization of mathematical learning does not necessarily occur after the discussion of the task. In some cases, as the lesson unfolds, the systematization of mathematical ideas may be simultaneous with the discussion of the task. In addition, there may be several moments of discussion/systematization during the development of the task. For example, if there is a generalized question or mistake, the teacher may stop the students' activity in order to discuss the issue before resuming the task.

Inquiry-based mathematics teaching is underpinned by a conception of communication as social interaction. Thus, it presents the teacher with a set of challenges with regard to the management of his own discourse and that of the students. Questioning is an important element of the teacher's discourse (and also a challenge), and this is what we address next.

QUESTIONING: A FACET OF THE TEACHER'S DISCOURSE

Discourse can be seen as language in action, that is, the usage of a linguistic system in real contexts with the goal of communicating (Sierpinska, 1998). According to Searle (1984), «speaking a language is performing acts according to rules, acts, acts such as making statements, giving commands, asking questions» (p. 26). The teacher is always a producer of discourse in the mathematics classroom. This discourse, which is substantiated through different communicative actions, may be of distinct nature according to the teacher's perspectives on mathematics teaching and learning and, in particular, on the role played by communication in these processes. Thus, «the discourse of the mathematics class reflects messages about what it means to know mathematics, what makes something true or reasonable, and what doing mathematics entails» (NCTM, 1991, p. 54). The teacher's communicative actions in a mathematics classroom may be quite varied: questioning, explaining, listening, responding (Nicol, 1999; Tomás Ferreira, 2005). In this paper, though recognizing the strong interrelationship amongst these actions, we focus on questioning, since it is a powerful promoter of student discourse.

We start by discussing the concept of questioning and its related terminology. Do the different terms *question*, *interrogation*, *query*, *demand*, *inquiry*, represent the same thing? As far as teaching practice is concerned, but also, to a great extent, in the field of mathematics education research, these terms are used interchangeably, describing an action by which one person asks information of another. Pereira (1991) sees the question as being an *interpellation*, which she defines as a «non-assertive enunciation – at least in its most common form – which corresponds, in some way, to the solicitation of a particular student or set of students who form a class» (p. 168). According to this author, interpellations may be questions demanding an answer, but they can also be orders or requests («Would you mind to step aside, so that your partner can look at the board?»), or an oral expression aimed at holding the students' attention («Ok?», «Isn't it?», «Right?»). Therefore, on the one hand, we have interpellations which, though formally interrogative, are not really questions since a verbal answer is not expected. On the other hand, we have interrogative enunciations, which we consider questions but which are, indeed, requests for information («Tell me what you are thinking») (Mason, 2010). In this paper, we consider all enunciations, interrogative or not, which reflect an actual request for information as questions. Thus, they are followed by a waiting time so that the answer may emerge.

Questioning has a strong presence in the practices of mathematics teachers. Yet, the teaching practices in general, and questioning practices in particular, of mathematics teachers with different perspectives of teaching and learning are, themselves, distinct. This distinction lies essentially on the purposes with which teachers ask questions of their students as well as on the moments in which that questioning occurs.

The roles played by students and teachers in direct or inquiry-based teaching are essentially different. In a mathematics classroom where the direct teaching approach has been adopted, all mathematical activities going on in the classroom are somehow focused on the teacher. Students, on the other hand, are supposed to listen to the teacher's explanations and reproduce his mathematical procedures. With this approach, questioning is an activity reserved for the teacher only and, in general, aims to test the students' knowledge.

On the contrary, in an inquiry-based mathematics classroom, it is the teacher's responsibility to propose learning situations that will help students to build their own knowledge. This is achieved not only by developing different actions aimed at promoting student learning but by placing the centre of mathematical activities in the hands of the students as a collective. Teacher and students question and listen attentively to each other, within a classroom culture that emphasizes sharing strategies and negotiating meanings. Therefore, questioning is an activity shared by all the classroom actors – teacher and students – who have different aims that go beyond testing the students' knowledge and scholastic achievement.

Hence, the relevance of questioning to the teacher's role in an inquiry-based approach to mathematics teaching is not surprising (Cengiz, Kline & Grant, 2011; Hufferd-Ackles, Fuson & Sherin, 2004). In particular, the teacher's questions challenge students to become active in the classroom through verbalization (presenting information orally or in written form) and reflection (analysing and weighing available information). As such, the teacher's questions may aim at either to verify the student's knowledge, or focus the students' attention on mathematical ideas or strategies, or even inquire them about how they are thinking. With these different purposes in mind, we can pinpoint three main types of questions: *verification*, *focusing*, and *inquiry* questions (Ainley, 1988; Mason, 1998, 2000).

Verification questions aim to test students' knowledge (which explains why they are also called *testing* questions), leading to short and immediate answers. These answers are previously idealised by the teacher, who believes that such questions contribute to regulate the way students learn mathematics. The teacher (the adult) builds a mental representation of the student's (the child) knowledge through verification questions, which test the knowledge supposedly acquired in the mathematics classroom. Verification questions also contribute to asserting the teacher's social control (Mason, 2010), namely when the teacher intends to regulate students' attitudes and behaviours in the classroom. However, when these questions have the latter aim, some authors refer to them as *pseudo-questions* (e.g., Ainley, 1988). In fact, with this type of query, no reply is expected, but rather some sort of enforced complicity (Mason, 1998).

Verification questions are quite common in the mathematics classroom. They play an important role in ascertaining knowledge acquisition, attesting to correctness, and articulating or interconnecting different ideas (Mason, 1998, 2000). This type of question may also include incomplete statements, made by the teacher, usually at the end of sentences, aiming to allow students to demonstrate what they know by completing the sentence (Menezes, 2004). This technique can promote the development of mathematical reasoning (by making sense out of the incomplete statement); but it can also result in mechanised routines (as with, for example, the recitation of multiplication tables).

Focusing questions aim to focus students' attention on a specific issue the teacher wants to underline. They can also be aimed to redirect the focus of attention to students' own reasoning. Such questions are specific to the educational arena and exhibit a strong formative intentionality. Hence, focusing questions do not usually appear in everyday life, as they are not genuine requests for information. In an educational context, however, focusing questions are a fundamental discursive tool that guides and supports student thinking, since they entail the effect of focusing or directing the attention of the audience (Mason, 2000, 2010). This sort of questioning may have a funnelling effect when students to not provide answers (Bauersfeld, 1994; Voigt, 1985), as they often generate a spiral of decreasingly difficult questions that only require of students short and quick answers. In the limit, this may be pushed to an unsatisfactory level with respect to the learning of mathematics, either when the teacher favours student involvement over fostering mathematical knowledge, or when the teacher maintains an excessively sophisticated level of mathematical discourse even if it is inaccessible to students. Focusing questions may assume a metacognitive dimension whenever they direct students' attention to their own thinking or when they deviate students' attention from the specificities of a task towards generalizing their mathematical ideas (Hufferd-Ackles *et al.*, 2004; Mason, 1998, 2000, 2010). In such cases, focusing questions promote the students' mathematical knowledge in a broad sense.

Inquiry questions are in fact the genuine questions a teacher asks of students when seeking information. This type of question is used in everyday life to obtain informative content, except in situations of a social nature (e.g., «How are you?», «How have you been?») where the answers may already be known and, therefore, have no informative value (they are likely to be questions of circumstance), despite their relevance as a communication tool (Tropea, 2007).

For the teacher, «it is difficult to enquire genuinely about the answer to problems or tasks which have well known answers» (Mason, 2000, p. 15). Actually, genuine inquiry aims essentially at accessing students' thinking, understanding their strategy use, and challenging them to build new mathematical knowledge. Sometimes an inquiry question may signal students that something is wrong with their performance, in which case the real motive of inquiry is distorted (Mason, 2010). However, when inquiry questions are absent from a mathematics classroom, it is likely that the teacher is omniscient and that the whole class has a non-questioning attitude. In such classrooms, students are unable to build their own mathematical knowledge through analysis, conjectures, and justification of properties and generalizations (Mason, 1998, 2000).

Looking globally at these three types of teacher questions, we realize that verification and focusing questions are closely linked to the didactical process, and are less common in everyday life, outside the school context. Indeed, such questions, especially verification questions, when used in daily interactions among adults (this is usually not the case when a child interacts with an adult), often lead to communication problems due to discomfort that can be created among people. Inquiry questions are most frequent in everyday life but their present in the mathematics classroom varies (Ainley, 1988). In fact, inquiry questions are common when the classroom culture emphasizes problem solving, reasoning and communication; such skills are regarded both as valuable tools for learning mathematics and as curricular goals.

Analysing the purposes of each of the three types of questions we have discussed in terms of their relationship to students' mathematical knowledge, we notice that verification questions are retrospective in nature because they target students' pre-existing knowledge. In contrast, focusing and inquiry questions are forward-reaching in that they focus on students' developing knowledge, with the support of the teacher and the collective classroom.

TEACHER'S QUESTIONING IN AN INQUIRY-BASED MATHEMATICS LESSON

As was previously discussed, the inquiry-based mathematics lesson may be seen as divided into four phases, distinguishing the experiences of collective discussion from systematization of mathematical learning. As the aim of this essay is to discuss the role of teacher questions in inquiry-based mathematics teaching, the subject becomes clearer if the four phases are analysed separately. Thus, we address the following phases: (i) introduction of the task; (ii) development of the task; (iii) discussion of the task; and (iv) systematization of mathematical learning. The teacher has an important presence in the discourse of all of these phases, namely through the questions he or she poses. The different types of questions we have addressed in this paper occur at different times and perform different aims throughout the lesson.

Next, we discuss the teacher's questioning in each of the phases of inquiry-based teaching. We use examples and classroom episodes from three different classes[1] to illustrate our claims and ideas. The classification of questions in the following episodes is the one that an independent, external observer can infer taking into account both the structure of the question and the associated contextual information. All episodes have been videotaped, so that the viewer can experience the verbal interactions, gestures, intonation, and the facial expressions of all the interlocutors.

TEACHERS' QUESTIONS IN THE INTRODUCTION OF THE TASK PHASE

When introducing the task, the teacher looks for familiarizing students with the task itself, while possibly referring them to resources they may use and explaining what is expected of them. Therefore, in this initial phase, the teacher may feel the need to pose a number of questions to check whether the students have understood what is being proposed and are ready to start working autonomously (Mason, 2000; NCTM, 1991).

Through *verification* questions, the teacher can *ascertain students' prior knowledge* that is necessary to accomplish the task. The teacher may ask several verification questions, namely questions *centred on concepts* that are explicitly present in the task, or related concepts; questions pertaining to the task's *context*, eliciting students' experiences; questions to assess the understanding of the task's *goals and language*, be they mathematical (notations and terminology) or natural (Ainley, 1988; Nicol, 1999).

In order to ensure that the understanding of the task is not an obstacle to its accomplishment, teachers often feel the need to ask questions like: «Is there any word or expression you do not know?» Canavarro *et al.* (2012) refer

1 The three lessons discussed in this paper are part of the multimedia cases developed in the context of task 3 of the research project P3M – Professional Practices of Mathematics Teachers. They exhibit the characteristics of inquiry-based mathematics teaching (Canavarro, 2011; Ponte, 2005), in the sense that they are built upon students' activities, triggered by tasks posed by the teacher towards the construction of mathematical knowledge in a process emphasizing the discussion of mathematical ideas. The three teachers involved have a long professional experience (15 to 20 years), and usually enact inquiry based teaching.
The 1st cycle (4th grade) lesson, conducted by Célia, develops around the task «Cubes with stickers», and aims at developing algebraic reasoning. In particular, it focuses on the recognition of a sequence and its variables, the identification of their relationship, and the development of a corresponding general expression, expressing it both in natural and mathematical language. The 2nd cycle (5th grade) lesson, conducted by Fernanda, aims to deepen the students' understanding of the concept of percentages and its connection to the notion of unit (in the context of the Rational Numbers theme). It is based on the task «The rise and fall of fuel prices». Finally, the task «The class president´s election» is proposed by Cláudia to a 7th grade class in order to enrich students' knowledge of first degree equations. The three tasks can be found in the appendix.

to these questions as «how?»-questions because they intend to «clarify the way the task is interpreted» (p. 9).

In the 7th grade mathematics lesson, students were expected to identify who had won the election for president in a class with 30 students. When, after reading the task aloud, the teacher asks: «Any doubts?», followed by «Do you know what blank or null votes are?» (episode 1), she is precisely verifying the students' prior knowledge.

EPISODE I

Teacher: Let's read the task «The class president´s election»: (…) Any doubts? Do you know what blank and null votes are?

Students: Yes.

Teacher: What do you mean by that?

Students: They didn't count…

Student: They didn't vote.

Teacher: It's not that they didn't vote. Voting blank means that they have not chosen any candidate. A null vote is when a vote is erased or somehow damaged (…) Now, you can use different strategies to solve the task; don't forget you have to show how you worked it out. You have 10 minutes to finish the task.

By asking «Any doubts?», the teacher aims to pinpoint students' difficulties in understanding the task. This question, which is equivalent to «Did you understand?», sometimes does not produce the expected outcome, thus missing its purpose (Menezes, 1996). Often, students simply do not answer because they are unable to process all the information or are unaware of possible difficulties. Thus, the goal of the teacher's verification questions is better achieved by testing some specific aspects of the task, which the teacher anticipates as potential sources of misunderstanding (Ainley, 1988; Cengiz *et al.*, 2011). This is exactly what the teacher did when, after asking «Any doubts?», she added the verification question «Do you know what blank and null votes are?».

When the teacher addresses blank and null votes, her intention is to verify whether students have assigned a meaning to the task statement «All 30 students in the class have voted and no blank or null votes were cast». Knowing that there are no blank or null votes is, in fact, an important piece of information, although it may seem, at first, unnecessary. However, without a grasp of this fact, the problem could have multiple solutions, instead of just one (as intended). Therefore, the teacher found it relevant to make sure that this knowledge was shared amongst the students. Similarly, the question «What do you mean by that?» reinforces the previous query while challenging students to justify their answer. Since they are verification questions, the answers tend to be quite brief, as can be seen in episode 1 (NCTM, 1991; Tomás Ferreira, 2005; Wood, 1995).

The teacher may follow different strategies in order to check whether students are able to understand the task, the main purpose of this phase of the lesson. For example, she may ask them to present the task in their own words, or to check if any concept is unfamiliar to them (Canavarro, 2011). Episode 2, drawn from a 4th grade lesson, illustrates the former strategy. The task «Cubes with stickers» aims to determine the number of stickers required to fill in the faces of rows of cubes, united by their faces. The teacher asks students to read the problem and explain it in their own words. In this way, she verifies if the problem is correctly understood, helps other students to clarify their own understanding of the problem by drawing on the interpretations of their classmates, and promotes the development of communication skills.

EPISODE 2

Teacher: Who is able to explain, in his own words, what this task is all about? João…

João: She is making constructions with the cubes and putting stickers in each visible face; but she didn't put any stickers in the middle; and they say there that she used 10 stickers and this is right because she didn't put any sticker in the middle of the cubes. They were together, so she can't… that is, she can but it wouldn't make much sense.

Teacher: Why wouldn't it make much sense?

Students: Because it couldn't be seen.

Teacher: So, in that construction, the one she made with 2 cubes, she used 10 stickers. I have one cube here, two cubes. Very quickly, let's see, how she would make this construction. Rita, do want to come here and help? How would Joana, we have a Joana here, make this construction? I have 2 cubes here, and glue …

In this episode, the teacher reinforces the initial proposal – «to explain in (…) own words» – with a direct question to the student: «Why wouldn't it make much sense?» This reinforcement is made even stronger since the teacher has decided to make the construction mentioned in the task statement herself by gluing the stickers with the help of a student.

Whenever tasks involve references to a real context, the teacher must also make sure the context is familiar to the students. For example, in the task dealing with the rise and fall of fuel prices (5th grade lesson), the teacher tries to draw the students' attention to the context by asking «Do you often put fuel in your parents' car?»

TEACHERS' QUESTIONS IN THE DEVELOPMENT OF THE TASK PHASE

The students work on their own to complete the task usually in pairs or small groups, since peer interactions and mutual support suit the cognitive level of challenging tasks. While monitoring students' activity, the teacher seeks to understand how they think, what meanings they ascribe to the mathematical ideas they are working with and the representations they are using, what justifications they present, and what difficulties they reveal in completing the task, etc. (Canavarro *et al.*, 2012). Thus, in this phase of the lesson, the purposes of the teacher's questions may include *verification*, *focalization*, and *inquiry*. Yet, it is likely that focusing and inquiry questions are predominant (Mason, 2000; Nicol, 1999).

One of the purposes of *inquiry* questions is to *access students' thinking and understand it*. Yet, we must bear in mind that these questions help to build a «*conjecturing* atmosphere» (Mason, 2010, p. 6), since the teacher deals with students' assertions or answers as if they were conjectures, not considering them right or wrong at the outset. One means of understanding students' thinking is to ask them to explain their ideas. In the following episode (episode 3), the 4th grade teacher asks Rita: «Why are you saying that?» In attempting to understand how Rita thought, she explicitly asks the student to explain her reasoning not only to herself but also to Diogo, who was working with Rita on the task. It is important to notice that the teacher also asks Diogo to pay attention to his classmate's explanation. In fact, as Rita verbalizes her thinking, the teacher's *hows* and *whys* emphasize the need for explanation and justification, while involving Diogo in the analysis of Rita's strategy.

EPISODE 3

Teacher: Why are you saying that? Explain it better, so that Diogo and I can understand it…

Rita: We took this one out and then we placed it here to make four cubes. Then we did like this: 4, 8, 12, 16, 17, and 18. 4 times 5 minus 2.

Teacher: 4 times 5. Why 5?

Rita: Because 1, 2, 3, 4, 5…

Diogo: Right, but then you're adding 2 cubes. So, you are saying that there are 2 cubes, one here, and another here.

Teacher: How do you think it should be?

Diogo: It should always be one less…

The following episode (episode 4), from a 7th grade lesson, involves the teacher posing inquiry questions. As in the previous episode, the ultimate purpose of these questions is to *understand how students think* while solving the task, which Nicol (1999) referred to as «posing questions to learn what students are thinking» (p. 53), or as Bishop and Goffree (1986) asserted: «the teacher's questions because he genuinely wants to know» (p. 329). We should realize that in episode 3 this goal is difficult for the teacher to achieve since students often fail to record the way they think or record it incompletely. The teacher tries to follow a group's strategy. Even though she recognizes a student's faulty reasoning, she asks her to explain the strategy that was followed by the whole group, insisting: «Okay… and so what?» As the student explains again her thinking, she realizes the flaw in her reasoning: «Ah, I know what was wrong!» Bishop and Goffree (1986) point out the explanatory value of this form of questioning: «This use of questioning by the teacher shows us that what is important about explaining is (…) that the *connections* get exposed – not that it is the teacher who necessarily does the exposing» (p. 334). In the end, the teacher realizes that, as she suspected, the students have followed a trial-and-error strategy.

EPISODE 4

Teacher: But you are not presenting your reasoning here! Okay, so you begin with 10, 10, 10. Is that it? Ten votes each. And then what?

Student: Then we know that Lucas got 2 votes less than Francisca, so we subtracted 2 from Lucas and added 2 to Francisca.

Teacher: Okay…

Student: Francisca gets 12 votes…

Teacher: Okay… and then what?

Student: Then Sandra got twice as many votes as Lucas, so what we did with Lucas… Ah, I know what was wrong! Okay… we need twice as many votes as Lucas'... twice as many votes as Lucas times 2... That was Sandra's number, Sandra's votes.

Teacher: So you're using trial and error, right? But show all the hypotheses. But you are thinking correctly.

Besides accessing students' thinking, inquiry questions also have another purpose, which entails *challenge* (Martinho & Ponte, 2009): to help students go further. For example, the 5th grade teacher stimulates students' thinking when she asks them: «Is it always going to be like that? Even with other prices? Affordable prices? Do you think so? Will it always happen like that?»

Focusing questions may cover several aspects. Some of them are *directly related to the task*, others to the *group dynamics* while students solve the task (Guerreiro, 2011). As for the aspects related to the task requiring the teacher's intervention, we find those dealing with difficulties in grasping the data of the problem, finding a solution process, or using language or representations. For example, the teacher may feel the need to focus students' attention on the task's wording. Without an accurate grasp of the specific statements used they may be unable to consider all the conditions needed to solve the problem and thus, hit a dead end (Menezes, 2004). In this sense (focusing on the data and problem conditions), the 7th grade teacher asks focusing questions of her students, such as «How do we use this 30?» and «What is this information for?», highlighting specific data in the mathematical problem students were working on.

Focusing questions naturally put emphasis on students' *mistakes* when solving mathematical tasks (Mason, 2000). In inquiry-based teaching, the goal of the teacher's questions is not to correct mistakes, but rather to help students identify and correct them by themselves (Wood, 1995). Having this in mind, Ms. Cláudia (7th grade teacher) goes through the students' notes to analyse their solution processes. As she identifies a processing error (such as $2 \times (-2) = 4$), she focuses the students' attention on the wrong answer to the multiplication, without immediately revealing the right answer, using a focusing question centred on the error «2 times – 2 ... is equal to 4?» The students analyse and correct the product «No, it is –4».

In episode 5, 5th grade students working in groups believed they were thinking correctly: it was clear to them that if the price of fuel had risen by 10% and then fallen by another 10%, it would return to the original value.

EPISODE 5

Teacher: Tell me what you've done so far.

Student 1: (…) plus 10%, it will return to the previous price.

Teacher: Really? Have you tried it out?

Student: No.

Teacher: Yes or no?

Student 1: If the fuel is at a given price, then goes up by 10%... and then it goes down by 10%...

Student 2: It reaches a certain price, then the price is taken, it goes back to the same…

Teacher: But, have you checked that using a particular price? So, you think that if the price goes up by 10% and then falls by 10%, it will go back to its original value, is that it? Is everybody in the group thinking in the same way? And have you tried that with a specific price?

Student 2: We have to do it now…

The teacher pinpoints this error in the students' reasoning and asks them to focus their attention on the process they were using. She challenges them to check their conjecture with a specific price per litre. Although they acknowledge they had not worked it out using any specific price, it took them a while to be convinced that trying out a few prices would be a good idea. The teacher re(emphasizes) the focusing question when suggesting the students to experiment with a few specific prices. Yet, before posing the question, she made sure no one in the group disagreed with the initial (erroneous) conjecture.

In the two previous episodes (4 and 5), the teacher focuses the students' attention on the shortcomings of their reasoning. There are other situations in which the teacher questions the language (terms and notation) used by the students, both orally and in written form. In the 7th grade classroom, the teacher focuses students' attention on the inadequate use of certain terms. She revoices, in an interrogative way, a student's statement that incorrectly uses the word «annul» (regarding the parentheses) – «Do we have to annul?», concerning the discarding of parentheses in the simplification of an algebraic expression.

In episode 6, the teacher focuses her 4th graders' attention on the inadequacy of their *written mathematical language*.

EPISODE 6

Fábio: Yes, and then we did 4 times 5 equals 20, minus 2.

Teacher: Pay attention, 4 times 5 is equal to?

Marco: 20.

Teacher: So, is it equal to 20 minus 2? Does 4 times 5 equal 20 minus 2? Can you leave it like this?

All: No.

Teacher: No. 4 times 5 equals 20; 3 times 5 equals 15, right? But it is not 20 minus 2, I mean, 4 times 5. You have to separate them, don't you? How can you do that?

Fábio: By putting it in brackets…

Teacher: In brackets? 4 times 5 equals 20… then, what do you want to do?

Fábio: 20 minus 2.

Teacher: So, write that down here, below: 20 minus 2, which equals…

Fábio: … 18.

Teacher: 18, isn't it? You cannot write everything in the same line.

While students work cooperatively, the teacher realizes that they are writing down the process incorrectly. The students did not write correct numerical expressions for each of the actions made (4‘5 = 20, 20 - 2 = 18); instead, they wrote one single expression reflecting the set of those actions in sequence (4‘5 = 20 - 2 = 18), thus creating an incorrect and meaningless numerical expression.

The teacher's questions can also be viewed in the light of their two main goals in an inquiry-based lesson: to *promote learning* and to *manage interactions* (Menezes, Canavarro & Oliveira, 2012). In episode 7, the 7^{th} grade teacher notices that the two elements of a pair are not really working together on the task (the election of the class president). Each of the students reveals different difficulties and makes different mistakes, though they each hold ideas that complement the other's. These ideas potentially help both students to correct each other and to find a solution to the problem together, provided they work cooperatively. The questions the teacher asks of them aim to not only focus the students' attention on their own mistakes and encourage them to identify and overcome their errors, but also help them to manage their joint interactions, while showing them that interacting has the potential to improve their performance.

EPISODE 7

Teacher: Do you agree, Beatriz? That the total of votes, plus Lucas's, plus Sandra's… this is what you wrote here…

Pedro: Plus Sandra's votes.

Teacher: Is that equal to Francisca's votes?

Beatriz: Wait a minute, I didn't get it…

Teacher: What he wrote here is that the total of votes, plus Lucas's … and there's something missing here, in the middle, plus Sandra's equals Francisca's. Do you agree with this equation?

Beatriz: No, I don't think so.

Teacher: So, how do you think you could write this in an equation?

As previously mentioned, in the development of the task phase, there is a prevalence of inquiry and focusing questions. Verification questions serve as supports for other types of questions, and sometimes help to resolve certain deadlocks. In this phase of an inquiry-based lesson, the role of verification questions is not so much one of testing or verifying, but rather one of supporting interactions of inquiry or focusing nature (Guerreiro, 2011). Thus, at a macro and more holistic level, the teacher's discourse in the development of the task phase is characterized mainly by inquiry and focusing questions; but it is also marked by some verification questions, especially when we look at the teacher-student interactions at a micro level.

TEACHER'S QUESTIONS DURING THE DISCUSSION OF THE TASK PHASE

In an inquiry-based mathematics lesson, the discussion of students' productions achieved during autonomous work, and the strategies and ideas employed requires the teacher to manage the students' discourse allowing everyone, including herself, to understand what is shared among the whole group (Cengiz *et al.*, 2011; Ruthven, Hofmann & Mercer, 2011; Stein *et al.*, 2008). The question is of paramount importance in attaining such goals as it serves to regulate discourse, leading the students to present information that the others do not know, which is one of the purposes of inquiry questions (Mason, 2000, 2010; Nicol, 1999). *Inquiry* questions are associated with *requests for explanation or justification* (Stein *et al.*, 2008; Yackel & Cobb, 1996) in the discussion of the task phase of an inquiry-based lesson. In this sense, the teacher challenges a group of 4^{th} graders to explain how they managed to find the pattern involved in placing a number of stickers on the cubes. She asks the following inquiry question: «What about for any given number of cubes? How would you find the number of stickers?»

Inquiry questions that elicit justification for ideas or procedures, allowing the teacher to learn about students' reasoning, are very common in inquiry-based mathematics teaching (Mason, 2010). Typically, these questions start with a «Why», and follow the students' own statements. In the 4th grade class (episode 8), the teacher repeatedly asks this type of question, seeking to gather information that will enable her to understand the students' thinking.

EPISODE 8

Caleça: If you remove plus 2, this is the 4 times table.

Teacher: Why is it always plus 4?

Caleça: Because you always do 4 times…

Teacher: But why?

Carolina: 9 times 4 equals 36. Then plus 2 makes 38. 10 times 4, 40; you add 2, 42. 2 is the number causing this…

Teacher: Number 2 is causing this. But why did you say … You have little arrows there, plus 4. But why plus 4 and not plus something else?

Carolina: Because the difference of 4…

Teacher: Why?

Focusing questions occur during the discussion of the task, usually when, during the explanation and justification of their ideas, the students display errors, imprecisions or lack of clarity (Guerreiro, 2011). In such situations, the teacher chooses to question the students, rather than point directly to the mistakes. Her intention is to have all the students re-examining their discourse (Nicol, 1999; Tomás Ferreira, 2005), acting in a similar way to that during the development of the task phase. In episode 9, the 5th grade teacher asks questions to clarify a student's explanation («Which, in your opinion, was how much?»; «Right, is it because otherwise 10% would still be 45 cents?») or to lead the student to conclude that the value found was incorrect («But, Rute, but the 45 cents that you are taking away from it, what is that?»).

EPISODE 9

Teacher: Rute, would you mind clarifying what you've just explained a bit more? Explain it better to us.

Rute: This A stands for the amount…

Teacher: Which, in your opinion, was how much?

Rute: It was € 4,50.

Teacher: Right, is it because otherwise 10% would still be 45 cents?

Rute: No.

Teacher: Bear in mind that you're adding 45 cents here. Okay, explain that to us.

Rute: Then, the previous amount of € 4,50 plus the 45 cents, which was 10% of that amount, would give us the price, a certain price, which is how much the fuel would cost including the increase. Then, that price together with the increase minus 45 cents would fall back to the same amount…it would lead to the same value…

Teacher: But, Rute, but the 45 cents that you are taking away from it, what is that?

Rute: Right, we made a mistake here because the 45 cents is related to € 4,50…

Verification questions have little weight during the discussion of the task phase. They usually occur when the teacher wishes to test the students' understanding of what has been presented, and often lay the grounds for the systematization of mathematical learning.

Teacher's questions during the systematization of mathematical learning phase

In this phase, the teacher combines the synthesis of the task's solutions, highlighting the appropriate usage of mathematical language (terms and notation), with possible extensions of the results obtained, often having in mind their mathematical generalization (Canavarro *et al.*, 2012). It is the moment at which mathematics learning becomes *institutionalized*, going beyond the task that has just been accomplished and attempting to systematize and to represent mathematical knowledge. At this phase of an inquiry-based lesson, which is not as rich in teacher questions as the previous phases, the teacher uses *verification* questions whose answers may indicate how well the students have understood the concepts or mathematical procedures involved. Such questions promote also the use of appropriate mathematical language. In episode 10, a 4th grade teacher uses a verification question («Is it 4 times 52 or 52 times 4?») to clarify the meaning of the order of the multiplication factors, reinforcing the concepts of multiplying and multiplier, and the proper use of mathematical language with understanding.

EPISODE 10

Teacher: Excuse me. Now I'm standing here thinking ... is it 4 times 52 or 52 times 4?

Fábio: It is 4 times 52.

Students: No...

Rita: No, it is 52 times 4.

Teacher: What is being repeated in the cubes?

The divergency in the students' responses causes a further intervention of the teacher through a *focusing* question («What is being repeated in the cubes?»). This question directs students to the context of the mathematical task, suggesting that they reanalyse the problem. Focusing questions encourage students to return to the task, so that they can reflect on what they did, systematize what was learned, and use mathematical language appropriately.

The generalization of mathematical results is a common purpose of the systematization of mathematical learning phase, in order to construct mathematical knowledge (Canavarro *et al.*, 2012). The teachers of the multimedia cases we have used to illustrate the ideas we have put forward have significant concerns about generalization, particularly the algebraic generalization of numerical results, sometimes without resorting to algebraic notation. In the 5^{th} grade class, the teacher tries to negotiate the generalization of mathematical results with the students, going beyond the situation of the mathematical task to other contexts, in order to enhance the students' understanding of mathematical generalization. This concern leads her to use focusing questions («We've worked with many different values, and haven't we reached the same conclusion?») centred on the mathematical solutions. The 7^{th} grade teacher assumes that through focusing questions she will help students generalize mathematical results and make mathematical connections (episode 11).

EPISODE 11

Teacher: Exactly. So, the big difference between this strategy and this one is that if we'd change the number of votes to 7653, it'd be enough to match the first expression to 7653, whereas using the previous strategy, what would happen?

Student: We would be trying, and trying, and trying...

Teacher: Exactly, we would be here ... in a much more complicated process.

The teacher asks the students to compare two mathematical strategies, algebraic modelling through equations and recognition of an algebraic pattern using numerical sequences, as a way of systematizing mathematical learning.

In episode 12, the mathematical connection between the two strategies – equations and sequences – is addressed by the teacher with focusing questions centred on data and procedures.

EPISODE 12

Teacher: Now my question is: look now at our general terms, those of these sequences, and look at the equation Mariana and David have written.

Students: It's the same.

Teacher: Okay. In other words, using trial and error...

Student: It's the same thing...

Emphasizing the connection between different mathematical strategies allowed students to get a better grasp of the algebraic relationship of the numerical sequences underlying the situation at hand. The teacher uses a verification question to ensure that students make sense of the algebraic expression for the general term of the numerical sequence and that they know what mathematical procedures are necessary to find the order of the term: «General term, and from here, if I had 7656 votes, what would I have to do to find the order?»

Thus, the systematization of mathematical learning phase is characterized by verification questions of acquired (institutionalized) mathematical knowledge, and by focusing questions centred on situations of mathematical incorrectness or difficulties evidenced by students. This phase develops around the reanalysis of data, procedures and mathematical strategies, with the ultimate goal of systematizing mathematical learning (Stein *et al.*, 2008).

CONCLUDING REMARKS

Dialogue, both amongst students and between the teacher and the students, is a significant feature of inquiry-based mathematics teaching. Such dialogue emerges, to a great extent, from students' mathematical activity which, in turn, is based on challenging tasks posed by the teacher (Ponte, 2005; Stein *et al.*, 2008). Though dialogue may spontaneously arise among students, it can be significantly enhanced by the teacher when inviting students to participate by requesting information. This may happen at any stage of the lesson (Hufferd-Ackles *et al.*, 2004).

As we have seen, one of the main purposes of teachers' questions is to gather information they do not possess, in order to access students' knowledge and thinking (through inquiry questions) or to assess students' knowledge (via verification questions). Focusing questions, while also generating dialogue, fulfil a specific purpose. They usually lead students to rethink their oral or written answers, focusing on particular aspects that the teacher deems relevant (Mason, 2000; Nicol, 1999).

This connection between the teacher's questions and the creation of opportunities for dialogue, which is not a direct relationship, is particularly evident in the practice of inquiry-based mathematics teaching. As we advocate, and as we have tried to illustrate through episodes of mathematics lessons involving students of various grade levels, the teacher's question is a discursive act that plays a fundamental role in inquiry-based mathematics teaching (Guerreiro, 2011; Hufferd-Ackles *et al.*, 2004). Thus, the questioning of the mathematics teacher, which is seen as a professional practice, is an important and hardly replaceable piece of inquiry-based teaching. Therefore, rather than talking about *good questions* in a mathematics classroom, it is more apposite to focus on good questioning *practices*, i.e., the appropriate use of questions, in a particular context, taking into account the goals one wants to achieve (Aizikovitsh-Udi, Clarke & Star, 2013).

It is natural for the teacher to pose questions throughout the various phases of an inquiry-based mathematics lesson. These questions may be of different types (verification, focusing, and inquiry questions), whether, in each phase of the lesson, the teacher needs to check, focus, or inquire her students' mathematical knowledge. However, given the different nature of the work students do throughout an inquiry-based lesson (as opposed to lessons guided by a direct teaching approach), some types of questions may predominate during each phase.

Thus, in inquiry-based mathematics teaching, verification questions are predominant (i) at the beginning of the lesson, in the introduction of the task phase, when the teacher verifies students' mathematical knowledge and their understanding of the task; and (ii) at the end of lesson, when the teacher aims to *institutionalize* learning, ensuring that students have developed new knowledge. Although verification questions are also common in a direct teaching approach to mathematics, in an inquiry-based approach these questions aim essentially to support assessment for learning and assist the teacher in deciding what she will do next (Mason, 2010).

Focusing questions act as indirect aids for the students. They focus the students' attention on errors, misunderstandings and alternative strategies, allowing them to build on their own reasoning and develop their autonomy. Cengiz *et al*. (2011) identify *extending episodes* during a mathematics discussion when, by means of focusing questions, the discussion moves to a different mathematical idea. However, focusing questions are also important in helping students understand and connect different ways of thinking or new mathematical ideas; thus, focusing questions are relevant when systematizing mathematical learning. As such, in inquiry-based mathematics teaching, focusing questions are suitable when the teacher monitors students' autonomous work, and also when she orchestrates collective discussions and brings the whole mathematical activity to a close. Thus, a significant presence of focusing questions makes sense during all the phases of an inquiry-based lesson except the first.

Inquiry questions make particular sense during the wintermediate phases of an inquiry-based mathematics lesson, i.e., during the development and the discussion of the task phases. When students are undertaking the task, the teacher's monitoring of their work stems largely from inquiry questions, which help the teacher gain understanding of the students' thinking. When students are undertaking the task, the teacher's monitoring of their work stems largely from inquiry questions, which help the teacher in gaining understanding of students' thinking. During the discussion phase, inquiry questions are especially relevant. They trigger students' explanations and justifications, fostering

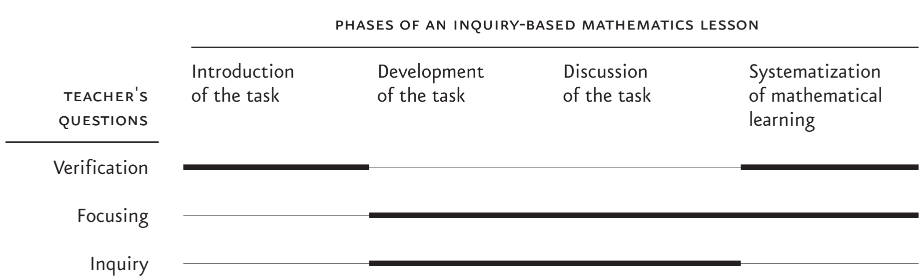


FIGURE I – THE TEACHER'S QUESTIONS IN AN INQUIRY-BASED MATHEMATICS LESSON the emergence of mathematical concepts, their terminology and their forms of representation (Ruthven *et al.*, 2011; Stein *et al.*, 2008). Figure 1 shows the phases of an inquiry-based mathematics lesson in which the different question types play a greater role.

The teacher's questions, as presented in this essay, play a central role in a mathematics teaching approach in which students develop their mathematical knowledge in interaction with one another, through negotiation of meanings, and not (exclusively) by direct transfer from the teacher.

Finally, we believe that this reflection on the role of teachers' questions in inquiry-based mathematics teaching opens several avenues for further research and poses various challenges to teacher education. The teachers in the episodes presented are experienced teachers and skilful questioners, focusing their questioning on mathematical learning. But what happens with less experienced teachers in an inquiry-based approach? Research has pointed out the significant influence of the teacher's knowledge, particularly content knowledge, in her suitable use of questions (e.g., Ball, 1991; Kahan, Cooper & Bethea, 2003; Ma, 1999; Mason, 2010). The issue rises: how does the teacher's mathematical knowledge influence her use of questions when teaching in an inquiry-based approach? These are some issues that require further research.

The complexity of inquiry-based mathematics teaching and, in particular, the role played by the teacher's questions, pose challenges to teacher education (pre and in-service teacher education). How to foster teachers' awareness of the role of questions as tools for the teaching and learning process?

ACKNOWLEDGMENTS

We wish to thank the three teachers (Célia Mestre, Fernanda Tavares, and Cláudia Nunes) who allowed us to use recordings of their classes in this paper to illustrate the use of teachers' questions in an inquiry-based mathematics teaching approach We also want to thank the authors of the *multimedia cases*, under the P3M project, for granting access to all the material pertaining to the three cases: tasks, transcripts and lesson videos, lesson plans and reflections, and students' written production.

This work has been supported by national funds through FCT – Fundação para a Ciência e a Tecnologia, under the project *Professional Practices of Teachers of Mathematics* (Grant PTDC/CPE-CED/098931/2008).

REFERENCES

Ainley, J. (1988). Perceptions of teachers' questioning styles. *Proceedings of PME XII* (Vol. I, pp. 92-99). Veszprém, Hungary.

Aizikovitch-Udi, A., Clarke, D., & Star, J. (2013). Good questions or good questioning: An essential issue for effective teaching. Paper presented at *CERME8: 8th Congress of the European Society for Research in Mathematics Education*. Antalya, Turkey.

Ball, D. L. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. *Advances in research on teaching* (Vol. 2, pp. 1-48). Greenwich, CT: JAI Press, Inc.

Bauersfeld, H. (1994). Theoretical perspectives on interaction in the mathematics classroom. In R. Biehler, R. Scholz, R. Sträßer & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 133-146). Dordrecht: Kluwer.

Bishop, A., & Goffree, F. (1986). Classroom organization and dynamics. In B. Christiansen, A. Howson & M. Otte (Eds.), *Perspectives on mathematics education* (pp. 309-365). Dordrecht: D. Reidel.

Bitti, P., & Zani, B. (1997). *A comunicação como processo social*. Lisboa: Estampa.

Brendefur, J., & Frykholm, J. (2000). Promoting mathematical communication in the classroom: Two preservice teachers' conceptions and practices. *Journal of Mathematics Teacher Education, 3*, 125-153.

Canavarro, A. P. (2011). Ensino exploratório da Matemática: Práticas e desafios. *Educação e Matemática, 115*, 11-17.

Canavarro, A. P., Oliveira, H., & Menezes, L. (2012). Práticas de ensino exploratório de Matemática: O caso de Célia. In L. Santos (Ed.), *Investigação em Educação Matemática 2012: Práticas de ensino da Matemática* (pp. 255-266). Portalegre: SPIEM.

Cengiz, N., Kline, K., & Grant, T. J. (2011). Extending students' mathematical thinking during whole-group discussions. *Journal of Mathematics Teacher Education, 14*, 355-374.

Gall, M. (1970). The use of questions in teaching. *Review of Educational Research, 40*, 707-721.

Godino, J., & Llinares, S. (2000). El interaccionismo simbólico en educación matemática. *Educación Matemática, 12*(1), 70-92.

Guerreiro, A. (2011). *Comunicação no ensino-aprendizagem da matemática: Práticas no 1.º ciclo do ensino básico*. (Doctoral Dissertation, University of Lisbon). Lisboa: APM.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education*, 35(2), 81-116.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education*, 6(3), 223-252.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the US*. Mahwah, NJ: Lawrence Erlbaum associates, Publishers.

Martinho, M. H., & Ponte, J. P. (2009). Communication in the classroom: Practice and reflection of a mathematics teacher. *Quaderni di Ricerca in Didattica*, Supplemento n.4 al n. 19, 35-43.

Mason, J. (1998). Asking mathematical questions mathematically. *Actes du Colloque DIDIREM, Réussites et/ou apprentissages Nouvelles technologies; Les mathématiques en premier cycle universitaire, où en est-on?*. Université de Versailles. Retrieved January 12nd, 2013 from http://www.math.jussieu.fr/~jarraud/colloque/mason .pdf.

Mason, J. (2000). Asking mathematical questions mathematically. *International Journal of Mathematical Education in Science and Technology*, 31(1), 97-111.

Mason, J. (2010). Effective questioning and responding in the mathematics classroom. Retrieved November 15th, 2012 from http://mcs.open.ac.uk/jhm3/Selected%20Publications/Effective%20Questioning%20&%20Responding.pdf.

Menezes, L. (1995). *Conceções e práticas de professores de matemática: Contributos para o estudo da pergunta* (Master's Dissertation, University of Lisbon). Lisboa: APM.

Menezes, L. (1996). A importância da pergunta do professor na aula de Matemática. In J. Ponte, C. Monteiro, M. Maia, L. Serrazina & C. Loureiro (Eds.), *Desenvolvimento profissional dos professores de Matemática: Que formação?* (pp. 105-116). Lisboa: SPCE.

Menezes, L. (2004). *Investigar para ensinar Matemática: Contributos de um projeto de investigação colaborativa para o desenvolvimento profissional de professores* (Doctoral Dissertation, University of Lisbon). Lisboa: APM.

Menezes, L., Canavarro, A. P., & Oliveira, H. (2012). Teacher practice in an inquiry-based mathematics classroom. *HMS i JME - International Journal for Mathematics in Education*, 4, 357-362.

ME – Ministry of Education. (2007). *Programa de Matemática do ensino básico.* Lisboa: ME.

NCTM – National Council of Teachers of Mathematics. (1991). *Professional standards for* teaching. Reston, VA: NCTM.

Nicol, C. (1999). Learning to teach mathematics: Questioning, listening, and responding. *Educational Studies in Mathematics, 37*(1), 45-66.

Pereira, A. (1991). *Comunicação e ensino das ciências: Contributo para o estudo da pergunta no discurso da aula de ciências do ensino básico* (Unpublished Master's Dissertation, University of Lisbon).

Ponte, J. P. (2005). Gestão curricular em Matemática. In GTI (Ed.), *O professor e o desenvolvimento curricular* (pp. 11-34). Lisboa: APM.

Ponte, J. P. (2012). Estudando o conhecimento e o desenvolvimento profissional do professor de matemática. In N. Planas (Ed.), *Educación matemática: Teoría, crítica y práctica* (pp. 83-98). Barcelona: Graó.

Ruthven, K., Hofmann, R., & Mercer, N. (2011). A dialogic approach to plenary problem synthesis. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 81-88). Ankara, Turkey: PME.

Searle, J. (1984). *Os actos de fala*. Coimbra, Portugal: Almedina.

Sierpinska, A. (1998). Three epistemologies, three views of classroom communication: Constructivism, sociocultural approaches, interactionism. In H. Steinbring, M. G. B. Bussi & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 30-62). Reston, VA: NCTM.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*, 313-340.

Tomás Ferreira, R. A. (2005). *Portuguese mathematics student teachers' evolving teaching modes: A modified teacher development experiment* (Unpublished Doctoral Dissertaion, Illinois State University, USA).

Tropea, F. (2007). O bit e a formiga. Especificidades e interferências entre o interpessoal e o mediático na comunicação. In J. Tornero (Ed.), *Comunicação e educação na sociedade da informação: Novas linguagens e consciência crítica* (pp. 85-106). Porto: Porto Editora.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactique des Mathématiques, 6*(1), 69-118.

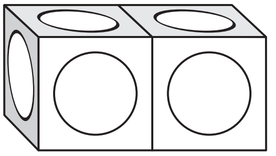
Wood, T. (1995). An emerging practice of teaching. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 203-227). Mahwah, NJ: Lawrence Erlbaum.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

APPENDIX

TASK OF THE 4TH GRADE LESSON

CUBES WITH STICKERS



Joana is building a game with cubes and stickers. She connects the cubes through one of their faces and forms a queue of cubes. Then she glues a sticker in each of the cube's faces. The figure shows the construction that Joana did with 2 cubes. in that construction she used 10 stickers.

1. Find out how many stickers Joana used in a construction with:
1.1 three cubes 1.2 four cubes 1.3 ten cubes 1.4 fifty two cubes

2. Can you find out what is the rule that allows you to know how many stickers Joana used in a construction with any given number of cubes? Explain how you thought.

TASK OF THE 5TH GRADE LESSON

THE RISE AND FALL OF FUEL PRICES

As you probably have noticed by now, fuel prices vary a lot, according to the price of the oil barrel. Petrolex Lda. pump stations have increased the fuel price by 10%, giving rise to a choir of protests by car drivers. As a reaction, the Director of Petrolex Lda. decided to lower ce by 10%. Did the fuel price return to its previous value? Justify your answer.

TASK OF THE 7TH GRADE LESSON

THE CLASS PRESIDENT'S ELECTION

The head teacher of the class coordinated the whole process for electing the class president. After the voting process, she told the class that:

1. all 30 students in the class have voted; no blank or null votes were cast;
2. only three students received votes: Francisca, Lucas and Sandra;
3. Lucas got two less votes than Francisca;
4. Sandra got twice as many votes as Lucas'.

Who won the election? With how many votes?
Do not forget to present and explain your reasoning.

PROFESSIONAL NOTICING: A COMPONENT OF THE MATHEMATICS TEACHER'S PROFESSIONAL PRACTICE

Salvador Llinares
sllinares@ua.es | Universidad de Alicante, Spain

ABSTRACT

In this paper, I characterize the notion of «teaching competency» as being able to use knowledge in a pertinent way to carry out mathematics teaching tasks. One aspect of teaching competency is the teacher's «professional noticing» of students' mathematical thinking. This feature of teacher competency involves the cognitive ability to identify and interpret the salient features of the students' output in order to make informed decisions. This construct is illustrated using a context in which the prospective teachers are learning to interpret the students' answers to linear and nonlinear problems (in order to recognize the students' development of proportional reasoning). In this context, when prospective teachers use increasingly more explicit mathematics elements and features of the development of mathematical understanding to describe and interpret the students' mathematical thinking is considered as evidence of the development of professional awareness.

KEY WORDS

Professional practice; Professional noticing; Teaching expertise; Mathematics teaching; Teachers' tasks in mathematics teaching.

SISYPHUS
JOURNAL OF EDUCATION
VOLUME 1, ISSUE 3,
2013, PP. 76-93

Professional Noticing: A Component of the Mathematics Teacher's Professional Practice[1]

Salvador Llinares

INTRODUCTION

The mathematics teacher's skill of identifying the relevant features in teaching situations and interpreting them from the learner's perspective in order to make decisions about what course the lesson should take is seen as an important component of the teaching practice (Mason, 2002; Sherin, Jacobs & Philipp, 2010). The teacher's knowledge of mathematics and the didactics of mathematics are central in this skill of mathematics teaching expertise. Indeed, the relationship between the different components of the knowledge mathematics for teaching has led some researchers to try to clarify them in order to understand the relation between knowledge and practice (Ball, Thames & Phelps, 2008). This is ultimately linked to what the teacher needs to know to solve professional problems (mathematics teaching and learning situations).

Here, we complement this perspective by identifying the mathematical knowledge that enhances the teacher's ability to «professionally notice» the students' mathematical thinking. We aim to reflect on the teacher professional practice and knowledge in order characterize the teacher's professional noticing examining the role played by mathematical knowledge in the profes-

1 This work has received support from the project I+D+i of the Plan Nacional de Investigación del Ministerio de Ciencia e Innovación, Spain (EDU2011-27288).

sional activities. We shall look at data from a teacher education program to gain a better understanding of this teaching skill in relation to how prospective teachers interpret the development of proportional reasoning. By examining these situations we can gain a better awareness of how to develop this skill in teacher education programs (Llinares, 2012).

THE MATHEMATICS TEACHER: KNOWLEDGE AND PROFESSIONAL PRACTICE

How to characterize what the mathematics teacher knows and how this is put into practice in the classroom is a topic in mathematics education (Ponte & Chapman, 2006). The teacher's knowledge and using this knowledge are dependent constructs. The mathematics teacher's professional knowledge is characterized by how to apply this knowledge in mathematics teaching contexts. This idea presupposes that the contexts in which one acquires knowledge and where one uses have a didactic relationship (Escudero & Sánchez, 2007a, 2007b).

Another element that goes into shaping the mathematics teacher's professional profile is the complementarity between the knowledge from research (knowledge that can be found in books and scientific journals) and the knowledge acquired through experience. In the long run, the professional's performance as a practitioner depends on the way teacher gathers, selects, integrates and interprets his/her experience. The practice of mathematics teacher's involves a number of professional tasks (Figure 1). One of these professional tasks is the adaptation of mathematical activities to support the students' learning (e.g., Gafanhoto & Canavarro, 2012; Morris, Hiebert & Spitzer, 2009); others aim to guide mathematical discussion in class (e.g., Fortuny & Rodríguez, 2012; Ponte, Quaresma & Branco, 2012); and yet others are to analyse student's mathematical thinking (e.g., Fernández, Llinares & Valls, 2013; Sánchez-Matamoros, Fernández, Valls, García & Llinares, 2012).

The identification of tasks which constitute the mathematics teacher's professional practice is relevant because it allows to relation the teacher's knowledge and «the use of knowledge in context.» In this sense, the focus on «the use of knowledge to resolve professional tasks» is relevant to better understand the practice and professional knowledge of mathematics teacher.

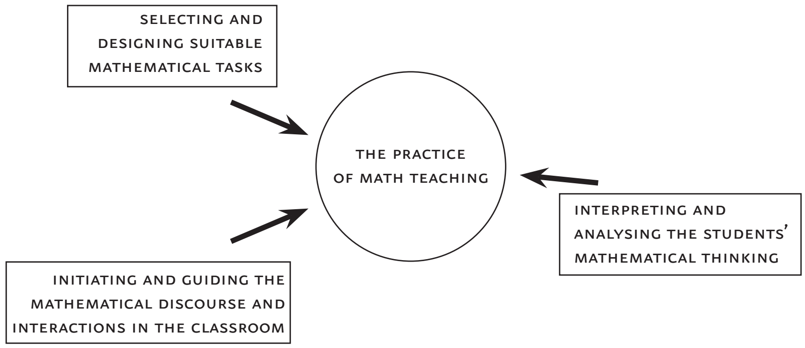


FIGURE I — SYSTEM OF ACTIVITY IN THE MATHEMATICS TEACHER PROFESSIONAL PRACTICE

«NOTICING» OF STUDENTS' MATHEMATICAL THINKING

The idea of «using knowledge to resolve professional tasks» is a core component of the teaching competency. This competence is knowing what, how and when to use specific knowledge to solve the mathematics teaching tasks. The skill «notice professionally» requires that the teacher be able to: identify relevant aspects of the teaching situation; use knowledge to interpret the events, and establish connections between specific aspects of teaching and learning situations and more general principles and ideas about teaching and learning (Jacobs, Lamb & Philipp, 2010; Mason, 2002; Sherin, Jacobs & Philipp, 2010). This way of understanding the construct of «noticing» considers that the teacher's identification of the mathematical elements which are relevant in the problem that the pupils have to solve and in the solution they might produce, allows the teacher to be in a better position to interpret their learning and to take relevant instructional decisions. Specifically, mathematics knowledge for teaching (Ball, Thames & Phelps, 2008; Hill *et al.*, 2008) allows the teacher to identify what is relevant and support her interpretation of these facts and evidences deemed relevant. In this sense, the role played by the teacher's mathematics knowledge for teaching in the resolution of professional tasks defines some aspects of his/her teaching competency (An & Wu, 2012; Sánchez-Matamoros *et al.*, 2013; Zapatera & Callejo, 2013).

One particular aspect of teacher notice is the ability to be attuned to the students' mathematical thinking. Being able to understand and analyse the students' mathematical reasoning involves the «reconstruction and inference» of the students' understanding from what the student writes, says or does. The teacher's skill of noticing the students' mathematical thinking demands more than just pointing out what is correct or incorrect about their answers. It requires determining in what way the students' answers are or are not meaningful from the mathematics learning standpoint (Hines & McMahon, 2005; Holt, Mojica & Confrey, 2013). In the following examples we illustrate some features of the skill of «professionally noticing» or being aware of the students' mathematical thinking. We will exemplify these features in the context of students' proportional reasoning.

NOTICING THE DEVELOPMENT OF PROPORTIONAL REASONING

During a teacher training course in which the prospective teachers were to develop their ability to «professionally notice» the pupils' mathematical output, the future teachers had to:

i) describe some pupils' solutions to proportional and non-proportional problems, and then;
ii) interpret the pupils' mathematical understanding from the evidence supplied in their answers (i.e., the way the pupils dealt with the problems reflected their mathematical understanding).

One of the topics in the course was proportional reasoning (Fernández & Llinares, 2012). To describe pupils' responses and interpret their mathematical understanding future teachers must be able to identify the mathematical elements of problems that foster proportional reasoning by interpreting the multiplicative relationship between quantities. In other words, the future teacher must «break down» the mathematics that define the problem and recognize the manner in which the mathematical elements that characterize the problem are present or not in the pupil's answer. In the development of proportional reasoning as a component of multiplicative structures, these mathematical elements are (Lamon, 2007; Vergnaud, 1983):

- The difference between linear and non-linear situations

- The scalar ratio (relationships between corresponding elements [$a/b=f(a)/f(b)$; within – internal – ratios, or comparisons within measure space)
- The constancy of the functional ratio ($a / f(a)=k$; between – external – ratios or comparisons between measure spaces)
- The constructive nature of the multiplicative relationship between two magnitudes: $f(ka+pb)=k.f(a)+p.f(b)$

The identification of the relevant mathematical elements in a problem and the interpretation of how they are present in the students' answers allow future teachers to be in better conditions to make relevant instructional decisions and help students develop their proportional reasoning. In this sense, the knowledge of mathematics (Hill *et al.*, 2008) allows that the teacher identify and interpret how the students use the mathematical elements when solving proportional problems. For example, recognition of the mathematical elements in the answer given by pupil 1 (figure 2) allows the future teacher to determine whether or not the procedure used is suitable. Moreover, identifying the mathematical elements which give sense to this procedure allows teacher to justify the way this procedure can be generalized (i.e., it is independent of the numbers used).

«If I multiply the number of metres covered by Sofia by 3, then this also corresponds to the triple of the metres covered by Sara (hence the multiplications 20×3 and 50×3).»

«If 20 corresponds to 50 metres, then half (10) corresponds to half (25, which is half of 50).»

«The total of two quantities of metres covered by Sofia corresponds to the total of the «respective quantities» of metres covered by Sara (therefore, 60 plus10 corresponds to 150 plus25).»

These three points in the process used by the pupil show his recognition of:

$f(20)=50$, so $f(3 . 20)=3 . f(20)=3 . 50$

$f(60 + 10)= f(60) + f(10)$,

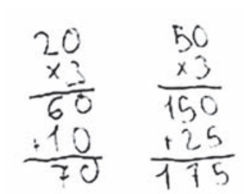
which is the breakdown of the mathematical elements in the problem that define the linear situations

$f(a+b)=f(a)+f(b)$

$f(k.a)=k.f(a)$

On the other hand, the knowledge of mathematics and students, as another component in the mathematics knowledge or teaching (MKT, Hill *et al.*, 2008), it is necessary to recognize that using linear in non-linear situations (which happens in pupil 2's answer (Figure 2) is an erroneous approach fairly common (De Bock *et al.*, 2007; Fernández & Llinares, 2012).

PUPIL 1	PUPIL 2
Sofia and Sara are walking through a field. They began at the same time but Sara is faster. When Sofia has walked 20 metres, Sara has walked 50 metres. When Sofia has walked 70 metres, how many metres will Sara have walked?	Juan and Carolina are driving a car around a track. They are driving at the same speed but Juan started later. By the time Juan completed 20 laps, Carolina had completed 60. When Juan has completed 100 laps, how many laps will Carolina have completed?



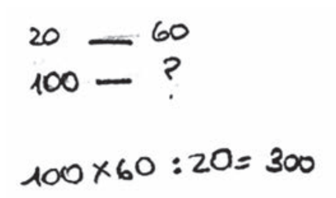


FIGURE 2 – SOME OF THE ANSWERS GIVEN TO FUTURE TEACHERS TO INTERPRET THE STUDENTS' MATHEMATICAL LEARNING

Examining the student's answers in fig. 2, a future teacher gave the following explanation:

> [in relation to pupil 1]. The student realized that in the first time-period, Sofia had covered 20 metres and in triple that time she had covered 60 metres. If he added half of what was covered in one time-period to those 60 metres (20=> 10) he would obtain the total metres we are told Sofia covered. Therefore, you would infer that the distance was covered in 3 and a half time-periods and you would have calculated the distance covered by Sara in that same time frame. (Emphasis added).
>
> [in relation to pupil 2]. In this problem the pupil thinks that the relationship of laps Juan and Carolina do, with respect to one another, is proportional. He does not realize that one started later than the other and that they are going at the same speed; consequently, they <u>have to</u> complete the same number of laps in a certain time, once they have begun (Emphasis added).

This future teacher's discussion shows that he deems relevant the relationship between the operations carried out by the students and the different relationships between the quantities. Describing pupil 1's answer, he mentions how the student has identified the multiplicative relationship between the quantities and has translated this relationship into the operations he carries out («in the first time-period Sofia had covered 20 metres and *in triple that time* she had covered 60 metres [triple the distance]»). Moreover, the fact that he has pinpointed the multiplicative relationship between the quantities of the two magnitudes (the distance covered by both Sofia and Sara) can be seen when he justifies the students' operations in the sense that adding 10 metres covered by Sara corresponds to adding 25 metres covered by Sofia. This manner of describing pupil 1's answer shows that the future teacher was able to recognize the way in which the mathematical elements of the proportional and non-proportional situations were present in the pupils' answers. The identification of these mathematical elements is the first step in the teacher's ability to correctly interpret the students' level of proportional reasoning.

When the future teacher then describes pupil 2's answer, he recognizes the discrepancy between the relation between the quantities in the problem and the operations the student is carrying out. He notes that pupil 2 is carrying out operations that do not suit the structure of the problem (the relationship between quantities). In other words, this future teacher can differentiate between the proportional situation (the problem of Sofia and Sara) and the non-proportional situation (the situation with Juan and Carolina), and he can recognize when they are - or are not – being picked up by the students.

This way of «noticing» the students' answers allows the future teacher to glean evidence from the students' answers and interpret them in light of the mathematical understanding they reflect. In this case, pupil 1's answers reflect his knowledge of proportional relations $f(k.x)=k.f(x)$ and $f(a+b) = f(a) + f(b)$. Whilst at the same time, the teacher is aware that student 2 does not recognize the additive relationships between quantities and therefore does not discriminate between proportional and non-proportional situations (saying «and he does not realize that one has started later than the other and that they are going at the same speed») making the student apply inappropriate proportional procedures. These examples illustrate how the future teacher can «professionally notice» or be attuned to the students' answers, which in turn allow the teacher to interpret student learning styles. We can see why being attuned/aware is such an important component of math teaching. They are examples of how the knowledge is being used to successfully carry out a professional task.

In this sense, *identify* relevant aspects of the students' output in order to *interpret* their mathematical understanding are teacher's cognitive activities that set professional noticing as a component of his/her teaching competence. That is to say, identifying and interpreting are cognitive actions the future teacher has to undertake in which he/she is using his/her mathematics knowledge for teaching. In this example, the prospective teacher is demonstrating that he/she is aware of the differences between proportional and non-proportional situations and how these differences impact the pupils' solutions; he/she is also aware of the student's misuse of linearity in non-linear situations.

In short, professional noticing or being attuned is a component of the mathematics teacher's professional practice and can be characterized by the teacher's

- possessing mathematical knowledge that facilitates identifying what is relevant from the perspective of learning mathematics in a teaching context, and
- using it to interpret the evidence according to the goals desired.

In other words, to become attuned or aware, the teacher not only needs to have an interpretive viewpoint toward math teaching and learning, but theoretical knowledge as well. Having the theoretical background that allows one to interpret or «professionally notice» is what justifies the use of the word »professional.» In this context, the teacher must assess to what extent her knowledge is relevant to the professional task at hand. In order for knowledge to become «relevant» to a professional task, the teacher must be aware of how his/her own knowledge dovetails with the task to be carried out (Mason, 2002).

We use the term «professionally» aware because this skill may not be innate in the math teacher. Research into the development of this awareness in teachers has shown how complex it is (An & Wu, 2012; Fernández, Llinares & Valls, 2011; Fernández, Llinares & Valls, 2013; Prediger, 2010; Prieto & Valls, 2010; Sánchez-Matamoros *et al.*, 2012; Spitzer, Phelps, Beyers, Johnson & Sieminski, 2010). For example, faced with the same task as described earlier, (Figure 2) another future teacher remarked:

> [In relation to pupil 1]. The student tried to solve the problem without using proportions. From 20 he tries to arrive at 70 using multiplications and addition.

He knew he had to go from 20 to 70. He did this by multiplying the first number by 3 and adding 10. Applying the same operation to 50 he arrived at 175.
[In relation to pupil 2]. He applied the proportions method, although he did not write 20:100=60: x, but what he did do was directly write down the formula $100 \times 60/20$.

This future teacher describes the operations that appear in the student's answer but he is not capable of giving them meaning in relation to the two structures of the situations (one situation is equivalent to the type «SOFIA'S metres=20/50 x SARA'S metres» and the other is equivalent to the type: «Carolina's laps=Juan's laps + 40»).

With regard to problem 1, this future teacher seems to recognize that the student is translating the operations (multiply by three and add half) between the two magnitudes. In describing student 1's solution, we can see that he recognizes the correct relationship between the operations carried out and the relationship between the quantities. However, when discussing problem 2, this future teacher only describes the operations carried out by the pupil, without establishing any relationship with the structure of the quantities involved. His description, which centres on the operations, not the relationship between the quantities, reveals that the future teacher is not capable of recognising that problem 2 is a situation with an additive structure. Therefore, what he calls «the proportions method» is not applicable. The future teacher describes the solving of the problem in terms of the operations carried out, but he does not relate these operations to the structure of the problem. He therefore does not recognize that the solution was incorrect for this type of problem. In this case the future teacher was not able to «break down» the relevant mathematical elements in this situation with the aim of targeting what was relevant in order to pinpoint the student's difficulties.

Answers of this type have been obtained from teacher education programs that are designed to enhance the teachers' ability to «professionally notice» or become attuned. Yet, at the same time, they reveal how difficult it is for some future teachers at this early stage to go further than just describing the operations used to solve a problem. It is indeed hard to make accurate inferences about the pupils' mathematical thinking by offering more than just a superficial description.

In teacher education programs, prospective teachers are usually able to offer different interpretations of the pupils' answers and manage to pinpoint the mathematical elements used in their answers. On the other hand, interpreting the mathematical reasoning in the students' answers is far more complex and future teachers should use as a reference from the mathematics education research. For example, as a reference, future teachers could use the information collated on the levels of development of proportional reasoning (figure 3).

Level 0. Non-proportional reasoning	Level I - Illogical	· Incapable of recognizing multiplicative relations. · Uses numbers and does procedures without sense · Applies proportional strategies to non-proportional situations
	Level A – Addition	· Uses addition relations between numbers. Applies addition strategies in a systematic way (in proportional and non-proportional problems)
Level 1. Informal reasoning in proportional situations		· Uses drawings or manipulatives to give sense to situations · Carries out qualitative comparisons
Level 2. Quantitative reasoning		· Uses constructive strategies · Identifies and uses functional ratios when the ratios are whole numbers
	Level T – Transition	· Begins to constructively use multiplicative relations between the quantities
Level 3. Proportional reasoning	Level R - Reasoning	· Identifies and uses functional ratios when the ratios are NOT whole numbers · Identifies and uses scalar ratios when the reasons are NOT whole numbers · Understands the constancy of scalar ratios · Understands that functional ratio is constant

FIGURE 3 – LEVELS OF DEVELOPMENT OF PROPORTIONAL REASONING WHICH CAN BE USED AS A REFERENCE TO INTERPRET PUPILS' MATHEMATICAL UNDERSTANDING, AS GLEANED FROM THEIR ANSWERS TO PROBLEMS.

The examples presented in this section illustrate that the teacher needs to have mathematical knowledge of the topic (domain-specific mathematical knowledge) and knowledge of mathematics and student in order to identify and interpret the students' mathematical thinking. Thus, one can see how important it is in this context to recognize, for example, the characteristics of proportional and non-proportional situations, the role of different contexts and the relationship between numbers when considering whole and non-whole number ratios.

KNOWLEDGE OF MATHEMATICS AND PROFESSIONAL NOTICING

Professional noticing as a component of teacher teacher's professional competence allows to mathematics teacher «notice» the mathematics teaching situations differently from another person that not is mathematics teacher. Although in recent years this skill has been conceptualized from different perspectives, the common approach involves highlighting the way in which teachers interpret mathematics teaching situations.

Mason (2002), in discussing this particular teaching skill, says that the teacher should be aware of how he/she interprets teaching and learning situations, by taking a structured view of what is relevant to his/her students' learning objectives. According to Mason (2002), one way of noticing in a «structured manner» is to be aware of *how* you are «noticing». The more explicitly future teachers use the mathematical elements of the situation to analyse teaching and learning situations, the more actively they are using specialized knowledge of mathematics (Llinares & Valls, 2009, 2010; Sánchez-Matamoros, Fernández, Llinares & Valls, 2013; Zapatera & Callejo, 2013). The difference in the level of explicitness with which future teachers use relevant mathematical elements to analyse the pupils' work determines to what extent they can develop this skill. Some of the different levels of development have been discussed in previous examples.

However, research results (Fernández *et al.*, 2011; Sánchez-Matamoros *et al.*, 2012) indicate that although future teachers may have adequate background preparation in mathematics, some find it hard to describe the students' solutions using relevant mathematical elements and identifying the features of the students' mathematical understanding. This demonstrates how important it is for future teachers to develop an explicit «awareness» of the mathematical elements involved in solving problems and their role in determining the students' reasoning.

In the previous examples the mathematical elements regarding the proportional situations would be $f(k.x)=k.f(x)$, $f(a+b)=f(a)+f(b)$ and in the non-proportional, $f(x)=ax+b$, with $b \neq 0$ and $a=1$. Another mathematical element to bear in mind is the type of scalar or functional ratio between the quantities, as well as the numerical relationship between the scalar ratios (relationships between quantities of the same magnitude, 20/70 in problem 1 or the ratio 20/100 in problem 2) and the functional ratios (relationships between quantities of a different magnitude, 20/50 in problem 1 or the ratio 20/60 in problem 2), which can be whole numbers or not and therefore facilitate the pupil's recognition of multiplicative and additive relationships. The future teacher needs to realize that the type of relationship between the quantities (whole numbers or not) introduces different levels of difficulty for students and therefore influences the development of proportional reasoning.

This is an example of the mathematical knowledge for teaching the teacher should use to professionally notice teaching and learning relating to ratio and proportion. It is an example of «knowledge in use» and a characteristic of the teacher's skill at becoming attuned to the students' mathematical thinking. It shows the meaningful use of mathematical knowledge, especially about the different meanings of mathematical objects. Being able to analyse the students' mathematical thinking allows the teacher to build his/her mathematical knowledge for teaching (MKT). Thus, issues of mathematical knowledge that teachers need to teach are linked to the knowledge of mathematics that teachers need to understand the students' mathematical thinking.

THE DEVELOPMENT OF TEACHERS' PROFESSIONAL NOTICING

Some research supports the hypothesis that the teacher's ability to «professional noticing» can be developed (Holt *et al.*, 2013; Llinares, 2012; Schack, Fisher, Thomas, Eisenhardt, Tassel & Yoder, 2013). On the other hand, the learning trajectories of currently practicing and future teachers is now being conceived as a process of enculturation that involves consider the nature and extent of the teacher's professional knowledge and how the teacher uses the knowledge in teaching practice. The challenge for teacher education programs is to coordinate the integrate nature of the knowledge (for example the relationship between the knowledge of mathematics and knowledge of the students' learning) (Hill *et al.*, 2008) and how teacher identify and interpret relevant elements of mathematics teaching (Fernández, Llinares & Valls, 2012; Penalva, Rey & Llinares, 2013; Roig, Llinares & Penalva, 2011).

However, our experience with pre-service teacher education has shown that analysing the students' work in order to infer levels of mathematical understanding is a difficult task. Currently, research centred on the development of this skill in initial teacher education has enabled us to begin to provide information about as we can characterize this development (Fernández *et al.*, 2013; Holt *et al.*, 2013; Schack *et al.*, 2013). Figure 4 shows some of the results obtained with regard to developing the teacher's ability to «professionally notice» when it comes to the subject of proportionality.

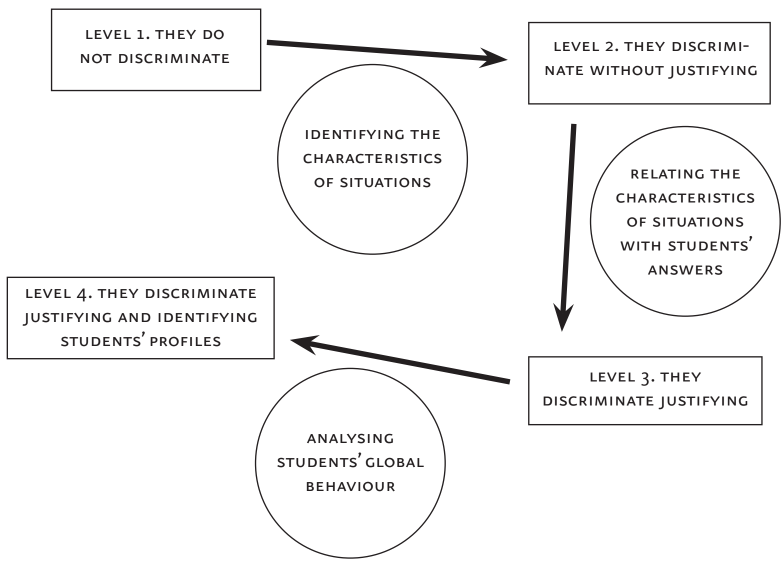


FIGURE 4 – LEVELS OF DEVELOPMENT OF THE TEACHER'S «PROFESSIONAL NOTICE» THE STUDENTS' MATHEMATICAL THINKING IN THE CONTEXT OF PROPORTIONALITY (FERNÁNDEZ, LLINARES & VALLS, 2013, P. 459).

Since it is difficult to develop this skill during teacher training, teacher educators have had to create opportunities – learning environments – for future and practising teachers to acquire new knowledge and skills and enhance their ability to learn through teaching (Llinares, 2012).

CONCLUSION

The emphasis placed recently on a teacher's professional notice or become aware of her students' thought processes is posited on the belief that this skill has a relevant impact on the teaching of mathematics. Some research has proven that when teachers bring enhanced awareness to their teaching, actual teaching practice was enhanced. But what it means to be aware or to «notice professionally» has to be clear to the prospective teachers and also new teaching methods should be introduced in teacher education programmes to improve this skill.

This work has discussed the role played by mathematics knowledge in articulating cognitive actions such as identifying and interpreting the manner in which the student is solving a problem. Hopefully, in the extent in which we as teacher educators link mathematical knowledge with the findings from the findings of mathematical education research on how students learn in different domains, we will be better positioned in order to help prospective teachers to develop this skill.

Research on the development of this teaching skill has allowed us to identify certain characteristics that, in turn, have enabled us to begin to define learning trajectories to describe how this skill evolves (Fernández *et al.*, 2013; Sánchez *et al.*, 2013). However, additional research is needed on the factors that constrain and/or promote this development, while better theoretical models must be developed that enable us to understand it.

REFERENCES

An, S., & Wu, A. (2012). Enhancing mathematics teachers' knowledge of students' thinking from assessing and analyzing misconceptions in homework. *International Journal of Science and Mathematics Education, 10*, 717-753.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-407.

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2007). *The Illusion of Linearity. From Analysis to Improvement*. New York: Springer.

Escudero, I., & Sánchez, V. (2007a). A mathematics teachers' perspective and its relationship to practice. *International Journal of Science and Mathematics Education, 6*, 87-106.

Escudero, I., & Sánchez, V. (2007b). How do domains of knowledge integrate into mathematics teachers' practice? *Journal of Mathematical Behavior, 26*(4), 312-327.

Fernández, C., & Llinares, S. (2012). Características del desarrollo del razonamiento proporcional en la educación primaria y secundaria. *Enseñanza de las Ciencias, 30*(1), 129-142.

Fernández, C., Llinares, S., & Valls, J. (2011). «Mirando con sentido» el pensamiento matemático de los estudiantes sobre la razón y proporción. *Acta Scientiae, 13*(1), 9-30.

Fernández, C., Llinares, S., & Valls, J. (2012). Learning to notice students' mathematical thinking through online discussions. *ZDM – Mathematics Education, 44*, 747-759.

Fernández, C., Llinares, S., & Valls, J. (2013). Primary teacher's professional noticing of students' mathematical thinking. *The Mathematics Enthusiast, 10*(1&2), 441-468.

Fortuny, J. M., & Rodríguez, R. (2012). Aprender a mirar con sentido: facilitar la interpretación de las interacciones en el aula. *AIEM – Avances de Investigación en Educación Matemática, 1*, 23-37.

Gafanhoto, A. P., & Canavarro, A. P. (2012). A adaptação das tarefas matemáticas: Como promover o uso de múltiplas representações. Comunication presented at the Seminar *«Práticas Profissionais dos Professores de Matemática»*. Lisboa, Portugal.

Hines, E., & McMahon, M. (2005). Interpreting Middle School Students' Proportional Reasoning Strategies: Observations From Preservice Teachers. *School Science and Mathematics, 105*(2), 88-105.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*, 430-511.

Holt, P., Mojica, G., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *Journal of Mathematical Behavior, 32*, 103-121.

Jacobs, V. R., Lamb, L. C., & Philipp, R. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.

Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-668). Charlotte, NC: Information Age.

Llinares, S. (2012). Construcción de conocimiento y desarrollo de una mirada profesional para la práctica de enseñanza matemáticas en entornos en línea. *AIEM Avances de Investigación en educación Matemática, 2*, 53-70.

Llinares, S., & Valls, J. (2009). The building of pre-service primary teachers' knowledge of mathematics teaching: interaction and online video case studies. *Instructional Science, 37*(2), 247-271.

Llinares, S., & Valls, J. (2010). Prospective primary mathematics teachers' learning from on-line discussions in a virtual video-based environment. *Journal of Mathematics Teacher Education, 13*(2), 177-196.

Mason, J. (2002). *Researching your own practice: The discipline of noticing.* London: Routledge Falmer.

Morris, A. K., Hiebert, J., & Spitzer, S. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education, 40*(5), 491-529.

Penalva, M. C., Rey, C., & Llinares, S. (2013). Aprendiendo a interpretar el aprendizaje de las matemáticas en educación primaria. Características en un contexto b-learning. *Educación Matemática, 25*(1), 7-34.

Ponte, J. P., & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 461-494). Rotterdam/Taipei: Sense Publishers.

Ponte, J. P., Quaresma, M., & Branco, N. (2012). Práticas profissionais dos professores de Matemática. *AIEM – Avances en Investigación en Educación Matemática, 1*, 65-86.

Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: The case of meanings of the equal sign. *Journal of Mathematics Teacher Education, 13*, 73-93.

Prieto, J. L., & Valls, J. (2010). Aprendizaje de las características de los problemas aritméticos elementales de estructura aditiva en estudiantes para maestro. *Educación Matemática, 22*(1), 57-85.

Roig, A. I., Llinares, S., & Penalva, M. C. (2011). Estructuras argumentativas de estudiantes para profesores en un entorno en línea. *Educación Matemática, 23*(3), 39-65.

Sánchez-Matamoros, G., Fernández, C., & Llinares, S. (2013). Developing pre-service Teachers' noticing students' understanding of the derivative concept. *International Journal of Science and Mathematics Education.*

Sánchez-Matamoros, G., Fernández, C., Valls, J., García, M., & Llinares, S. (2012). Cómo estudiantes para profesor interpretan el pensamiento matemático de los estudiantes de Bachillerato: La derivada de una función en un punto. In A. Estepa, Á. Contreras, J. Deulofeu, M. C. Penalva, F. J.

García & L. Ordóñez (Eds.), *Investigación en Educación Matemática XVI* (pp. 497-508). Jaén: SEIEM.

Schack, E., Fisher, M., Thomas, J., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective elementary school teachers' professional noticing of children's early numeracy. *Journal of Mathematics Teacher Education, 16*, 379-397.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.) (2010). *Mathematics teacher noticing: Seeing through teachers' eyes*. New York, NY: Routledge.

Spitzer, S., Phelps, C., Beyers, J., Johnson, D. Y., & Sieminski, E. (2010). Developing prospective elementary teachers' abilities of identify evidence of student mathematical achievement. *Journal of Mathematics Teacher Education, 14*, 67-87.

Vergnaud, G. (1983). Mujltiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). New York, NY: Academic Press.

Zapatera, A., & Callejo, M. L. (2013). Cómo interpretan los estudiantes para maestro el pensamiento matemático de los alumnos sobre el proceso de generalización. In A. Berciano, G. Gutiérrez, A. Estepa & N. Climent (Eds.), *Investigación en Educación Matemática XVII* (pp. 535-544). Bilbao: SEIEM.

facilitating reflection and action: the possible contribution of video to mathematics teacher education

David Clarke
d.clarke@unimelb.edu.au | University of Melbourne, Australia

Hilary Hollingsworth
hilary.hollingsworth@acer.edu.au | Australian Council for Educational Research, Australia

Radhika Gorur
radhika.gorur@vu.edu.au I Victoria University, Australia

abstract

In the Interconnected Model of Teacher Professional Growth (Clarke & Hollingsworth, 2002), change in teacher beliefs, knowledge and practice is mediated by either enaction or reflection. The stimulus for change can be provided by an external source such as a professional development program or it can result from the teacher's inevitable classroom experimentation and reflection on the consequences of that experimentation. This paper explores the role that video can play in catalysing change and facilitating teacher reflection. In particular, we examine: (i) international research employing video and the capacity of such research to inform practice and stimulate teacher reflection in both pre-service and in-service settings; (ii) the use of video in professional development programs and the choice between exemplary and problematic practice as catalysts for teacher reflection in both pre-service and in-service programs; and (iii) teacher agency and the catalytic role of video in supporting teachers' reflection on their own practice, through the use of video as the communicative medium to sustain a professional community of reflective practitioners.

key words

Mathematics Teacher education; Reflection; Action; Video.

SISYPHUS

journal of education

volume 1, issue 3,

2013, pp. 94-121

Facilitating Reflection and Action: The Possible Contribution of Video to Mathematics Teacher Education

David Clarke | Hilary Hollingsworth | Radhika Gorur

INTRODUCTION

Ideas of «teacher change» are open to multiple interpretations, and each interpretation can be associated with a particular perspective on teacher professional development. Clarke and Hollingsworth (1994) described six perspectives of teacher change:

- Change as training – change is something that is done to teachers; that is, teachers are «changed»
- Change as adaptation – teachers «change» in response to something; they adapt their practices to changed conditions
- Change as personal development – teachers «seek to change» in an attempt to improve their performance or develop additional skills or strategies
- Change as local reform – teachers «change something» for reasons of personal growth
- Change as systemic restructuring – teachers enact the «change policies» of the system
- Change as growth or learning – teachers «change inevitably through professional activity»; teachers are themselves learners who work in a learning community

It should be noted that these alternative perspectives on change are not mutually exclusive, and that many are in fact interrelated. Recent decades have witnessed a shift in conceptions of teacher change from professional development programs designed to «change teachers» to programs designed to facilitate teacher professional learning (Clarke & Hollingsworth, 1994, 2002; Fullan & Stiegelbauer 1991; Guskey, 1986; Hall & Loucks, 1977; Johnson, 1996). The key shift is one of agency: from programs that change teachers to teachers as active learners shaping their professional growth through reflective participation in professional development programs and in practice. Recognition of the need to contextualize teaching and teacher development has led to the advocacy of approaches to professional development that employ cases, including video cases (Clarke & Hollingsworth, 2000), as a means to situate the professional development of teachers in realistic contexts. This contextualization of teaching was also advocated in proposals for the «authentic» assessment of teaching (Darling-Hammond & Snyder, 2000).

Fundamental to 'new' perspectives on teacher change and teacher professional development that have learning as their core are views of teachers as learners and schools as learning communities. In this paper, we examine video as a medium for facilitating both teacher reflection and teacher action and thereby as a key tool for the promotion of teacher learning and teacher professional growth. In particular, we examine: (i) international research employing video and the capacity of such research to inform practice in both pre-service and in-service settings; (ii) the use of video in professional development programs and the choice between exemplary and problematic practice as catalysts for teacher reflection in both pre-service and in-service programs; and (iii) teacher agency and the catalytic role of video in supporting teachers' reflection on their own practice, through the use of video as the communicative medium to sustain a professional community of reflective practitioners. Specific research projects provide the examples of each of the three roles.

THE INTERCONNECTED MODEL OF TEACHER PROFESSIONAL GROWTH

Professional growth is an inevitable and continuing process of learning. By acknowledging professional growth as a form of learning, we become inheritors of a substantial body of learning theory and research. The application of contemporary learning theory to the development of programs to support teacher professional growth has been ironically infrequent. In particular, models of teacher professional development have not matched the complexity of the process we seek to promote. Clarke and Hollingsworth (2002) outlined an empirically grounded model of professional growth that incorporated key features of contemporary learning theory (Figure 1).

The Interconnected Model (as shown in Figure 1) suggests that change occurs through the mediating processes of reflection and enactment in four distinct domains which encompass the teacher's world: the Personal Domain (Teacher Knowledge, Beliefs and Attitudes), the Domain of Practice (Classroom Experimentation), the Domain of Consequence (Salient Outcomes), and the External Domain (Sources of Information, Stimulus or Support). The four domains are analogous (but not identical) to the four domains identified by Guskey (1986). The mediating processes of reflection and enaction are represented in the model as arrows linking the domains. This model recognizes the complexity of professional growth through the identification of multiple growth pathways between the domains. Its non-linear nature, and the fact that it recognizes professional growth as an inevitable and continuing process of learning, distinguishes this model from others identified in the research literature. This model also identifies the mediating processes of reflection and enactment as the mechanisms by which change in one domain leads to change in another. Any processes of Professional Growth represented in the model occur within the constraints and affordances of the enveloping Change Environment (Hollingsworth, 1999).

The model locates «change» in any of the four domains. The type of change will reflect the specific domain. For example, experimentation with a new teaching strategy would reside in the Domain of Practice, new knowledge or a new belief would reside in the Personal Domain, and a changed perception of salient outcomes related to classroom practice would reside in the Domain of Consequence. Change in one domain is translated into change in another through the mediating processes of «reflection» and «enaction». The term «enaction» was chosen to distinguish the translation of a belief or a pedagogical model into action from simply «acting», on the grounds that acting occurs in the Domain of Practice, and each action represents the enactment of something a teacher knows, believes or has experienced. The empirical basis of the model has been outlined in some detail in Clarke and Hollingsworth (2002).

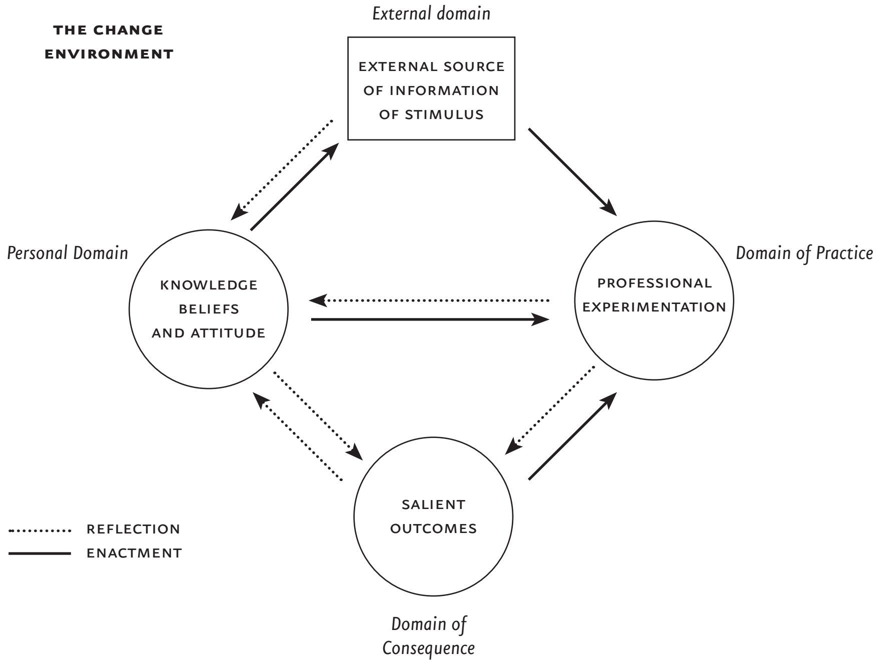


FIGURE I – THE INTERCONNECTED MODEL OF TEACHER PROFESSIONAL GROWTH
(CLARKE & HOLLINGSWORTH, 2002, P. 951)

One consistent challenge for theorists has been how to account for the demonstrable diversity of individuals' knowings within the evident commonalities of action associated with participation in a common social setting. Various theoretical positions have been constructed from which to resolve this tension. A focus on learning as a form of incrementally increasing, but differentiated, participation in an existing body of social practice has provided one useful lens (Lave & Wenger, 1991). This identification of learning with social practice is an important advance from notions of learning as simply occurring in social settings. Specifically, «learning is an integral part of generative social practice in the lived-in world» (Lave & Wenger, 1991, p. 35). The social 'situatedness' of learning can then enter the equation through consideration of the extent to which features of the social setting constrain or afford particular practices associated with learning and thereby constrain or afford the learning itself (Greeno, Collins & Resnick, 1996), delineating socially enacted tolerances within which individual idiosyncrasy can develop.

This is the description of learning that we find in closest accord with the Interconnected Model. Such a description gives, in our opinion, due recognition to situated practice and to the development of individual practice and individual theories of practice within an environment that both constrains and affords such individual variation. The two mediating processes, enaction and reflection, usefully connect to practice and to cognition and identify both activities as mediators of change.

The Interconnected Model of Teacher Professional Growth takes teacher change to be a learning process and suggests the possible mechanisms by which this learning might occur. The non-linear structure of the model provides recognition of the situated and personal nature, not just of teacher practice, but of teacher growth: an individual amalgam of practice, meanings, and context. Our support for the process of teacher growth must offer teachers every opportunity to learn in the fashion that each teacher finds most useful. If our professional development programs are to recognize the individuality of every teacher's learning and practice, then we must employ a model of teacher growth that does not constrain teacher learning by characterizing it in a prescriptive, linear fashion, but anticipates the possibility of multiple change sequences and a variety of possible teacher growth networks. Professional development programs that prioritise teacher agency are needed. Such programs require tools that inform teacher action and facilitate teacher reflection on that action. We suggest that video is such a tool.

VIDEO-BASED INTERNATIONAL CROSS-CULTURAL RESEARCH

Of all data sources currently available to researchers in education, video data seems most amenable to multiple analyses. The richness and complexity of video records of social interactions provide opportunities for reinterpretation, recoding, and for re-presentation of what is captured in the video records of social settings. Increasingly, research designs are anticipating multiple analyses of the complex data sets generated from educational settings (Clarke, Mitchell & Bowman, 2009; Clarke *et al.*, 2012). Research studies with which we have been involved have collected and configured data in anticipation of the use of such multiple analyses to realise the potential of classroom video data. We suggest that it is through multiple analyses of the same educational settings that research can come closest to matching in its findings the complexity of the situations and practices in those settings.

The *Learner's Perspective Study* (LPS) (Clarke, Keitel & Shimizu, 2006), for example, is predicated on the principle that the complexity of educational settings such as mathematics classrooms can only be studied through research approaches that match that complexity with (i) adequate recognition of the perspectives of all participants and specific embodiment in the data generation of those perspectives, (ii) deliberate utilisation of multiple analyses to provide a wide range of theoretical perspectives on the social setting and situations being studied, (iii) an acceptance from the outset of the obligation to anticipate and enact the synthesis of the multiple analyses into an integrative amalgam of interrelated complementary accounts (Clarke, 2006), and (iv) the development of «practical explanatory theory» that would provide «knowledge about the ways in which classroom activities, including teaching, affect the changes taking place in the minds of students: what students know and believe and what they can do with their knowledge» (Nuthall, 2004, p. 295).

The challenge confronting classroom researchers has always been to make confident connection between classroom activities and learning outcomes in order to optimize classroom learning environments and promote learning. We believe that serious research addressing this issue cannot be restricted to a single analytical frame, but must take a programmatic approach, where a well-equipped research team, combining a range of methodological and theoretical expertise, undertakes careful parallel analyses of high-quality, complex data. Advances in technology and particularly the growing sophistication in the research use of video bring us ever closer to the realisation of this vision.

The example of LPS illustrates one way in which video-based research can generate findings that catalyse teacher professional learning. The complete LPS research design is set out elsewhere (Clarke, 2006). For the analysis reported here, the essential details relate to the standardization of transcription and translation procedures. Three video records were generated for each lesson (teacher camera, focus student camera, and whole class camera), and it was possible to transcribe three different types of oral interactions: (i) whole class interactions, involving utterances for which the audience was all or most of the class, including the teacher; (ii) teacher-student interactions, involving utterances exchanged between the teacher and any student or student group, not intended to be audible to the whole class; and (iii) student-student interactions, involving utterances between students, not intended to be audible to the whole class or to the teacher. All three types of oral interactions were transcribed, although type (iii) interactions could only be documented for two selected focus students in each lesson. We distinguish *private* student-student interactions from whole class or teacher-student interactions, both of which we consider to be *public* from the point of view of the student.

Where necessary, all transcripts were translated into English. Transcription and translation were carried out by the local team responsible for data generation and were therefore undertaken by native speakers of the local language. The analyses reported here were undertaken on the English version of each transcript of public classroom dialogue. Analyses were conducted of 110 lessons documented in 22 classrooms located in Australia (Melbourne), China (Hong Kong and Shanghai), Germany (Berlin), Japan (Tokyo), Korea (Seoul), Singapore, and the USA (San Diego) (see Clarke, Xu & Wan, 2013a, 2013b). Figure 2 shows the number of public utterances per lesson averaged over five sequential lessons for each classroom, where an utterance is a single, continuous (uninterrupted) oral communication of any length by an individual or a group (choral). The average number of public utterances per lesson provides an indication of the public oral interactivity of a particular classroom. Lesson length varied between 40 and 45 minutes, and the number of utterances has been standardized to a lesson length of 45 minutes.

Figure 2 distinguishes utterances by the teacher (white), individual students (black) and choral responses by the class (e.g., in Seoul) or a group of

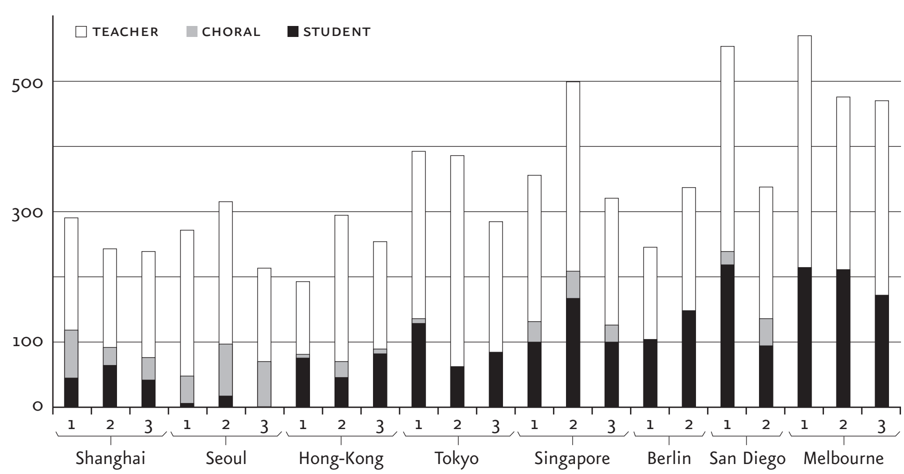


Figure 2 – The number of public utterances per lesson (averaged over five lessons) (Clarke, Xu & Wan, 2013a, p. 21)

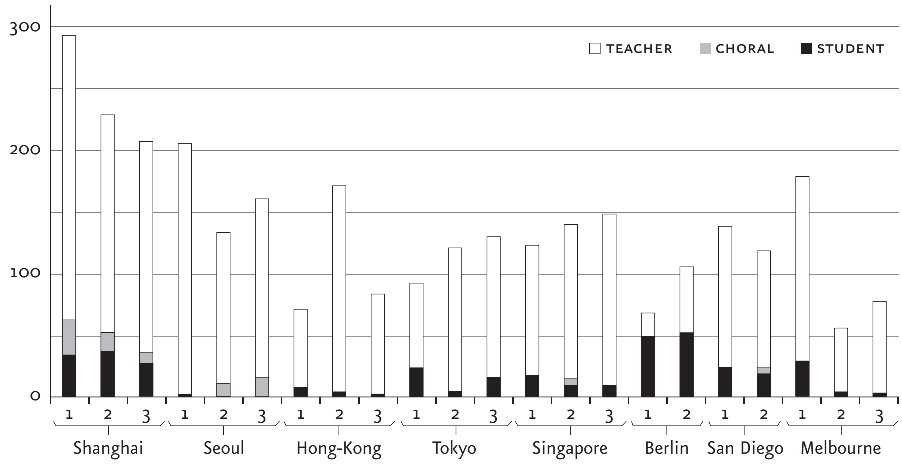


FIGURE 3 – THE NUMBER OF PUBLIC UTTERANCES CONTAINING MATHEMATICAL TERMS (AVERAGED OVER FIVE LESSONS) (CLARKE, XU & WAN, 2013A, P. 23)

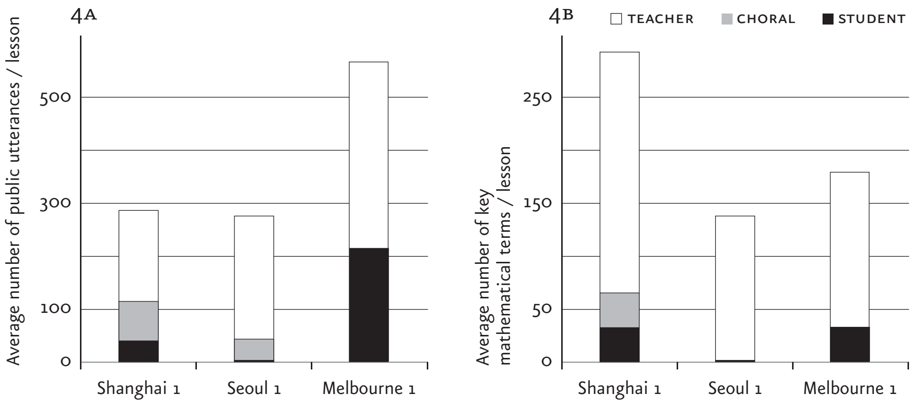
students (e.g., in San Diego) (grey). Any teacher-elicited, public utterance spoken simultaneously by a group of students (most commonly by a majority of the class) was designated a «choral response.»

It is of interest to know how many of these utterances made use of mathematical terms. Figure 3 shows the frequency of public utterances containing mathematical terms.

Shanghai 1 and the three Seoul classrooms were characterised by highly frequent choral utterances. By contrast, the classrooms in Tokyo, Berlin, and Melbourne did not appear to attach significant value to this type of utterance. The level of individual student contribution to the public classroom interactions also varied considerably.

It must be emphasised that Figures 2 and 3 refer only to what we called public speech. The comparison of three particular classrooms (Shanghai 1, Seoul 1 and Melbourne 1) makes clear just how profound were the differences in public discourse patterns between classrooms. Figures 4a and 4b focus attention on public utterances and the public use of mathematical terms in these three classrooms.

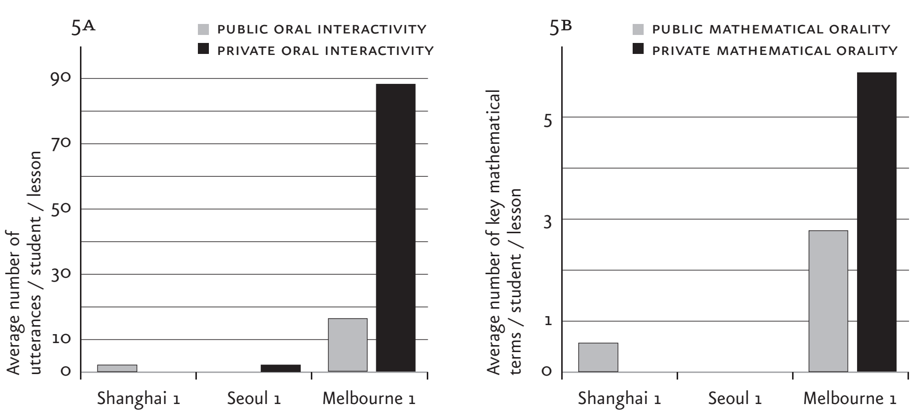
It was also possible to analyse student-student spoken interaction, where this occurred, and Figures 5a and 5b make comparison of the same three classrooms with respect to the frequency of public and private (student-student) utterances and the public and private spoken use of mathematical terms *per student per lesson*. Figures 5a and 5b show the frequencies per student averaged



FIGURES 4A AND 4B – COMPARISON OF PUBLIC UTTERANCES (4A) AND PUBLIC USE OF SPOKEN MATHEMATICAL TERMS (4B) IN THREE CLASSROOMS

over ten students (two different students recorded for each of the five consecutive lessons analysed).

It may be useful to note the number of students in each class: Shanghai 1 = 50 students; Seoul 1 = 36 students and Melbourne 1 = 25 students. The differences between the pedagogies and associated discourse patterns in the three classrooms are evident in these two sets of figures (4a, 4b and 5a, 5b). Student-student interaction is clearly a key mode of discursive exchange in the Melbourne classroom, where students discussed mathematical tasks both in mathematical and colloquial terms.



FIGURES 5A AND 5B – PUBLIC AND PRIVATE UTTERANCES (5A) AND PUBLIC AND PRIVATE MATHEMATICAL TERMS (5B) PER STUDENT PER LESSON (AVERAGED OVER TEN DIFFERENT STUDENTS – TWO PER LESSON)

In all three Shanghai classrooms and all three Seoul classrooms, there was no use of mathematical terms in private student-student interaction (Clarke, Xu & Wan, 2013b). This made it all the more remarkable that Shanghai Teacher 1 could assert in a post-lesson interview: «It is the students who have to think and talk about the problems by themselves. The role of the teacher is only to guide them. In other words, students are the active agent.» Figure 6 and Table 1 illustrate how this teacher employed whole class discussion to develop student fluency in spoken mathematics.

Studiocode, the video-coding software used, combines basic descriptive coding statistics with a capacity to reveal temporal patterns in a highly visual form (see Figure 6). *Studiocode* connects a time-coded transcript to the video record of a lesson and supports the coding of either events in the video record or the occurrence of specific terms in the transcript. Using *Studiocode*, a timeline display could be generated of the occurrence of selected mathematical terms throughout a given lesson. Figure 6 shows the occurrence of specific mathematical terms and phrases: *linear equations in two unknowns*; *equation*; *unknown*; *solution*; *integral solution*; and *solution set* in the public discussion occurring in one lesson in the classroom of Shanghai Teacher 1. We are employing 'public' in the same sense as previously: that is, spoken participation in whole class or teacher-student interaction. The occurrence of each distinct term or phrase is indicated here by a particular shade of grey. Within a shaded band, each line represents the use of a particular term, such as «equation,» by an

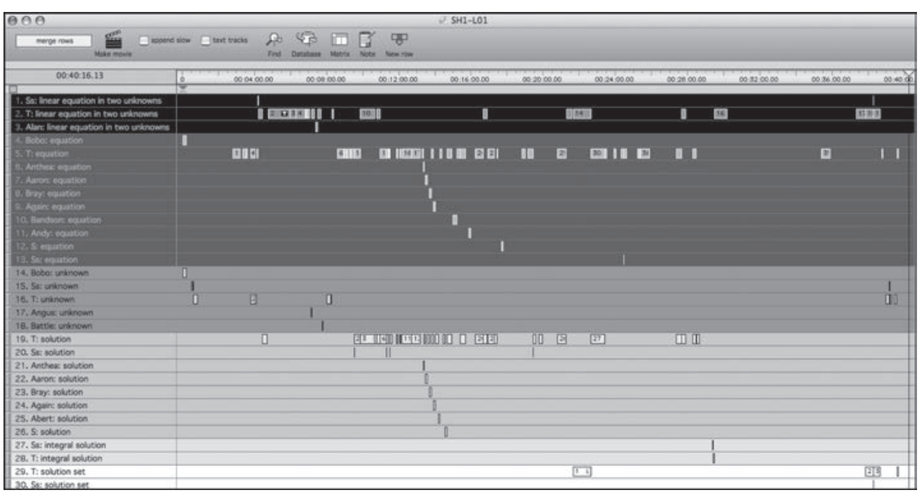


FIGURE 6. THE OCCURRENCE OF MATHEMATICAL TERMS AND PHRASES IN SHI-LOI (CLARKE, XU & WAN, 2013A, P. 20).

individual in the classroom discussion. The width of a shaded band is an indication of the number of individuals who made use of the term in public discussion. Not surprisingly, the teacher (signified by «T») made most frequent use of each term. All other timelines refer to student use of each term.

The highly visual nature of the timeline display can reveal temporal patterns in the occurrence of the coded terms. In the case of Shanghai Teacher 1, the solicited articulation of a key mathematical term (e.g., «equation» or «solution») from a sequence of students seems to be a distinctive characteristic of that teacher's practice. Once identified, such distinctive patterns can be examined in more detail. Below is the transcript of a one-minute interaction (min: sec) focusing on the term «solution.»

TABLE I – ELICITED PUBLIC REHEARSAL OF «SOLUTION» – CLASSROOM TRANSCRIPT (SHI-LOI)

12:42(m:s) T:	So let's read ... ah, let's read question one, question one. It says... in the following pairs of number value, each of them can be matched with a pair of x and y. So, let's read this. It is asking, which of them are the **solutions** of the equation two x plus y equals three? Which are the **solutions** of the equation three x plus four y equals two? Come on, have a try.
13:10 T:	So, let's take a look. How about the first one? Oh, ok, you.
13:14 Anthea:	x is equal to zero, y is equal to three. It is.
13:17 T:	It's an equation. That means, x is equal to zero, y is equal to three. It is... ?
13:21 Anthea:	It is a **solution** of the equation two x plus y equals three..
13:24 T:	A **solution**. Okay, sit down please. How about you, Aaron?
13:28 Aaron:	x equals zero and y equals one over two is a **solution** of the equation three x plus four y equals two..
13:35 T:	Ah, a **solution** of this. Sit down please. Let's continue. Question three, question three. Come on, (...) [Apollo and Amanda raising their hands]
13:41 Bray:	If x equals negative two, y equals two, it is the **solution** of the equation three x plus two y equals two.
13:48 T:	Oh,...... it's a **solution** of the equation three x plus four y equals two. A **solution**, right? Ok, sit down please. Let's continue. Come on.
13:55 Again:	When x equals one over two, y equals two, it is the **solution** of the equation two x plus y equals three.
14:00 T:	Okay, it is a **solution** of two x plus y equals three. Okay, sit down please. So now, x equals one, y equals one over two, come on, (...) Tell me.
14:12 Albert:	When x equals one, y equals negative one over two, it is a **solution** of three x plus four y equals two.

[STUDENTS WHOSE NAMES ARE GIVEN IN FULL WERE SUBSEQUENTLY INTERVIEWED; T = TEACHER, THROUGHOUT]

This level of frequency of student spoken articulation of key mathematical terms was evident in all five lessons analysed from this Shanghai classroom. The pattern of elicited rehearsal of a key term, so visible in Figure 6 and Table 1, was also clearly evident in the practice of Shanghai Teacher 2 and Shanghai Teacher 3.

It has been our experience that consideration by practising teachers of the distribution of opportunities for 'spoken mathematics' in the various classrooms has served as a powerful catalyst for teacher discussion in pre-service and in-service settings. Prompts such as «Which classroom most resembles your own?» have generated lively and fruitful discussion. In terms of the Interconnected Model displayed in Figure 1, the preceding findings from the Learner's Perspective Study constitute an «External Source of Information or Stimulus» and may prompt teacher reflection leading to the reconstruction of knowledge and beliefs in the Personal Domain or action leading to some form of classroom experimentation in the Domain of Practice.

VIDEO CASES AND TEACHER PROFESSIONAL LEARNING

It is important to clarify what is meant by a case, as this term is used in professional development situations. Cases, for the purposes of teacher professional development, are candid, dramatic, accessible representations of teaching events or series of events. Barnett (1999) has recently provided an extremely practical introduction to narrative-based cases.

Other professions (such as law, medicine, social work) make extensive use of the study of cases for professional development. Most people would have some idea of the function served by «cases» in such professions. Teaching has now adopted the strategy of case-based professional development (Barnett, 1991, 1999; Louden & Wallace, 1996; Merseth, 1991; Wasserman, 1993). Whether we are dealing with the professional development of practicing teachers or pre-service teachers, cases offer identifiable benefits. In particular, a case-based approach should be contrasted with a principles-based approach. Every profession has principles of good practice, and it is tempting to see professional development as consisting of experienced practitioners passing on these principles to novices or less experienced colleagues through either formal lectures or through some variation on the apprenticeship/internship model.

We would, however, question the value of any professional development program based solely on the communication of such principles.

In the case of novices, «principles alone» tend to confirm the beginner's already oversimplified notion of what teaching is all about. In the case of more experienced practitioners, an in-service program restricted to the communication of principles implicitly disregards the expertise of the practising teacher, offering little opportunity for the teacher or the group to benefit from the accumulation of practical wisdom present in any gathering of professionals. For both groups, beginners and veterans, principles alone minimize the opportunity for participants to relate the content of the professional development program to their existing practice or to classroom or school settings with which they are familiar. By contrast, cases connect teachers to professional practice. In the remainder of this section, we focus on the use of Video Cases for professional development.

It is a key feature of cases that they offer a common point of reference for practitioner collegial reflection. Asking practitioners to reflect on specific instances of professional practice, captured anecdotally in text form or visually through the use of videos, ensures that the resultant discussion will be firmly grounded in a shared familiarity with a particular incident in a particular educational setting. It has become common in professional development programs to have participating teachers share good practice and to reflect on their classroom experimentation. This approach affirms the expertise of the participants and can create a collegial environment for the sharing of good practice. A disadvantage, however, is that discussion centres on individual participants' accounts of their experiences and practice. The discussion of these accounts is coloured by one teacher's ownership of the recounted incident and constrained by the group's sensitivity to the personal nature of the accounts. One virtue of a case discussion is that the situation being discussed is held in common by the group. While each teacher will interpret the case in their own terms and focus on different aspects of the case, the case itself serves as a common reference point and a shared «experience.» One teacher's interpretation of the case can be evaluated by other group members in terms of its fidelity to a situation familiar to all. Since the case is held jointly rather than by one individual, discussion is unrestrained by any identification with one particular group member.

Case discussions are intended to develop practical knowledge that allows a teacher to judge a situation or context and take prompt action on the basis of knowledge gained from similar situations in the past. In this, the case methods approach bears strong similarity to some programs seeking to develop problem solving skills through expanding participants' repertoire of problem situations and associated actions, rather than through the accumulation of decontextualized general problem solving strategies. On this basis, case methods can appeal to the logic of situated cognition for theoretical support (Lave, 1988; Lave & Wenger, 1991).

Given the lack of prescription offered by the case methods approach, it is interesting to examine the research on the consequent practice of teachers with extended experience of case discussions. Teachers involved in case discussions appear to move toward a more student-centred approach. These teachers appear to learn to adapt and choose materials and methods that reveal student thinking, and anticipate and assume rationality in students' misunderstandings (Barnett & Friedman, 1996).

Recently several professional development programs have included video recordings of classroom incidents as a catalyst to discussion, rather than the narrative vignettes that characterize the text-based case methods approach. The use of such Video Cases has taken many forms.

1. *Cross-cultural Video Cases*

When teachers view videotapes of classrooms the familiarity of the classroom setting can reduce the power of the video clip to catalyse teacher reflection. However, if the videotaped lessons are taken from a very different culture, the teacher's assumptions about accepted and expected practice no longer apply. In this situation, teachers are more inclined to interrogate the videotape and, by implication, their own practice. The unfamiliarity of what they are viewing challenges their assumptions about what is acceptable, competent teaching practice. In our experience, experienced teachers, in particular, find video clips of lessons in other countries interesting. Teachers interact with such video clips by either challenging the legitimacy of the less familiar practices of another country or by justifying their own practice, where this is different from the teacher actions captured in the video clip. Videotapes of classrooms from different countries, such as those in the *TIMSS Video Study* public access collection (www.timssvideo.com) offer opportunities for such teacher interaction.

2. *Examples of Practice*

In a Californian program directed by Nanette Seago, American teachers are guided through a discussion of video recordings of American classrooms (typically the greater part of a lesson). Familiarity with the socio-cultural context of the lessons enables the discussion groups to undertake fine-grained interpretation of the teacher's and students' actions. In the discussions, teacher interaction with the video material is mediated by the teachers' construal of the video recorded practice as either exemplary or problematic. The immediacy of the video record can encourage the objectification of the teacher and the discussion can take on an evaluative tone that is less concerned with exploring and understanding classroom practice and more concerned with identifying shortcomings in the teacher's approach. However, as has been shown by Nanette Seago, in the hands of a good Case Discussion Facilitator, teachers can focus on «what could have been done?» rather than «what should have been done?» and the video clip can stimulate group participants to share their own teaching practices and beliefs and relate these to those evident in the video clip and those of the other group members.

3. *Structured Illustration*

Collated video examples of different teaching approaches are in widespread use in pre-service and in-service teacher education programs. For example, a two-DVD set of video material, entitled *Effective Mathematics Teaching: Algebra and Fractions*, was commissioned by the Victorian Department of Education and Early Childhood Development.. Short video clips were organised into categories of activity types and distributed to schools to illustrate different approaches to the teaching of algebra and fractions. In another initiative, video resources were developed to support the education of pre-service teachers at the University of Melbourne. The material was presented as an interactive DVD and prospective teachers were guided through structured interactions with video clips of elementary and high school classrooms and videotaped interviews with teachers and students. The video clips were clustered into nine «Focus Areas» such as «Student Learning and Teaching Purposes», «Individual and Group Differences» and «Evaluating Teaching.» The interaction of prospective teachers with this material was guided by questions and tasks integrated into the program and supported by linked interviews with teachers and students, frequently discussing the video clip just viewed. Electronic notebook facilities were provided within the program environment and an Audit Trail was built into the program so that a student's interactive pathway through the material could be reviewed by students and lecturers.

4. *Structured Investigation*

MILE (Multimedia Interactive Learning Environment) is a highly structured, interactive learning program implemented at the Freudenthal Institute in the Netherlands, whereby pre-service (or in-service) teachers are assisted to utilize classroom video to undertake guided investigations related to issues of pedagogy and learning. Within MILE, prospective teachers can view and review fragments of lessons. The selection of lesson fragments for inclusion in the MILE data bank was based on criteria related to subject matter and pedagogical and educational points of view that were felt to be on display in the chosen video clips. The full MILE database consists of more than three thousand five hundred video clips or lesson fragments. Each fragment is a solitary case, but at the same time is related to the lesson as a whole. Prospective teachers can carry out full text retrieval searches of the class dialogue (transcribed) and of synopses of the lessons and lesson fragments. In addition, some preparatory coding has already been carried out on the lesson fragments and prospective teachers can search the lesson fragments using these codes. The intention is that the video material provides a vehicle for prospective teacher investigation of professional activity and thereby stimulates their reflection on the nature and optimisation of that activity.

5. *Problematic cases*

Scripted videos could be used to illustrate either exemplary practice or problematic situations. In the example with which we are most familiar problematic classroom situations were simulated using the students and facilities at a local high school. Each situation was scripted, each was intended to be problematic in some way, and the scripted scenario and several alternative teacher strategies for each situation were acted out and recorded on videotape. The resultant video clips were clustered into thematic groups such as classroom management, content-related problems, pedagogical problems, and so on. The set of video clips was used in a teacher training program at Monash University in Australia to promote discussion (Clarke, 1986). Scripted videos of problematic cases have the virtue of not contravening good ethical practice since the competence of neither teacher nor students is in question. In contrast, the use in professional development programs of actual video clips of problematic classroom situations runs the risk of showing either the teacher or the student(s) in a bad light, with possible negative consequences for reputation and career.

Video cases allow participants to construct their own interpretations of the classroom depicted and to attend to those aspects they consider important. While this holds the potential for greater participant interest, it also holds the threat of a discordant, unfocussed discussion in which a variety of personal agendas compete for discussion time. The role of a case discussion facilitator in framing the group's discussion assumes new significance as the variety of possible themes for discussion expands. Conversations that we have had with those using classroom video clips suggest an inclination on the part of teachers to be immediately critical of the teacher depicted in a video. Again, the role of the case discussion facilitator is critical. The distinction between «should» and «could» is particularly useful, and we paraphrase this approach as: «Focus on what the teacher *could* have done, not what they *should* have done.» It seems to us that this distinction is at the heart of a productive case discussion.

VIDEO AS A TOOL TO SUPPORT TEACHERS' REFLECTION ON THEIR OWN PRACTICE

In conventional models of professional development, the university academic is positioned as 'outside expert' with the role of sharing knowledge and expertise with the community of teachers who are consequently positioned as 'needy', lacking the academic's knowledge or expertise. In the last decade, research on professional development focused on bringing together science and classroom practice, for example with a focus on professional communities (Lachance & Confrey, 2003) or communities of practice (Krainer, 2003; Zaslavsky & Leikin, 2004). These efforts of fusing teacher education and research are mostly intervention research; that is, the same people responsible for the intervention do the research. In neither situation, in-service professional development or research, can the relationship between academic and teacher be described as a partnership.

Recently developed programs in several countries have contested this positioning and constructed programs in which significant agency resides with the participating teachers. In the instance reported here, a partnership was established between university academics and teachers for the purpose of utilising video vignettes of the teachers' classroom teaching to catalyse the group's collective learning about classroom practice (Gorur, 2007). The Case-Based Learning (CBL) group discussions provided a forum for a process of collaborative reflection on the stimulus video material (see below for an outline of the procedure).

In this section of the paper, we explore the possibilities for teacher professional growth through academic-teacher collaboration using video case data generated in the classes of the participating teachers. Video case studies capture the 'visual, nonverbal, physical, tactile, and verbal elements of teaching', and 'bring together both teaching action and space for reflection' (Harris, Pinnegar & Teemant, 2005). Further, such records of everyday teaching practice, when used skilfully by collaborative teams of teachers and academics, afford the possibility of building theory and couching such theory in the language of teacher learning and everyday classroom practice (Shulman, 1992; Shulman & Shulman, 2004).

Cases often serve to focus attention on particular issues or dilemmas that may be encountered in 'real' classrooms (Harris *et al.*, 2005). When cases are specifically written for professional learning, the focus of learning and the 'content' to be learned become pre-determined, at least in intent, with pre-defined outcomes. This approach we term 'embedded' – the content is embedded in the cases. Previously, the case method approach to teacher professional development has typically consisted of narrative instantiations of classroom situations (Barnett & Friedman, 1996). In our approach, practicing teachers use video tapings of their own classroom practice and select and share excerpts that they think would provide stimulus for useful discussion (Clarke & Hollingsworth, 2000). This results in open-ended exploration of issues, what we call the 'encounter' approach. The cases become 'boundary objects' that provide multiple points of entry and broker connections between theory and practice (Yoon *et al.*, 2006).

THE CASE-BASED LEARNING (CBL) PROCESS

The CBL program involved a dozen teachers from different schools in Melbourne, Australia; two university academics; and two observers with backgrounds in teaching and research. Each teacher was invited to have a lesson of choice videotaped, and a DVD of the video footage was provided to the

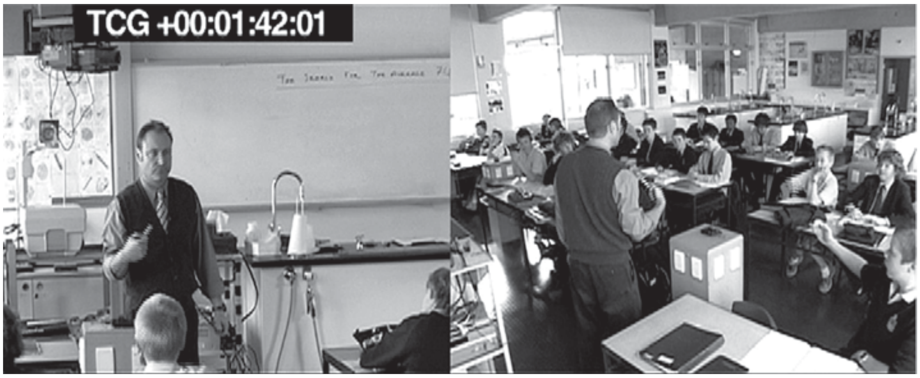


FIGURE 7 – STANDARD SPLIT-SCREEN VIDEO RECORD USED AS CBL STIMULUS

teacher. The methods of videotaping varied according to the budget available, and ranged from sophisticated, professional jobs with two or even three camera formats, with high quality microphones that picked up student group work discussion as well as teacher voice and whole group discussions and split-screen presentation, to more modest one-camera recordings. Figure 7 shows the most common split-screen display, in which two images (one from the «teacher camera» and one from the «whole class camera») were combined in a single, synchronised record of the classroom.

The teacher involved was then required to select a five minute segment of their choice, which represented either a puzzle that needed explication, an interesting or unexpected learning or teaching event or outcome, an instance of trying out a particular method, such as a thinking routine, or simply one that would provoke discussion and lead to new insights. The presenting teacher introduced the segment with some background information about the lesson/unit in general, and then focused the audience by providing the reason the particular segment had been chosen for discussion.

The discussion that followed was facilitated by one of the university academics. The structure of discussion was deliberately loose, rather than structured, so that new directions could emerge and new discussion points be raised. The focus of discussion was not whether the lesson had been taught well, or whether the teacher had 'done the right thing'. Rather, the focus was on issues in student learning, teacher learning and teaching practice, and on the possible consequences for student learning with alternate teacher choices.

The meetings were each about two hours in duration and were held monthly. Participation protocols were developed to ensure the discussion remained fruitful and the focus did not deviate toward what the teacher 'ought' to have done. Notes from the session were posted on the group's electronic discussion forum, which helped to continue discussion between meetings. Members of the group shared papers or articles or opinions that spoke to some of the discussion via this forum. Observer's notes are reproduced below to provide insight into the nature of the group discussion.

> Example: Gary (précis from notes taken by observer)
> Gary has chosen to share a segment that shows a group of his students in a passionate discussion for several minutes without progressing far in their thinking. Gary wonders if he should view this part of his lesson as a success or a waste of time. Was there merit in allowing the group to continue the discussion when they were off the track? Should he have insisted on equity in participation? Should he have intervened with a question that might have nudged their thinking out of their current quagmire, or given them more time to work it out for themselves? What, he asks, do members of the case-based learning group think he could have done? The group discuss various approaches that Gary could have taken at this point in the lesson, carefully considering the potential outcomes of each one. They recognise that some of the approaches suggested have more merit than others, and decide to discard those that they agree are less helpful. It's astonishing that a five-minute piece of recording yields such rich material to ponder. The two hours set aside for discussion seem to evaporate all too soon. Questions and ideas remain active in the participants' minds even after the meeting closes, and the discussion spills over into the group's wiki where the dialogue continues.

The Participants' Perspective

The participants in this event were invited on the basis of their past interest in professional learning, and were an enthusiastic and dedicated group. While the style of discussion did not uniformly suit everyone (one participant stated that she was unable to think and speak on the spot, and felt she was not contributing adequately to the discussion), all participants felt that the discussions spoke directly to their classroom experience, and that they found themselves reflecting on the issues raised in the discussion long after the meeting sessions. In the words of one participant:

> The themes of conversations really stick in my mind, I think because during the CBL session I had to make active links with my classroom. [For example]

> after we spoke about the challenges of grouping students, for weeks I deliberated on the way I grouped my students.

The teachers found this form of professional learning to be radically different from other forms of professional development opportunities they had encountered. Many commented on the validation of the importance of the ordinary and everyday aspects of teaching that this forum provided. The focus on the nitty-gritty and the mundane – the daily business of classroom life spoke directly teachers' everyday practice.

> CBL has ordinary teachers as the focus. Normal teachers doing what they do every day. It is not whiz bang «Here's a set of tasks for next week's maths lessons», it is far deeper and [more] meaningful. Real issues about real classrooms from real teachers.

The heterogeneity of the group meant that important insights were gained about (and from) the different assumptions about learners and learning made in different classrooms – prompting reflection on their own assumptions as reflected in their classrooms.

> [It was] intriguing to see classes in different levels to the one I teach in operation. In that regard, [it was] very interesting also to see the way that students at the different levels express their thinking and do their thinking.

The nature of video data was also an important material actor, since it could bring the classroom to the discussion, be viewed over and over, and 'saw' things literally from different perspectives. Participants reported that once they had overcome the initial resistance to being filmed, they found the footage to be very valuable. They noticed things about themselves, their students and the environment of which they were previously unaware. As one participant noted,

> One of the most significant outcomes is the variety of classroom issues for consideration which have arisen: from the arrangement of furniture in different levels of classrooms to deep considerations of student thinking. All of these discussions have served to open up my thinking about my classroom and the many aspects of it, which I should be thinking about or at least be aware of.

Teachers also reported becoming more aware of their own decision-making processes and the impact their decisions had for students and student learning.

> To view the body language and the way we carry ourselves as educators- what we say and do and how that may influence our students... It [p]oses more questions: hidden curriculum, values, ethics, standards, etc.

Some continued to have their lessons taped for their own reflection and growth, beyond the requirement of the CBL participation. One teacher reported that the experience of having a camera in the classroom as a 'seeing eye' has prompted her to develop a 'seeing eye' within herself, so that she is much more conscious of the goings-on in the classroom as the camera might see it.

With the teacher setting the focus for discussion at the time of presentation, issues raised were the stuff of practitioners' interests and dilemmas. Daily issues became matters worthy of discussion. Here the encounter nature of learning, where the agenda for discussion was not pre-determined by the academics but by the presenting teacher, and where discussion sometimes took unexpected turns, also served to provide opportunities for teachers to see themselves as experts and to validate their own questions, knowledge and experience. It was a recurrent experience of CBL discussions that an excerpt chosen by a teacher would trigger associations among the group and catalyse discussion that ranged far beyond the features or issues that had initially prompted the teacher to select that excerpt. In the language of the model in Figure 1, the salient features of the excerpt were a matter for individual interpretation and group discussion and led to a stimulating negotiation of the participants' meanings and values.

CONCLUDING REMARKS

This paper was intended to illustrate some of the ways in which video can be used to facilitate teacher reflection and enaction. In all of the examples provided in this paper, the video material has provided an explicit or an implicit bridge between the contexts portrayed in the stimulus material and the classrooms of the teachers participating in the in-service or pre-service programs. In the case of the use of cross-cultural research, the video material provides a warrant for the legitimacy of the shared findings; a warrant that encourages teacher engagement with the shared data as fundamentally grounded in the practice of actual classrooms. This immediacy and connection with practice is even more apparent in the discussion of video cases and teachers' reflection on their own practice. Our concluding remarks will focus on the conditions under which video might best support case-based approaches to teacher professional learning.

If video cases are to stimulate productive teacher reflection, then the discussion of such cases must be carefully framed, centring on possibility rather than prescription. Our increasing use of video material to facilitate teacher reflection on classroom practice may (i) render visible, for the first time, some of the unnoticed practices of teachers[1]; and (ii) facilitate the development among the teaching community of a new vocabulary by which we might describe teaching practice. Both these developments are important. Many of the practices of our most capable teachers have a subtlety that renders them effectively invisible to casual observation. Frequently this will be because the teacher's actions carry a significance or meaning that is shared by teacher and class but not readily apparent to an outside observer. Videotape, which lends itself to re/view, can facilitate the sort of fine-grained «data-driven» discussion likely to reveal the nature and significance of such practices. Some of these practices are not even represented in our discourse. These might include the strategies by which a teacher signifies a willingness to accept student contributions to class discussion or through which student-student interaction is sanctioned or promoted. It may consist of oral inflections signifying invitation or non-verbal acts, gestures, body posture or physical location in the classroom that serve to signify to students that «the floor is yours.» Such strategies may not yet have labels within the profession and may only become part of the discourse of the teaching profession through the provision of the opportunity for teachers to discuss video records of classroom practice.

The other product that may emerge from teacher discussions of video material is the body of «principles or theories of practice» lying behind teacher classroom decision-making. The existence of such theories of practice has already been postulated in Shulman's conception of the wisdom of practice (Shulman, 1987). Using video material as a catalyst for discussion, we can facilitate the articulation of teachers' theories of practice and construct their professional development experiences on that basis. Video material can provide open-ended, minimally cued stimulus more likely to facilitate the articulation of the teachers' actual theories of practice.

One of the important objectives of the CBL project was to promote and refine teacher professional dialogue (including the development of protocols for discussion). This appears to have been achieved most successfully, as one of the observers noted:

> The role of the discussion facilitator is vital in protecting the presenting teacher from thoughtless, tactless, repetitive and overlong comments from other group members. The discussion rules have to be clearly set out. I noticed that respectful collegiality grew as people worked together over time and the facilitator [took] a less dominant role. Teachers need time to learn how to 'look and talk' in the CBL context.

In the words of one participant, his experience in CBL brought home to him what professional learning teams could achieve:

> [T]he strength and effectiveness of a group of teachers meeting regularly, coming to at least know each other on a professional level and the depth and frankness of discussion which consequently follows...

Several members of the CBL group are now setting up and facilitating similar CBL forums within their schools. We consider this development to be the most compelling endorsement of the value of video as a significant facilitator of teacher reflection and enaction.

REFERENCES

Barnett, C. (1991). Building a case-based curriculum to enhance the pedagogical content knowledge of mathematics teachers. *Journal of Teacher Education*, *42*, 263-272.

Barnett, C. (1999). Cases. *Journal of Staff Development*, Summer Issue, 26-27.

Barnett, C., & Friedman, S. (1996). Mathematics case discussions: Nothing is sacred. In E. Fennema & B. Scott-Nelson (Eds.), *Mathematics teachers in transition* (pp 381-399). Hillsdale, NJ: Lawrence Erlbaum.

Clarke, D. J. (2006). The LPS Research Design. In D. J. Clarke, C. Keitel & Y. Shimizu (Eds.), *Mathematics Classrooms in Twelve Countries: The Insider's Perspective* (Chapter 2, pp. 15-37). Rotterdam: Sense Publishers.

Clarke, D. J. (2013). Contingent conceptions of accomplished practice: The cultural specificity of discourse in and about the mathematics classroom. *ZDM: The International Journal in Mathematics Education 45*(1), 21-33.

Clarke, D. J., & Hollingsworth, H. (1994). Reconceptualising teacher change. In G. Bell, B. Wright, N. Leeson & J. Geake (Eds.), *Challenges in mathematics education: Constraints on construction. Proceedings of the Seventeenth Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 153-164). Lismore, NSW: Southern Cross University.

Clarke, D. J., & Hollingsworth, H. (2000). Seeing is understanding: Examining the merits of video and narrative cases. *Journal of Staff Development, 21*(4), 40-43.

Clarke, D. J., & Hollingsworth, H. (2002) Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*(8), 947-967.

Clarke, D. J., Keitel, C., & Shimizu, Y. (Eds.) (2006). *Mathematics classrooms in twelve countries: The insider's perspective.* Rotterdam: Sense Publishers.

Clarke, D. J., Mitchell, C., & Bowman, P. (2009). Optimising the use of available technology to support international collaborative research in mathematics classrooms. In T. Janik & T. Seidel (Eds.), *The power of video studies in investigating teaching and learning in the classroom* (pp. 39-60). New York, NY: Waxmann.

Clarke, D. J., Xu, L. H., Arnold, J., Seah, L. H., Hart, C., Tytler, R., & Prain, V. (2012). Multi-theoretic approaches to understanding the science classroom. In C. Bruguière, A. Tiberghien & P. Clément (Eds.), *Proceedings of the 2011 Biennial Conference of the European Science Education Research Association* (Part 3, pp. 26-40). Lyon, France, September 7 to 11, 2011 (e-Book published March 23, 2012).

Clarke, D. J., Xu, L., & Wan, V. E. V. (2013a). Spoken mathematics as an instructional strategy: The public discourse of mathematics classrooms in different countries. In B. Kaur, G. Anthony & D. Clarke (Eds.), *Student Voice in Mathematics Classrooms around the World* (pp. 13-32). Rotterdam: Sense.

Clarke, D. J., Xu, L., & Wan, M. E. V. (2013b). Students speaking mathematics: Practices and consequences for mathematics classrooms in different countries. In B. Kaur, G. Anthony, M. Ohtani & D. J. Clarke (Eds.), *Student Voice in Mathematics Classrooms around the World* (pp. 33-52). Rotterdam: Sense.

Clarke, D. M. (1986). *Critical incidents in the mathematics classroom: Teacher notes to accompany video.* Clayton, Victoria: Mathematics Education Centre, Monash University.

Darling-Hammond, L., & Snyder, J. (2000). Authentic assessment of teaching in context. *Teaching and Teacher Education*, *16*, 523-545.

Fullan, M., & Stiegelbauer, S. (1991). *The new meaning of educational change.* (2nd Edition). New York, NY: Teachers College Press.

Gorur, R. (2007). Making a Case for Learning. *Professional Educator*, *6*(4), 10-12.

Greeno, J. G., Collins, A. M., & Resnick, L. (1996). Cognition and learning. In D. C. Berliner & R. C. Calfree (Eds.), *Handbook of educational psychology* (pp. 15-46). New York, NY: Simon & Schuster Macmillan.

Guskey, T. R. (1986). Staff development and the process of teacher change. *Educational Researcher*, *15*(5), 5-12.

Hall, G. E., & Loucks, S. (1977). A developmental model for determining whether the treatment is actually implemented. *American Educational Research Journal*, *14*, 263-276.

Harris, C. H., Pinnegar, S., & Teemant, A. (2005). The case for hypermedia video ethnographies: Designing a new class of case studies that challenge teaching practice. *Journal of Technology and Teacher Education*, *13*(1), 141-161.

Hollingsworth, H. (1999). *Teacher professional growth: A study of primary teachers involved in mathematics professional development*. PhD Thesis, Deakin University, Burwood, Victoria, Australia.

Johnson, N. (1996). Reconceptualising schools as learning communities. *Reflect*, *2*(1), 6-13.

Krainer, K. (2003). Teams, communities & networks. *Journal of Mathematics Teachers Education*, *6*, 93-105.

Lachance, A., & Confrey, J. (2003). Interconnecting content and community: A qualitative study of secondary mathematics teachers. *Journal of Mathematics Teacher Education*, *6*, 107-137.

Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life.* Cambridge: Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge: Cambridge University Press.

Louden, W., & Wallace, J. (1996). *Quality in the classroom: Learning about teaching through case studies*. Rydalmere, NSW: Hodder Education.

Merseth, K. (1991). *The case for cases in teacher education*. Washington, DC: American Association of Higher Education and the American Association of Colleges for Teacher Education.

Nuthall, G. (2004). Relating classroom teaching to student learning: A critical analysis of why research has failed to bridge the theory-practice gap. *Harvard Educational Review, 74*(3), 273-306.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Shulman, L. S. (1992). Toward a pedagogy of cases. In J. H. Shulman (Ed.), *Case methods in teacher education* (pp. 1-32). New York, NY: Macmillan.

Shulman, L. S., & Shulman, J. H. (2004). How and what teachers learn: A shifting perspective. *Journal of Curriculum Studies, 36*(2), 257-271.

Wasserman, S. (1993). *Growing teachers: Learning about teaching from studying cases*. New York, NY: Teachers College Press.

Yoon, S., Pedretti, E., Bencze, L., Hewitt, J., Perris, K., & Van Oostveen, R. (2006). Exploring the use of cases and case methods in influencing elementary preservice teachers' self-efficacy beliefs. *Journal of Science Teacher Education, 17*, 15-35.

Zaslavsky, O., & Leikin, R. (2004). Professional development of mathematics teacher educators: Growth through practice. *Journal of Mathematics Teacher Education, 7*(1), 5-32.

MATHEMATICS TEACHERS' LEARNING THROUGH INQUIRY

Olive Chapman
chapman@ucalgary.ca | University of Calgary, Canada

ABSTRACT

Inquiry-based teaching strives to engage students in learning mathematics with understanding in the classroom. Therefore, there is great interest in supporting teachers to meet this pedagogical challenge by developing practices that promote such an educational environment at different school levels. A powerful way for teachers to learn and transform their teaching is through teacher inquiry. This paper presents a model for inquiry into mathematics teaching based on the perspectives of theorists directly associated with teacher education. This model is described as an overarching inquiry cycle in which teachers begin with practice, pose a pedagogical problem, understand a key construct in the problem, hypothesize an inquiry-teaching model, test/apply it, and finally revise/apply this model. This approach is illustrated with a self-directed professional development process aimed at helping elementary teachers to develop understanding of inquiry-based teaching of mathematics.

KEY WORDS

Inquiry-based learning; Mathematics teacher learning; Noticing; Dialogic inquiry; Inquiry stance.

SISYPHUS
JOURNAL OF EDUCATION
VOLUME 1, ISSUE 3,
2013, PP. 122-150

Mathematics Teachers' Learning Through Inquiry

Olive Chapman

INTRODUCTION

Current learner-focused perspectives of mathematics education require teachers to use effective pedagogy that will actively engage students in learning mathematics with understanding. Inquiry-based teaching offers opportunities to achieve this in the mathematics classroom. This makes inquiry an important consideration in mathematics teachers' learning and practice. For teachers facing new pedagogical challenges, teacher inquiry can be a powerful vehicle for their learning and transformation of their practice. This paper discusses inquiry from the perspectives of theorists who deal directly with teacher education and the use of these perspectives to frame mathematics teachers' learning. It examines how inquiry has been interpreted and used in studies of practicing mathematics teacher education. Finally, it discusses a self-directed professional development process aimed at helping elementary teachers to develop an understanding of inquiry-based mathematics teaching, how it is related to the different perspectives of inquiry and the implications for the development of an inquiry stance.

PERSPECTIVES OF INQUIRY IN TEACHER LEARNING

Dewey's (1933/1971, 1938) work has provided a foundation for current perspectives of inquiry. For Dewey, what distinguishes inquiry from the trial and error that people are continually engaged in as they transact with their environment is that inquiry is «*controlled* or *directed* by means of reflection or thinking» (Biesta & Burbules, 2003, p. 58). Thus, *reflective thinking* is «central to all learning experiences enabling us to act in a deliberate and intentional fashion (…) [to] convert action that is merely (…) blind and impulsive into intelligent action» (Dewey, 1933/1971, p. 212). Dewey defines reflective thinking as an «active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends» (Dewey, 1933/1971, p. 9). He argued that encountering problems impels us to reflective thought, the essential characteristic of which is inquiry and that «We inquire when we question; and we inquire when we seek for whatever will provide an answer to a question asked» (Dewey, 1938, p. 105). Thus, for him, there is a direct relationship among questioning, reflective thinking, and inquiry. «Thinking is inquiry, investigation, turning over, probing or delving into, so as to find something new or to see what is already known in a different light. In short, it is questioning» (Dewey, 1933/1971, p. 265).

Dewey's (1933/1971) inquiry process begins when one encounters a puzzling situation, i.e., «a state of doubt, hesitation, perplexity, mental difficulty» (p. 12); «an entanglement to be straightened out, something obscure to be cleared up» by thinking (p. 6) and then entails the following phases or states of thinking:

1. Suggestions in which the mind leaps forward to a possible solution. If the solution seems feasible, it is applied, and full reflection does not occur. Otherwise, these phases take place:
2. Intellectualization of the difficulty or perplexity into a specific problem to be solved or question to be answered (i.e., placing the perplexity into a relevant context)
3. Development and use of a hypothesis to initiate and guide observation and other processes in the collection of empirical data (e.g., «searching, hunting, inquiring, to find material that will resolve the doubt, settle and dispose of the perplexity» [p. 12])

4. Elaboration of the hypothesis
5. Testing the hypothesis, either by overt action or thought experiment (imaginative action).

Dewey notes that «the sequence of five phases is not fixed» (p. 115). They also form a continuous process.

There are clear links to Dewey's view of reflective thinking in Schön's (1983) notion of *reflection-on-action* as the way practitioners focus on problem posing (questioning) to inquire into practice and meaningful situations. For Schön, reflection-on-action involves looking back at an event. It takes into consideration the context of the event by:

- analysing the circumstances of the event, including personal biases or misunderstandings
- planning actions based on careful consideration of all the information
- guiding future actions

This form of inquiry, according to Schön (1983, 1987), involves a process of posing and exploring problems or dilemmas identified by the practitioners themselves in order to examine their practice by analysing, adapting, and always challenging their assumptions in a self-sustaining cycle of reflecting on their theory and practice. This cycle allows them to learn from one problem to inform the next. This process of reflection (inquiry) enables practitioners to assess, understand and learn through their experiences. It is, therefore, a process that starts with their own experiences.

While Dewey's notion of inquiry is oriented towards a cognitive perspective, Wells' (1999) approach is oriented towards a socio-cultural perspective in which a «community of inquiry» is central. As Wells noted, «The construction of understanding is a collaborative enterprise» (p. 125). Wells (1999) defines *dialogical inquiry* as «a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them» (p. 122). He represents this as a «spiral of knowing» consisting of: experience, information, knowledge building, and understanding. He considers the relationship among experience, discourse, and the enhanced understanding to be the goal of all inquiry. He explains that each cycle of the spiral starts from past personal experience and new information is added from the current environment. The goal of each cycle is enhanced understanding that is reached through knowing in action a specific situation and almost always involves dialogic knowledge-building with others. This goal can be achieved through telling stories, developing explanations, making connections, and testing conjectures through action. The critical aspect of the spiral of knowing is interpersonal and collaborative and is always aimed at enhancing the understanding of both the group and participating individuals.

The importance of beginning with one's own experience and reflecting on it is also characteristic of the view of inquiry embodied in Mason's notion of noticing (Mason, 2002). Noticing, as a basis of teachers' learning, «is a collection of practices both for living in, and hence learning from, experience and for informing future practice» (p. 29). It is «a reference to lived experience through an invitation to check something out in your own experience» (p. xi). Mason defines it specifically as a collection of systematic practices consisting of four interconnected actions: preparing and noticing; systematic reflection; recognizing and labelling choices; and validating with others. This process is informed by research and shared practice through «*introspective observation* (in which an inner witness observes the self caught up in the action…); and *interspective observation* (in which people share observations as witness to each other, yielding objectivity from negotiated subjective information)» (Mason, 2002, p. 85). «The core of researching from the inside is attending to experience (…) so as to develop sensitivities to others and to be awake to possibilities» (Mason, 1994, p. 180).

Table 1 provides a summary of the *key components* of the preceding ways of viewing inquiry. Each column represents a complete inquiry cycle. However, it is not intended to represent a linear process with a definite end point. The relationship among components can be dynamic and cyclic. More importantly, the end point of each inquiry cycle (each column of Table 1) is an actual or potential beginning of a new cycle. In addition, each cycle begins with experience.

These approaches to constructing knowledge have been directly linked to the way teachers can learn and change. For example, Dewey (1933/1971) called for teachers to engage in inquiry or «reflective action» (action based upon thoughtful deliberation; intelligent action) that would transform them into inquiry-based, classroom practitioners. Inquiry provides teachers with a way to better understand their own practices, so that they can ultimately «transform actions into *intelligent* action» (Biesta & Burbules, 2003, p. 38) that result in growth. According to Biesta and Burbules (2003), the outcomes of this

REFLECTIVE THINKING DEWEY	A puzzling situation Generation of questions or specific problems Formulation of a solving suggestion (initial hypothesis) Transformation of hypothesis into possible lines of action Experimental testing of hypothesis in the concrete situation
REFLECTION-ON-ACTION SCHÖN	Problem or dilemma Analysing, adapting, and challenging assumptions Action plan Application to future actions
DIALOGIC INQUIRY WELLS	Personal experience Information Knowledge building Understanding
NOTICING MASON	Preparing and noticing Systematic reflection Recognizing and labelling choices Validating with others

TABLE I – SUMMARY OF KEY COMPONENTS OF PERSPECTIVES OF INQUIRY

process are changes in the way the teachers and students think and know and in the situation, which includes the way the curriculum gets enacted. Similar to Dewey, Schön (1983) argued that teachers could orchestrate their own change if they are helped to develop an inquiry (reflective) stance of looking at their own practice. This stance usually results in changes in their perspectives of a situation or new learning, which, if applied to practice, can result in improvement.

Wells (1999) made a case for teachers to engage in inquiry as a way to systematically investigate their own practice to find out what approaches, choice of activities and patterns of organization are most successful in their own particular situations. The outcome of this investigation is the improvement of both their pedagogical understanding and their practice. Mason (2002) directed his process of noticing specifically to mathematics teachers. He explained that a goal of this process is for teachers to examine their own experience of work on themselves, informed by research and shared practice, while addressing how to help their students to learn mathematics.

Table 1 also presents ways of interpreting inquiry that are consistent with «inquiry as stance» (Cochran-Smith & Lytle, 1999, 2009). This notion states that it is important for inquiry to be about teachers' learning as opposed to the tasks in which they engage. As Cochran-Smith and Lytle (1999) explained, «Inquiry as stance is distinct from the more common notion of inquiry as time-bounded project or discrete activity within a teacher education course or professional development workshop» (p. 289). Instead, it is about teachers

> working together in communities (…) [to] pose problems, identify discrepancies between theories and practices, challenge common routines, draw on the work of others for generative frameworks, and attempt to make visible much of that which is taken for granted about teaching and learning. From an inquiry stance, teachers search for significant questions as much as they engage in problem solving. They count on other teachers for alternative viewpoints on their work (pp. 292-293).

In addition, «from the perspective of inquiry as stance, teacher learning is associated more with uncertainty than certainty, more with posing problems and dilemmas than with solving them, and also with the recognition that inquiry both stems from and generates questions» (Cochran-Smith & Lytle, 1999, p. 294). In the context of inquiry as stance, Cochran-Smith and Lytle (2009) have broadened the scope of inquiry from a study of classroom practice to a lifelong habit of mind wherein teachers use an inquiry lens to question any aspect of the educational system. This added dimension of inquiry has a social justice goal of more equitable outcomes for students.

Table 1, then, provides a basis for a theoretical framework to guide and interpret an inquiry perspective of mathematics teachers' learning. The four ways of viewing inquiry have common features. However, they also have particular features that can be combined to produce a framework that recognizes the cognitive perspective of reflective thinking, the sociocultural perspective of dialogic inquiry and the importance of noticing in both of these perspectives. Such a framework is consistent with the view that knowledge is both an individual and a social construction and that individual and social dimensions of learning complement each other. This framework also represents a perspective of inquiry as a fundamental principle and a way of being in mathematics teacher education. Thus, it provides a basis for inquiry to be a norm of practice through teachers' development of an inquiry stance. An example of this framework is illustrated after discussing how inquiry has been addressed in research on practicing mathematics teachers' learning.

INQUIRY IN PRACTICING MATHEMATICS TEACHER EDUCATION

The reform movement in mathematics education and the focus on constructivism have provided support for inquiry as a mathematical process, as a way of teaching mathematics and as a way of developing mathematics teaching. However, several obstacles can arise for teachers when they try to teach from an inquiry perspective because it requires skills that are unfamiliar in traditional mathematics classrooms. In addition to holding deep understanding of mathematics for teaching, teachers must possess, for example, the ability to embrace uncertainty, foster student decision-making by balancing support and student independence, recognize opportunities for learning in unexpected outcomes, maintain flexible thinking, and tolerate periods of disorganization (National Research Council, 2000). Teachers are more likely to develop an understanding of such behaviours, and inquiry in general, if they learn through inquiry. But more importantly, as previously discussed, it is important for them to learn in a way that will help them to develop an inquiry stance as a central aspect of being a teacher of mathematics.

Current professional standards for teaching and research in mathematics teacher education suggest approaches to teachers' learning that have potential to help teachers to develop an inquiry stance. For example, the National Council of Teachers of Mathematics [NCTM] (2000) Teaching Principle states:

> Opportunities [for teachers] to reflect on and refine instructional practice – during class and outside class, alone and with others – are crucial in the vision of school mathematics. (…) To improve their mathematics instruction, teachers must be able to analyze what they and their students are doing and consider how those actions are affecting students' learning. (…) Collaborating with colleagues regularly to observe, analyze, and discuss teaching and students' thinking or to do «lesson study» is a powerful (…) form of professional development (p. 19).

This perspective is reflected in practice-based learning communities, a current trend in mathematics teacher education. Practice-based learning communities are now viewed as a more desirable and meaningful way to facilitate mathematics teachers' learning and have been increasingly used in studies of teachers' professional development (e.g., Even & Ball, 2008; Krainer & Wood, 2008; Ponte *et al.*, 2008; Tirosh & Wood, 2008). A core feature of these learning communities is having teachers work collaboratively on a variety of activities linked to the context of their teaching. These activities are purposefully connected to their mathematics curriculum, their students' learning or work, and their classroom pedagogy. Thus, a common feature of this approach is to provide realistic or actual events and contexts of classroom situations that enable teachers to explore important mathematical and pedagogical ideas that relate to their own teaching.

The following examples of current studies on mathematics teacher education suggest some ways in which practicing teachers' learning has been facilitated through teachers working in groups over an extended period of time, investigating and discussing situations directly related to classroom teaching. Some studies engaged teachers in a collaborative process that included analyzing self-created videos of their teaching or researcher-created videos of teaching or students at work in the classroom. For example, Maher (2008) discussed a process of facilitating teachers' learning that included the use of researcher-created video recordings. This process involved:

> (1) teachers studying mathematics by working on a strand of tasks; (2) teachers collectively studying their own solutions; (3) teachers viewing and analyzing video recordings of children working on the same or similar tasks; and, (4) teachers implementing and analyzing together, the same or similar lessons in their own classrooms (p. 71).

van Es and Sherin (2010) also discussed a model of professional development called "video clubs" in which teachers watched and discussed excerpts of videos from their classrooms. In both studies, the approaches influenced teachers' thinking and teaching in positive ways.

Some studies involved the use of cases, as in Markovits and Smith (2008), who engaged teachers in a process that included:

> Solving and discussing the mathematics task on which the case is based, reading the case guided by a framing question, engaging in small and whole group discussions of the case centered on the framing question, and generalizing beyond the case to one's own teaching practice and to a larger set of ideas about mathematics teaching and learning (p. 47).

Other studies involved using students' work. For example, Kazemi and Franke (2004) initiated and organized a monthly work group of 10 teachers at an elementary school. Students' work from the teachers' classrooms guided the content and direction of discussions at each work-group meeting. Prior to the meetings, teachers used a common problem that they could adapt to their classes. For each meeting, teachers selected samples of students' work to share with the group. Work-group discussions centred on the students' work those problems generated. The mathematical domains the researchers chose to focus on during the work group reflected those that the teachers were working on in their classrooms. The approach helped the teachers to become more attentive to the details in the students' thinking.

In another example, the teachers' group work was based on observing students in an actual classroom. Francisco and Maher (2011) reported on the experiences of a group of elementary and middle school teachers who participated as interns in an after-school, classroom-based research project on the development of mathematical ideas for middle-grade students. For one year, the teachers observed the students working on well-defined mathematical investigations during research sessions in which the researchers taught the classes. In these classes, the researchers encouraged students to work collaboratively and justify their solutions, received their contributions positively, and gave them extra time to work on tasks and opportunities to refine and make connections between mathematical ideas. The teachers, in groups of two or three, observed a different group of four to six students in different sessions and occasionally followed the same group of students over several sessions. They received instructions about what to focus on in their observations and were told to refrain from interacting with students. This approach enabled the teachers to gain insights into the students' mathematical reasoning.

In these studies, inquiry is implied as consisting of situations or tasks for teachers to explore as they worked in groups. For the most part, this type of inquiry is influenced by the intentions and expectations of the researchers (the professional development leaders) and constrained by pre-set activities and goals. While such types of learning communities offer opportunities for teachers to construct knowledge about mathematics pedagogy, they are less likely to help them to develop an understanding of inquiry as a way of being and to adopt it as a way of framing their teaching. They do not offer the key aspects of the inquiry perspectives in Table 1 or the Cochran-Smith and Lytle

(1999) perspective of inquiry stance that are important for teachers to be able to develop an inquiry stance.

Lesson study, as practiced in Japan, also involves learning communities and has exerted an influence in other countries (e.g., Lewis, Perry & Hurd, 2009). In this approach, a small group of teachers works together to plan, teach, observe, and analyse the lessons. They start by identifying a goal or problem they want to explore. This is followed by a four-phased cycle: collaboratively developing a lesson plan, implementing the lesson with observation by colleagues and other experts, analytically reflecting on the teaching and learning that occurred, and revising the lesson for re-implementation (Curcio, 2002; Shimahara, 2002; Stigler & Hiebert, 1999). During each cycle of implementation, a different teacher teaches the lesson to his or her students in a normal classroom setting, while the other group members observe, taking notes on how it is being implemented. In the end, the teachers produce a report of what they learned, particularly with respect to their goal. This approach has the potential for teachers to engage in and develop an inquiry stance. However, the tendency is for it to be more theoretical than about personal experience, to have a specific purpose or outcome that is not about inquiry as a way of being, and to be based on a predetermined pre-learned process. As Yoshida (2008) pointed out, teachers who engage in lesson study need to learn how to investigate, plan a research lesson, observe it, and discuss it; and they need to receive strong support from other knowledgeable persons such as teacher educators.

In contrast to the preceding examples of the use of learning communities in practicing mathematics teacher education, Jaworski (2004) made a case for the use of «community of inquiry» based on Wells' perspective of inquiry, instead of «community of practice». As she stated, «In a community of inquiry, inquiry is more than the practice of a community of practice: teachers develop inquiry approaches to their practice and together use inquiry approaches to develop their practice» (p. 25). Jaworski (2004, 2006) discussed such an inquiry community in which teachers viewed themselves as researchers. In this community, teachers and didacticians/researchers worked together in a way that supported each other's learning through inquiry. The didacticians drew the teachers into inquiry in a variety of ways, such as workshops that created opportunities to do mathematics together in inquiry mode and exploration into what inquiry looks like in mathematics learning. The teachers formed an inquiry group to discuss what their teaching might look like from an inquiry perspective and to plan classroom activities that encouraged students to get involved in inquiry in mathematics.

Jaworski (2004) also discussed a study that involved «learning study», which is different from lesson study in terms of its theoretical basis and its purposeful nature. She described learning study as

> a group of teachers designs innovative classroom activity, based on agreed theoretical principles, and explores the consequent teaching. *Design* and *innovation* offer purposeful directions. Teachers use inquiry as a tool to explore teaching, alongside didacticians who offer theoretical ideas and practical support and who research the processes of teaching development. Teachers develop their thinking and practice through successive cycles of inquiry. They each work in their own classroom, interpreting a design they have produced jointly. Observation of each other's teaching and group reflections lead to building of group and individual awareness through which *inquiry as a way of being* develops (p. 27).

Thus, Jaworski's work offers insights of a perspective of inquiry that can be related to key aspects of the perspectives in Table 1 and provide a basis to help teachers to develop an inquiry stance with regard to their teaching.

The preceding discussion provided a brief profile of the nature of inquiry in practicing mathematics teacher education based on examples of professional development situations involving community-of-learners. These examples suggest approaches that were effective in helping teachers learn specific aspects of pedagogical content knowledge. However, inquiry as an explicit focus was lacking, despite its importance to learning mathematics. More research attention is needed, as in Jaworski's case, where teachers' inquiry includes inquiry of inquiry as a basis of their learning and as a way of developing an inquiry stance in their teaching. The following section describes an example of such a study with practicing elementary teachers.

PRACTICING TEACHERS' SELF-DIRECTED INQUIRY-BASED LEARNING

This example draws on a study that focused on teacher learning *through* and *about* inquiry. In this study, the teachers engaged in a self-directed professional development process in which they decided what to do and how to do it. Chapman (2011) discusses the study from the self-directed aspect of the professional development experience. The focus here is to highlight key aspects of the process based on the theoretical perspectives in Table 1.

The participants were 14 practicing teachers with representation from grades 1 to 6 in the same elementary school. They had from 3 to 20 years of teaching experience; most had over 10. Teachers in Alberta are required to have a professional growth plan. Each school could choose its own way of implementing this. At the school of this study, the teachers were required to form disciplinary study groups of their choice. The teachers in this study chose the mathematics group because they thought mathematics was the area in which they needed the most help to bring their teaching more in line with curriculum expectations that fostered a constructivist or inquiry perspective. The curriculum was significantly influenced by NCTM (1989, 2000) standards. Although some teachers were beginning to make meaningful changes based on ideas in the textbooks linked to the curriculum, most were well behind in implementing the reform perspective of the NCTM standards in their classrooms. So, the participants' starting point was oriented towards a teacher-directed approach.

I was invited to join the group as an «expert-friend» and given consent to study the group's work. Since the teachers wanted to engage in a learning process based on their way of thinking, my role was to provide non-threatening, non-authoritarian support, by responding to their needs rather than imposing direction, and not deliberately influencing events by dictating what they should do or how they should do it. Therefore, the teachers' learning process was completely open-ended in that they controlled and made the decisions for every aspect of it.

Three of the teachers assumed the role of group leaders and were responsible for organizing the group's meetings and activities. The group met in the school once every three weeks for about one and a half to two hours after their last class. They were able to use one half day and one full day of their school's professional development days in each term for their group work. They also organized it so that they could take turns, in small groups, to observe their *research* lessons. They also sometimes met during lunch breaks to plan and reflect on the lessons. Although the study group continued beyond the first year, the focus here is only on year one because it consisted of the key activities in the self-directed approach that framed what occurred in subsequent years.

The actual process the teachers engaged in was too complex to describe here in detail because of its non-linear nature and multiple dimensions. It involved, for example, several layers of activities, multiple voices, negotiation of meaning and process, and inquiry within inquiry. Thus, only an overview of some key components of the process around which an inquiry stance unfolded over eight months during the school year is provided.

OVERVIEW OF THE TEACHERS' INQUIRY PROCESS

When I joined the teachers, they had already spent three of their group meetings sharing and reflecting on examples of what they were doing in their classrooms to engage students in learning mathematics. Based on this process, they had decided that they wanted to learn more about inquiry-based teaching and adopting it in their practice. Thus, their overarching *puzzling situation* was what it means to teach from an inquiry perspective and the best way for them to learn about it. Two parallel processes then emerged: learning about inquiry and pursuing an as-yet undefined path to achieve their aim. This allowed them to assume an inquiry stance as they embraced uncertainty in terms of the path they would take and what they would eventually learn. The following is an overview of key aspects of the resulting process based on the decisions they made beginning with when I joined the group:

Deciding on a Pedagogical Problem. The teachers began with the *puzzling situation* of what to do to get started. They discussed this by considering possibilities such as studying relevant theory, trying out and sharing ideas individually, and doing mathematics. They agreed that a process of trying out and sharing ideas made the most sense because it was practical. However, as they discussed how this process would work, they decided that being from different grades was an issue for it to be meaningful for all of them and if they divided up according to grades, they would lose the multi-grade community they wanted to maintain. One teacher suggested, «We should think of something we can all work with that cuts across the grades.» This resulted in a discussion of what topic of common interest would relate to everyone's teaching. Someone suggested working with the new curriculum, which they pursued, but were still unsure of what was common to all of them.

At this point, they asked what I thought. I asked if they were familiar with the «front matter» of the curriculum. They were not but became curious and decided to read it for homework. The «front matter» outlined the perspectives of mathematics and learning and the mathematical processes that were required to enact the curriculum as intended. In the following group session, after three weeks to read and think about the «front matter», the teachers shared and discussed what might be meaningful to explore in relation to their practice. Their focus was on the mathematical processes emphasized throughout the curriculum (i.e., communication, connections, estimation and mental mathematics, problem solving, reasoning, and visualization). They became more interested in communication, connection and problem solving. After examining and evaluating these processes in relation to their teaching (e.g., what they did and did not do), they concluded that communication was the most meaningful for them to start with to make changes to their practice. A key reason for this conclusion was that inquiry-based communication would improve students' engagement and how they learned the mathematics. As one teacher explained, and the others agreed:

> Our students and their parents were used to doing math calculations but did not always have the experience or understand the importance of explaining and thinking through math. So it seems like a logical starting point for all levels of our learning community and our teaching.

Thus, at the end of the second group meeting, their pedagogical problem became what it meant to use communication to facilitate inquiry teaching.

Interpreting Key Construct in the Problem. Focusing on communication as the key construct to understand in their pedagogical problem and starting with their experiences, the teachers shared the types of questions they used in their teaching and how they engaged students. Some of the teachers shared ideas about questioning that they had read about. They eventually decided that it would be helpful to see what communication looked like in an inquiry lesson. They asked me for suggestions of how they could do this. I suggested a video study, which they liked, and decided to try. Some of them were aware of a Marilyn Burns' mathematics book, so they selected the Burns videos «Mathematics with Manipulatives» (Burns, 1988) from what I had access to for them to use.

The set of videos consisted of constructivist lessons that included inquiry-based learning approaches and communication in the elementary mathematics classroom. The teachers chose two of these videos, «Pattern Blocks» and «Cuisenaire Rods». Each video consisted of six lessons that covered the elementary grades. While the videos came with suggestions for use in professional development, the teachers were not interested in those guidelines. Instead, they discussed what they thought they should look for in the videos in relation to their practice. They decided to focus not only on communication but also on what they could learn about inquiry teaching. They asked if I had any advice before they looked at the first lesson. I suggested that they focus on what they could learn and use and not on being judgemental about the lesson for the sake of being critical. After clarifying what this meant, they then used the first lesson to orient their observation and record what stood out for them.

While there were many similarities in what the teachers observed, there were also differences that contributed to the variety of factors they found meaningful in the lesson. This outcome enabled them to decide on a common set of factors to focus their observations of the other video lessons. I helped them to organize these factors under broad categories that included students' role, teacher's role, questions posed by the teacher to stimulate/provoke and extend students' thinking, nature of tasks, and inquiry features of the lesson. After each lesson, they shared and built on each other's observations and used this to reflect on their own teaching in terms of what was lacking and what might be easy to begin to change. Two approaches they identified as applicable for all the grade levels were the use of groups and requiring students to share their thinking and not just give answers.

Creating an inquiry-teaching model. While the video study gave the teachers many ideas about inquiry-oriented practice and communication, they still had to decide on how to integrate these ideas into their teaching, not solely as individual techniques, but as a way of transforming their teaching. They decided they needed «a plan» – a systematic way to do this. Influenced by the structure they perceived in the video lessons, they decided to create a similar structure to guide their teaching, which they later called the inquiry-teaching model. Based on the video study and their discussions, they hypothesized that a model of inquiry teaching should include the following seven features: *free exploration, focused exploration, discussions, predictions, applications, evaluation, and extension* of the concept being taught. These are facilitated through communication, in particular, student-focused questioning by the teacher (e.g., What did you notice?) and students collaborating in small groups. Free exploration allowed students to see what they know on their own, while focused exploration involved the teacher providing a specific inquiry task.

Testing the inquiry-teaching model. In order to test their hypothesized inquiry-teaching model, the teachers planned an experimental lesson, then conducted, observed, analysed, and evaluated it. A Grade 1 teacher volunteered her classroom. The topic «explore and classify 3-D objects according to their properties» from the curriculum was selected to correspond with this teacher's schedule for the class. Based on their experience and new knowledge about inquiry, the teachers first brainstormed in small groups then shared different approaches to teaching the topic. Group 1 would: have students observe objects in the classroom; discuss why these objects have certain shapes; post pictures of objects in the real world around the classroom and use them to identify shapes; name geometric objects; make links to objects in class; refer to a chart with formal names; and have students investigate attributes and relate them to the real world (e.g., why things have certain shapes). Group 2 would: have students describe geometric objects in groups/pairs; list names of objects students suggest and descriptive words on a chart; have students build a model of one object and discuss and compare the model and an actual object; and introduce formal names. Group 3 would: pose a problem (e.g., build a house with this object); discuss attributes; have students explore attributes and classify attributes; and describe common features. Reflecting on these approaches and the seven features of the hypothesized inquiry-teaching model, the teachers sketched out the plan in Table 2 for grade 1 students' engagement in the lesson.

Brief introduction to set the tone
Free exploration of eleven 3-D geometric objects (Talk/experiment/observe in small groups)
Whole-class *discussion* of what they noticed
Individual *prediction*: Will shapes roll or slide? (using worksheet with pictures of the eleven 3-D objects and columns for rolls only, slides only and rolls and slides)
Discussion with a partner
Prediction if all will agree
Whole-class discussion of an *application* (think of self as a builder; Suppose I want to build a house on a mountain, what would I need to know about shapes?)
Focused exploration to test predictions (check with objects)
Discussion of findings with others in groups
Whole-class discussion of findings with justification and building of Venn diagram on white board with pictures
*Evaluation/*generalization (Venn diagram to sort pictures of shapes and make general statements about «What I know about 3D shapes!» 3-D vocabulary of objects)
An application *(extension)* task for homework (Look for things at home and around school that roll or slide.)

TABLE 2 — EXPERIMENTAL LESSON

Evaluating and revising the inquiry-teaching model. Following the evaluation of the lesson, the teachers discussed how well the model worked based on the level of students' engagement and learning. The Grade 1 students were «natural inquirers» and readily embraced the level of engagement of the lesson. The teachers were amazed and impressed with what the children were able to do, the richness of their thinking, and the depth of their learning of the concept. This provided evidence to support the meaningfulness and effectiveness of their inquiry-teaching model and understanding of student-focused communication. Follow up hypotheses and testing of the model involved questions that included: Does sequencing of the components matter? Are all components necessary in a lesson? Will the model work for different grades and topics? How can each of them implement the model successfully?

Table 3 highlights the key components the teachers finalized for the model. They are situated in inquiry-oriented questions the teacher must pose to prompt or challenge students' thinking. Although the teachers described it as a teaching model, it focuses on learning and learners and not the teacher, representing a significant shift in their thinking.

Students: · reveal *prerequisite knowledge* · make *predictions* · engage in *free exploration* · engage in *focused exploration* · work on *applications* · engage in *discussion*, *comparison*, *evaluation* and *reflection* of their learning · work on *extension*

TABLE 3 – COMPONENTS OF THE INQUIRY-TEACHING MODEL

Applying the Inquiry-Teaching Model. The teachers determined that their inquiry-teaching model was flexible in terms of the components to be used and how they are to be sequenced in a lesson. This conclusion allowed them to personalize how they used the model in their teaching. Planning in teams according to grade levels, they started to adopt the model in their own ways to their teaching, reporting back to the whole group and reflecting on what worked and difficulties they encountered. The difficulties included lack of depth in understanding important aspects of the mathematics they were teaching, which along with problem solving, became the focus in the second year of their study group and the basis of ongoing inquiry cycles. By the end of the first year of the study group there were significant changes in the teachers' thinking and teaching. They did acknowledge, however, that this was just the beginning of an ongoing journey toward becoming an inquiry teacher. They summarized some of the key aspects of their learning at the end of the first year as a:

deeper and more meaningful understanding of: inquiry teaching; questioning techniques for student thinking; open ended, thought provoking questions to motivate students to discuss and understand mathematics at a deeper level; student-centered strategies for listening to students and observing their problem-solving behaviours; and strategies that allow students to assume ownership of their knowledge and knowledge construction.

Theoretical framework underlying the teachers' development process

The preceding section described key activities in the first year of the teachers' self-directed professional development initiative. As these activities indicate, the process the teachers went through was an inquiry in itself because it was not predetermined. Within this process was a parallel process of inquiry into their practice and how to make it more inquiry-based. Both were dependent on the experience and knowledge they brought to these processes and the questions that emerged as the processes unfolded. Thus, both can be linked to the perspectives in Table 1, which provide a theoretical framework for interpreting the inquiry orientation of the teachers' self-directed learning approach. To illustrate this relationship, the approach is considered as being composed of an overarching inquiry cycle (Table 4) and a series of inquiry cycles (Table 5).

In Table 4, the column «teachers» represents the key components (overarching cycle) of the teachers' learning process as described in the preceding section. The cycle was initiated by a «puzzling situation» about inquiry teaching that grew out of the teachers' own experiences (practice). Each component is linked to a phase in Dewey's and Schön's processes. More importantly, each involved reflective thinking (Dewey) and reflection on action (Schön). For example, as previously described, the teachers reflected on their teaching, their elementary mathematics curriculum and the mathematical processes to decide on a pedagogical problem. They analysed their own teaching and the teaching in a video to understand communication (a key construct in the problem) to further understand the problem and generate an initial hypothesis of inquiry-based communication and teaching. In relation to Wells' perspective, each component of the teachers' process involved dialogic inquiry, i.e., beginning with personal experience and using it to obtain information to build knowledge and understanding through discussion (dialogic discourse). Similarly, in relation to Mason's perspective, *noticing* was important in each

TEACHERS	DEWEY	SCHÖN	WELLS	MASON
Beginning with practice	Becoming aware of puzzling situation in experience	Reflecting on experience	Reflecting on experience	Reflecting on experience
Posing a pedagogical problem	Generating a specific problem	Identifying a problem		
Understanding key construct in the problem	Formulating an initial hypothesis	Analyzing practice		
Hypothesizing an inquiry-teaching model	Elaborating on the hypothesis	Planning action	Dialogic inquiry	Noticing
Testing/applying the hypothesized inquiry-teaching model	Testing the hypothesis	Applying results to future action		
Revising/applying the inquiry-teaching model	Generating new problems	Generating new problems		

TABLE 4 – TEACHERS' OVERARCHING INQUIRY CYCLE

component to enable the teachers to bring to the surface issues and ideas and to recognize, label and validate choices. In addition, introspective and interspective processes were also involved as they thought about their own practice and that of the other teachers.

Table 4, then, illustrates how Table 1 can be used as a theoretical framework for the teachers' learning process. However, as is required for Table 1, the learning process was not linear in terms of moving from one component to the next in an unproblematic way. Instead, each component can be viewed as an inquiry cycle, as illustrated for three cycles in Table 5.

While Table 4 presents the macro-level of the teachers' learning process, Table 5 represents the micro-level of the first three cycles within the macro-level. The column «theoretical framework» represents a combination of the different perspectives in Table 1. Each of the «teachers' cycle» columns highlights the key components of the sub-cycles the teachers went through as they navigated their way through an undefined process. Cycle 1 corresponds to «deciding on a pedagogical problem,» cycle 2 to «interpreting key constructs,» and cycle 3 to «creating an inquiry-teaching model», wich are the components of the macro-cycle as discussed in the section on overview of teachers' inquiry process. In some cases, there were abbreviated cycles within the micro-cycles as the teachers' discussions and reflections diverged from their intended topic/problem. Such cycles were based solely on dialogic discourse and may or may not have led to a resolution.

INQUIRY FRAMEWORK	TEACHERS' CYCLE 1	TEACHERS' CYCLE 2	TEACHERS' CYCLE 3
Encountering a puzzling situation based on interests, curiosities, thinking and experiences that initiates a process	Overarching puzzling situation about inquiry teaching; what inquiry means and how to transform practice	Puzzling situation resulting from cycle 1	Puzzling situation resulting from cycle 2
Identifying problem/ dilemma through reflective action, dialogic discourse and noticing	Problem of what topic to study to resolve puzzling situation	Problem of how to learn about communication in inquiry teaching	Problem of how to apply knowledge from video study to create an inquiry-teaching model to guide teaching
Creating and elaborating on hypothesis (plan of action) through reflective action, dialogic discourse and noticing	Hypothesis that the curriculum would be a good starting point in identifying a topic	Hypothesis that studying a video would be more meaningful as the basis for learning	Hypothesizing possible components for an inquiry-teaching model
Testing the hypothesis (applying the plan) and evaluating through reflective action, dialogic discourse and noticing	Studying «front matter» of the curriculum, focusing on mathematical processes	Studying videos of inquiry-based elementary school mathematics lessons	Planning experimental grade 1 lesson based on personal experience and hypothesized inquiry-teaching model
Drawing conclusions about the outcome (new knowledge created) through reflective action, dialogic discourse and noticing	Creating knowledge of mathematical processes that meant the most to the teachers' own teaching	Creating knowledge of inquiry tasks; student and teacher roles; inquiry questions/ prompts; inquiry lesson structure	Creating knowledge of possible ways to implement inquiry-teaching model to the teachers' practice
Generating a new problem/ dilemma through reflective action, dialogic discourse and noticing	New problem on how to learn about inquiry-based teaching focused on the communication process	New problem on how to apply knowledge from video study to the teachers' own teaching	New problem on how to test and observe experimental lessons in the classroom

TABLE 5 – TEACHERS' SERIES OF INQUIRY CYCLES

Tables 4 and 5 show how the perspectives in Table 1 can provide a theoretical framework for interpreting the teachers' learning process from an inquiry of inquiry perspective. They also demonstrate the complexity of the self-directed inquiry process in terms of the layers of inquiry that can emerge. These layers of inquiry were driven by the problems, challenges, and dilemmas the teachers encountered and their desire to pursue their interests and curiosities in ways that made sense to them as they tried to achieve their goal of engaging in a self-directed learning experience to understand inquiry teaching and transform their practice. The decisions they made in each cycle shaped the nature of their inquiry process, which in turn shaped the nature of the knowledge of inquiry teaching they constructed.

Although not elaborated on in describing the cycles, at both the macro and micro levels, reflection on experience and dialogic discourse played important roles in the teachers' learning. They often returned to experience to recall or detail salient events that resulted in new possibilities. In general, they began with self by examining what they knew, did not know, and wanted to know about a particular situation of interest. Their dialogical engagement opened possibilities for conducting the inquiry and creating a community of inquirers with shared goals. They shared stories of past and present experiences that formed a source and basis for their reflection. Their discourse took various forms, including telling stories of their classroom behaviours and the students' learning of mathematics, debating issues as they took sides, sharing and critiquing specific classroom experiences, sharing relevant experiences and knowledge from other subjects they taught, and sharing knowledge/thinking about mathematics and pedagogy.

An example of how the teachers shared and reflected on their experiences involves a session that was initiated by a puzzling situation some of them experienced while trying to get students to work in groups to solve a problem. They started with sharing situations/events involving the difficulties they experienced in getting students to share their group work (i.e., the puzzling situation). This evolved into the sharing of experiences about how their students' engaged with the problems. For example, Teacher L (a grade 3 teacher) shared:

> In the fractions [lesson], I did the ground work. (…) I knew they had some knowledge of fractions because when you brainstormed, they knew stuff. I asked them what they wanted to learn and they told me they wanted to add, subtract, do this. So I knew they knew what a fraction was. But it was interesting – out of all our talks, they did not know that the fraction needed equal parts. And those were my keen, keen ones [students]. So we actually cut things up into parts that were not equal for them to see how that would not represent a fraction. (…) Then we cut up into the equal. So that was really neat. So that was a good thing that came out of it. (…) Maybe I was wrong to expect that they would know that, I don't know. But I guess that's where my disappointment was. So maybe (…) that's not the best place to do it.

After Teacher L answered some questions about what she did, Teacher K (another grade 3 teacher who taught the same topic) then shared:

> Where I thought the fraction question was going to go, it didn't go there also. They all came up with pie charts and showed the 1/3. The question was: *If you put your hand into a bag of M & Ms and took out some M & Ms and 1/3 of them are red, what would that picture look like?* So I thought, «Oh, you can get some nice pictures here! Some of them might have 24 and some of them might have 12.» No! I got a pie chart divided into three equal pieces, (*laugh*), 1/3 red, and the other two coloured green or blue or whatever colours there were, right. (...) [One group said] «You should see the M & Ms. Let's draw a hand!» (...) So they drew the hand and they drew some M & Ms. [One student explained] «It's got to be three, and I don't know why, I don't know why exactly, but it's got to be three» because you are counting by threes, right? (...) This one group eventually came up with that. (...) But the others went to the pie chart.

Other teachers also shared related experiences. For example, Teacher B (grade 4) recounted the following:

> You know, I have to say that's what happened in the lesson that I did. They were to use equations that had their chosen number in it. So they chose like say, 25. And a lot of them had figured out the skip counting. So if it was 25 plus 37, they knew to jump down to 3 and then go over 7. But if it was a subtraction, they were okay minus-ing, but then they didn't know which direction to go, and so watching them struggle with that, you know, let me know where to go with the next lesson so that they knew where subtraction went on it. So the most valuable thing that I got out of it was not what they learned, but what they hadn't learned.

In this session, a key idea the teachers learned from their sharing, reflection and discussion was that, depending on how they listened to and observed their students, they could learn from the students' thinking and actions how or where to make changes in their teaching. For example,

> Teacher A: Listening to them you can find out 'Where do I need or how do I need to improve?' or 'Where do I need to go next?'
> (...)

> Teacher C: That was the biggest part for me. (…) Like, when they all said (…) «we need this more» then you'll need to do this, so like you say –
>
> Teacher L: It's a great indicator of that, what we need to do. Yeah. (…)
>
> Teacher B: Yeah, listening to what they think and also look at what they do and not what we want them to do or to say. It can help us to help them more with how they are understanding the math.

They also became aware of how their thinking and expectations regarding their students' work could differ from the actual situation in significant ways and that they needed to be more open and flexible. This enabled them to make connections to the initial puzzling situation of how to get students to share their group work. The concern was that students did not know what to share and would share very little even with prompting. But as this excerpt of their discussion indicates, they became aware of a different way of viewing this.

> Teacher K: We usually want for the sharing to be about what they did to get the answer, what they are able to do to get the answer. So the point is not that or whether you get an answer, but when it gets to sharing, could you talk – well –
>
> Teacher A: Explain your thinking –
>
> Teacher K: Yeah, talk about your thinking, and it could be about what they can't do or don't understand.

Based on this new understanding, two new «problems» emerged from their *reflection on experience* for further inquiry: (i) What does it mean to observe and listen to students in an inquiry classroom? Initially, prior to this session, they considered it to be about what they wanted to know; now they hypothesized that it should be about learning from the children. (ii) What does it mean for students to share their work? Initially, they wanted students to get to a correct answer and share how they got it; now they hypothesized it should be about explaining their thinking, regardless of how or whether they completed the task.

In relation to the framework in Table 4, in this session, the teachers engaged in Wells' dialogic process (Table 1) by sharing experiences of practice and obtaining information from it that led to their development of new knowledge and a different understanding of their teaching. They engaged in Mason's noticing (Table 1) by reflecting on and attending to significant actions and moments in their individual and collective experiences and validating their understandings (choices) with each other. They also engaged in Dewey and Schön-like reflective thinking by reflecting in and on action to analyze their teaching. This enabled them to identify specific «problems,» formulate hypotheses, and plan actions during the session, to test and apply the hypotheses in their teaching, and to follow up with discussions/reflection, thus completing an inquiry cycle. The excerpts of their sharing presented above also show how they were beginning to develop an inquiry stance, discussed in the next section, by reflecting, noticing, then acting.

THE TEACHERS' INQUIRY STANCE

By embarking on self-directed professional development to learn about teaching through inquiry, the teachers engaged in inquiry in a way that is consistent with developing an inquiry stance. It was a journey that challenged them to confront their practice and thinking in order to make changes. When the journey began, while they all participated in the discussions and decision-making, only a few seemed reflective and open to confronting their own teaching. This changed as they started to see themselves in each other's experiences in ways that resonated or conflicted with their own thinking and practices. This prompted them to share their own stories and open up their practice for examination by themselves and others. Comparing their practice to the «front matter» of their curriculum and to the teaching/lessons in the videos they studied, and planning and testing the experimental lessons were also instrumental in helping them to learn to reflect more deeply and notice aspects of their thinking and practice that they had taken for granted. The experiences prompted them to start making changes to their teaching throughout the journey prior to the completion of their inquiry-teaching model. For example, they started trying to get students to share and justify their thinking, to work in groups, and to explore with manipulatives.

In addition to the teachers' learning process being consistent with the inquiry perspectives presented in Table 1, it was consistent with Cochran-Smith and Lytle's (1999) perspective of the inquiry stance. For example, the teachers worked together to pose significant problems relevant to their teaching and their learning, challenge the status quo in their practice and engage in ways to bring about change. They reflected on each other's work and counted on each other for alternative viewpoints. They envisioned and theorized their practice, and interpreted and questioned the theory and research of others (e.g., the curriculum, the videos, and later readings from professional journals for mathematics teachers). They embarked on a process that involved uncertainty, posing problems and dilemmas, and recognized that inquiry both stems from and generates questions. This enabled them to learn to embrace uncertainty, to become flexible, and to notice. They embraced and learned from the process in a way that placed them on a path toward ongoing development of an inquiry habit of mind to question their practice and achieve the goals of becoming «life-long learners» as mathematics teachers and meaningfully engaging their students in learning mathematics.

Cochran-Smith and Lytle (1999) also noted that «teachers (...) who take an inquiry stance work within inquiry communities to generate local knowledge» (p. 289), which they considered «knowledge that may also be useful to a more public educational community» (p. 290), i.e., local knowledge with broad implications. The teachers' «local knowledge» included meaningful ways of engaging students in communication and an inquiry-based model for teaching mathematics. In addition, with encouragement from me, by the end of the study group's second year, some of the teachers presented at teachers' conferences and were invited to conduct workshops within and outside their school system. A couple of years later, a few of them accepted appointments to be «teacher leaders» in schools that were receiving professional development funds to start study groups. Cochran-Smith and Lytle (1999) also noted that «The most significant questions about the purposes and consequences of teacher learning are connected to teacher agency and ownership» (p. 293). Since the teachers engaged in a self-directed process, teacher agency and ownership were central to the process. The teachers also talked about how much they valued the collegiality of the learning community and learning an approach that they could use for ongoing learning and growth in their teaching. In general, the teacher's learning process was effective in helping them to develop an inquiry stance in relation to their practice.

CONCLUSION

If we accept that one aspect of being a teacher of mathematics is to develop an inquiry stance, then we need to think of inquiry as more of an ongoing, recursive process of learning than is generally reflected in studies of mathematics teacher education. Developing an inquiry stance requires an attitude of openness and acceptance of the idea that learning from inquiry is not only a path with no end, but one that is also a continual source of professional growth. Being able to accept this requires that teachers develop a willingness to participate in ongoing reflection and learning as part of their everyday practice. In sum, learning from inquiry requires an attitude of openness towards one's own teaching.

This paper illustrated one way in which mathematics teachers can engage in inquiry. It is based on a process of learning *through* and *about* inquiry, which in turn leads to the development of an inquiry stance. More importantly, the paper has illustrated how four interrelated inquiry perspectives can form a theoretical framework for mathematics teachers' learning. Such a framework requires that teachers engage in an open-ended process in which they determine – or play a key role in determining – the initial topic and questions to pursue. In this process, their personal experience and practice are crucial to their learning to «interpret and theorize what they are doing» (Cochran-Smith & Lytle, 1999, p. 291). The intent here is not to imply that teachers should embark on a self-directed learning process, but that whatever the approach, if it is from the perspective of the inquiry stance or the proposed theoretical framework, the teachers' perspectives and experiences are central. Research to further explore this framework should consider both self-directed situations with an «expert-friend» and situations supported by others as in Jaworski (2006) to shed more light on the roles of the teachers and «mentors» in creating an effective process involving the inquiry stance.

REFERENCES

Biesta, G., & Burbules, N. C. (2003). *Pragmatism and educational research*. Lanham, MD: Rowan & Littlefield.

Burns, M. (1988). *Mathematics with manipulatives*. New Rochelle, NY: Cuisenaire Company of America.

Chapman, O. (2011). Elementary school teachers' growth in inquiry teaching of mathematics. *ZDM – The International Journal of Mathematics Education*, *43*(6-7), 951-963.

Cochran-Smith, M., & Lytle, S. L. (1999). Relationship of knowledge and practice: Teacher learning in the communities. *Review of Research in Education*, *24*, 249-305.

Cochran-Smith, M., & Lytle, S. L. (2009). *Inquiry as Stance, Practitioner Research for the Next Generation*. New York, NY: Teachers College Press.

Curcio, F. R. (2002). *A users guide to Japanese Lesson Study: Ideas for improving mathematics teaching*. Reston, VA: National Council of Teachers of Mathematics.

Dewey, J. (1938). *Experience and education*. New York, NY: Simon and Schuster.

Dewey, J. (1933/1971). *How We Think: A Restatement of the Relation of Reflective Thinking to the Educative Process*. (2nd revised edition). Chicago: Henry Regnery.

Even, R., & Ball, D. L. (Eds.) (2008). *The Professional Education and Development of Teachers of Mathematics: The 15th ICMI Study*. New York, NY: Springer.

Francisco, N. P., & Maher, C. (2011). Teachers attending to students' mathematical reasoning: lessons from an after-school research program. *Journal of Mathematics Teacher Education*, *14*, 49-66.

Jaworski, B. (2004). Grappling with complexity: co-learning in inquiry communities in mathematics teaching development. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17-36). Bergen: PME.

Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, *9*(2) 187-211.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education*, *7*, 203-235.

Krainer, K., & Wood, T. (Eds.) (2008). *The international handbook of mathematics teacher education: Individuals, teams, communities and networks*. Rotterdam: Sense publishers.

Lewis, C. C, Perry, R. R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: a theoretical model and North American case. *Journal of Mathematics Teacher Education*, *12*, 285-304.

Maher, C. A. (2008). Video recordings as pedagogical tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *International handbook of mathematics teacher education: Vol. 2: Tools and Processes in Mathematics Teacher Education* (pp. 65-84). Rotterdam: Sense Publishers.

Markovits, Z., & Smith, M. (2008). Cases as tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *International handbook of mathematics teacher education: Vol. 2: Tools and Processes in Mathematics Teacher Education* (pp. 39-64). Rotterdam: Sense Publishers.

Mason, J. (1994). Researching from the inside in mathematics education:

Locating an I-you relationship. In J. P. Ponte & J. F. Matos (Eds.), *Proceedings of the 18th Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 176-191). Lisbon: University of Lisbon.

Mason, J. (2002). *Researching your own practice: The discipline of noticing.* New York: Routledge Falmer.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Research Council. (2000). *Inquiry and the National Science Education Standards: A guide for teaching and learning.* Washington, DC: National Academy Press.

Ponte, J. P., Zaslavsky, O., Silver, E., Borba, M., van den Heuvel-Panhuizen, M., *et al.* (2008). Tools and settings supporting mathematics teachers' learning in and from practice In R. Even & D. L. Ball (Eds.), *The Professional Education and Development of Teachers of Mathematics: The 15th ICMI Study* (Chapter 2.3). New York, NY: Springer.

Schön, D. A. (1983). *The reflective practitioner: How professionals think in action.* Aldershot Hants: Avebury.

Schön, D. A. (1987). *Educating the reflective practitioner.* Oxford: Jossey-Bass.

Shimahara, N. K. (2002) Teacher professional development in Japan. In G. DeCoker (Ed.), *National standards and school reform in Japan and the United States* (pp. 107-120). New York, NY: Teachers College Press.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in classrooms.* New York, NY: Free Press.

Tirosh, D., & Wood, T. (Eds.) (2008). *International handbook of mathematics teacher education: Vol. 2: Tools and processes in mathematics teacher education.* Rotterdam: Sense Publishers.

van Es, E. A., & Sherin, M. G. (2010). The influence of video clubs on teachers' thinking and practice. *Journal of Mathematics Teacher Education*, 13,155-176.

Wells, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education.* Cambridge: Cambridge University Press.

Yosida, M. (2008). Exploring ideas for a mathematics teacher educator's contribution to lesson study: Towards improving teachers' mathematical content and pedagogical knowledge. In D. Tirosh & T. Wood (Eds.), *International handbook of mathematics teacher education: Vol. 2: Tools and Processes in Mathematics Teacher Education* (pp. 85-107. Rotterdam: Sense Publishers.

LEARNING AND PROFESSIONAL DEVELOPMENT OF THE MATHEMATICS TEACHER IN RESEARCH COMMUNITIES

Dario Fiorentini
dariof@unicamp.br | Universidade Estadual de Campinas (UNICAMP), Brazil

ABSTRACT

The aim of this article is to identify, describe and understand the learning and professional development of teachers who participate in research communities. To analyse and interpret the process, we have chosen the case of a mathematics teacher who, throughout her career, participated in two professional and one academic research communities. The analysis is supported by the social theory of learning in communities, although there have been adaptations for professional and research communities and for teachers. We include descriptions of the learning contexts and a narrative analysis based on the teacher's history of participation and reification in those communities. The results show that as a result of her collaboration with critical partners in academic or professional research communities, the teacher developed professionally and gained a research-oriented attitude, constantly exploring new knowledge and opportunities regarding what is taught and learnt at school. She also, changed the way she worked and interacted with her pupils and dealt with mathematical and didactic knowledge, especially in classes of pupils with learning difficulties.

KEY WORDS

Professional learning; Mathematics teacher; Research communities; Professional development; Narrative analysis.

SISYPHUS
JOURNAL OF EDUCATION
VOLUME 1, ISSUE 3,
2013, PP. 152-181

Learning and Professional Development of the Mathematics Teacher in Research Communities[1]

Dario Fiorentini

INTRODUCTION

The aim of this article is to identify, describe and understand the learning and professional development of a particular mathematics teacher from her participation and reification in research communities.

This text begins with a brief description of the context – the research community – where the learning and professional development took place. We then present the theoretical basis of this study, highlighting the social theory of learning in communities of practice. We also present a bibliographical review on learning and professional development in professional and research communities.

Next, we broach the methodological side of this study, presenting the case of a teacher-researcher who participated in three research communities. We provide a narrative analysis of this teacher's learning process and professional development, based on her participation and reifications in the research communities.

1 This study is part of the author's Productivity Grant for Research in the CNPq (PQ-ID). A first version of this article was presented at the «Seminar for the Professional Practices of Mathematics Teachers», which took place at the University of Lisbon, in Feb/2013, having received important contributions from Luís Menezes (ESE, Viseu), João Pedro da Ponte (IEUL) and other anonymous reviewers from the *Sisyphus Journal*, whom I thank. I also thank Vanessa Crecci (Unicamp) for her collaboration.

Finally, we narrate the learning process and the ways in which the teacher developed as a result of her participation. From there in narrative style, we pinpoint and analyse certain episodes that took place in the communities.

THE LEARNING AND PROFESSIONAL DEVELOPMENT CONTEXTS OF THIS STUDY

Given the aims of this article, we shall first describe the three contexts of learning and professional development covered in this study. We have characterized these contexts as research communities, since they all carried out, in a collaborative environment, studies, analyses, research and the writing of articles on the process of teaching and learning mathematics in school.

The earliest of the three is the Research Group on Pedagogical Practices in Mathematics (PraPeM), which emerged, in 1995 as an academic research community associated with Unicamp's Post-Graduate Program in Education. Its aim was to offer theoretical-methodological support to Masters' and PhD students. It is a collaborative community with a university-school relationship, as it lends itself to the shared study of schoolteacher problems and demands. The group's research has centred on two main axes. One deals with teaching and learning mathematics in schools and includes ethnographical research on everyday schooling and/or the teachers' research into their own practice. The other centres on teacher training and professional development in a context of reflection, research and collaboration among educators and teachers.

The second context emerged in 1997 from a teacher in-service programme (Specialization Course) offered by PraPeM. At this time a group of five school teachers and two PraPeM educators was set up with the aim of collaboratively supporting teacher research into their own math teaching practices in school. The group continued after the course had ended until late 1999 when they published a book, entitled *Por Trás da Porta, que Matemática Acontece*? [What Kind of Mathematics Takes Place Behind Closed Doors?] (Fiorentini & Miorim, 2001a), containing the teachers' research in the form of narrative analyses.

In this research community, the collaborative process of researching one's own practice bears similarities with the Japanese *Lesson Studies* (Doig & Groves, 2011). As with the Japanese initiative, it included: an initial phase of group lesson planning for each teacher; a second phase of implementing the lesson plan, accompanied by written records, audio recordings of classroom activities and documentation of the written student output; and a third phase of group analysis of the activities developed, where the records of the activities and the students' output were evaluated and collectively analysed, initially to plan new activities and later, to write the book (Fiorentini & Miorim, 2001b).

The third context features the Grupo de Sábado [Saturday Group] (GdS) which emerged in 1999, bringing together schoolteachers interested in studying, reflecting and researching mathematics teaching in schools; and academics (university teachers, masters' and PhD students) interested in researching the in-service teacher education process and the professional development of teachers in a collaborative context of reflection and research into teaching practice. The GdS is so named as it meets every fortnight on Saturday mornings.

Although the GdS was a subgroup of PraPeM, it was always run autonomously. Both have sought to discuss and carry out studies and provide theoretical-methodological contributions, with a socio-cultural perspective, that deal with (1) math pedagogy as a complex, multi-faceted practice involving multiple, constantly changing, dimensions; (2) the mathematics' teacher as a subject capable of producing and giving new meaning, through her practice, to her knowledge of professional activity and to her own professional development; (3) teacher training as an ongoing and always inconclusive process, which begins before a scholar obtains her degree and continues throughout her life, gaining strength, mainly through shared processes of reflection and research (Carvalho & Fiorentini, 2013). By 2013, GdS had published, along with articles in periodicals and annals, five books containing stories and research, which were, for the most part, *narrative analyses* of mathematics lessons.

These groups have been analysed in various studies. Among these, we highlight Jiménez (2002), Fiorentini *et al.* (2005) and Fiorentini (2009) who researched the learning of the GdS participants over the 12 years of its existence. A common characteristic of these communities is their heterogeneity, as they rely on the participation of school teachers, educators and university academics. This heterogeneity which never became hierarchical or unbalanced, featured participants with different knowledge and *overviews* (Bakhtin, 2003).

In relation to the future teachers, the schoolteachers, with their *overviews*, bring with them the classroom instructor's math teaching experience and their knowledge of the conditions and possibilities offered by certain tasks and teaching practices. The knowledge they mobilize and produce is based on the complexity of their teaching practice. And their teaching experience is crucial to negotiating sense and meanings for the tasks they design, analysing episodes and situations of teaching-learning, appropriating and authenticating the knowledge gleaned from classroom practice and academic research.

The university educators, in turn, have *overviews* that feature theories and methodologies from which they produce analyses, interpretations and an understanding of actual classroom practice. Their aim is to question and break down these practices for analysis. Future teachers, who began participating in the GdS in 2003 display, more than any of the other participants, their skills at using information and communication technologies and a greater proximity to and understanding of the students' reference cultures.

Before going any further, we should clarify that our intention is not to question or defend learning and research communities. According to Hargreaves and Fink (2007), not every learning community brings about empowerment or greater professional autonomy for its participants; it depends on the reasons a community is set up and the activities it engages in. For example, communities, may be monitored, controlled or moved by external agents and/or by pragmatic motives that are contrary to the emancipation of students and teachers. On the other hand, communities of empowerment and of sustainable leadership tend to construct their own knowledge and motivations, moved by political-emancipatory principles or notions of inclusion and social justice, such as improving learning for all, that is, promoting inquisitive, wide-reaching, meaningful learning for all, not just a selected few young people.

These considerations shed light on the different kinds of teacher research communities, which can be academic, school-based or somewhere in between.

Academic research communities, which are monitored/governed institutionally by the university, may be endogenous, geared towards theoretical problems and unconnected to school practices. They may be colonizers of school practices, or collaborative, open to the problems and demands of school teachers and schools. They may be able to maintain a joint study agenda, as is the case of the PraPeM group and the collaborative group that emerged from the specialization course.

School-based communities, being governed from the schools themselves, may also be endogenous, open to collaboration and partnership with the university, or wish to benefit from university participation.

The *borderline* communities are on the border between school and university and normally have more freedom of action and ability to define their own work and study agenda, since they are not institutionally monitored by the school or university. The border is a free place where interested parties from different communities can meet, venture forward, construct and question knowledge, and also carry out research. The borderline, however, is also a place of danger, a locale to transgress, a place to defy that which has been established in schools and academia. Since its participants come from various origins, the meetings tend to be interspersed with narratives of events that have occurred in the original communities. Still, what is produced and learnt in the borderline communities ends up having an appreciable impact on the personal and professional lives of each participant.

The GdS can be considered a borderline community. Although the teachers meet at the University, the meetings take place on Saturdays, a day when there are no formal academic activities or control over who attends and what is discussed. There is, however, a mutual commitment to build a pleasant study and research environment, and the freedom to suggest agendas that reflect common interests.

LEARNING IN PROFESSIONAL COMMUNITIES

From the perspective of *social learning theory* (Lave & Wenger, 1991; Wenger, 2001), all learning is situated in social practice that occurs through active participation in social community practices and the construction of identities within those communities. Knowledge in a practicing community is produced and evidenced through the shared forms of doing and understanding within the community, which results from the dynamics of negotiation involving full *participation* or *legitimate peripheral* and *reification* in (or from) the community.

In our interpretation of the theory, participation is a process whereby the members of a community share, discuss and negotiate the meaning of what they are doing, saying, thinking and producing. To participate, however, means engaging in the activity of the community; appropriating its practice, knowledge and values; and also contributing to the development of the community, especially to its members and to its repertoire of knowledge (Fiorentini, 2009).

Reification, according to Wenger (2001), means *turning into a thing*, and does not only refer to material or concrete objects (texts, tasks, manipulative materials). It also includes concepts, ideas, routines, written records and theories that give meaning to the community's practices. *Participation* and *reification* are, therefore, interdependent and essential to the learning and constitution of identities in (or of) a community.

The theory of learning situated in a community of practice, according to Lave (1996, p. 8), can be supported by four knowledge and learning premises:

> (1) Knowledge always undergoes construction and transformation in use.
> (2) Learning is an integral aspect of activity in and with the world at all times. That learning occurs is not problematic.
> (3) What is learned is always complexly problematic.
> (4) Acquisition of knowledge is not a simple matter of taking in knowledge; rather, things assumed to be natural categories, such as «bodies of knowledge,» «learners,» and «cultural transmission,» require reconceptualization as cultural, social products.

Given this theory, we posed the question: What would a teacher's learning be like in a community of mathematics teachers working in a school? What practices would be formative in that community? Within this context, in-service programmes, which focus primarily on analysing and problematizing the teaching and learning practices of the teachers involved, seem to make sense. In these types of programmes, the educators and teachers, together and collaboratively, can design teaching tasks or analyse classroom episodes, which may be videotaped, orally narrated or written down by the teachers who are taking part. Such programmes are warranted, because everyday practices (with their procedures, discussions and knowledge) are fraught with values, finalities and know-how which may be relevant to personal development, but because of their routine nature – as Foucault (1977) highlights – often become valid in and of themselves and hide deviations, ideologies and power relationships.

In the process of problematizing and denaturalizing the everyday practices of classroom teaching and learning, heterogeneous professional learning communities may be useful, especially if people with different knowledge and social practices are involved. According to Ponte *et al.* (2009), although heterogeneous communities may find it harder to construct a common language and coordinate their ideas and work methods, the different points of view and the diversity of experiences and knowledge present may empower the community even further, by promoting understanding, identifying and analysing nuances, potentials and limits in the practices the group is examining. That community diversity may, therefore, provide opportunities for more intense and meaningful learning experiences.

Ponte *et al.* (2009, p. 202), analysing three studies[2] on learning in communities of mathematics teachers, confirmed that an important and significant «variety of a learning community occurs when that community establishes itself as a community of inquiry, that is, when inquiring on some issue becomes part of the purpose of the whole group». It is, therefore, a powerful way for a community to construct knowledge and learning, as we shall see later.

LEARNING IN RESEARCH COMMUNITIES

Every research community is also a community of learning and practice. But not every community of learning, even if it is reflective, is a research community. Reflective practice differs from research practice. The latter requires a systematic procedure for treating a phenomenon or educational problem. That is, the teacher's research practice, according to Beillerot (2001) and Cochran-Smith and Lytle (2009), presupposes a methodical process of collecting and treating information concerning the phenomenon. The teacher-researcher needs, from a certain perspective (snippet, focus or research question), to make written records, organize her ideas and revise and analyse her practices. By doing so, she is seeking and producing a better understanding of her teaching. At the end of the process, she should «publicly present a final written report on the study developed» (Fiorentini & Lorenzato, 2006, p. 75).

For Jaworski (2008), the teacher who participates in an *inquiry community* does classroom research and, as part of her work, questions, explores and analyses her own teaching practice. Jaworski is a mathematics teacher who, in her studies and research, often uses the term «inquiry community». However, she believes that the term *inquiry* in the field of mathematics education holds two meanings. In one, inquiry is a teaching and learning tool, as is the case of mathematical research, or of research-teaching. In the other sense, inquiry is a way of being, in that that the identity of the individual or of the group within a research community is rooted in a form of inquiry:

2 Presented at the 15th ICMI Study: Fiorentini *et al.* (2005); Ponte & Serrazina (2005); Van den Heuvel-Panhuizen & De Goeij (2005).

> Developing inquiry as a way of being involves *becoming, or taking the role of*, an inquirer; becoming a person who questions, explores, investigates and researches within every day, normal practice. The vision has much in common with what Cochran-Smith and Lytle (1999) speak of as «inquiry as stance» – the stance of teachers who engage in an inquiry way of being (Jaworski, 2008, p. 312).

The identity constructed by teachers in a community, who reflect together and research their own practice, approximates what we have called, teaching professionalism based on a research attitude, as was espoused by Cochran-Smith and Lytle (2009, p. 57),

> The work of practitioner inquiry assumes that practitioners generate local knowledge of practice by taking an inquiry stance on both the knowledge generated by those outside the local context and the knowledge constructed through the joint efforts of practitioners working together in research communities.

This research-teaching professionalism, in the present study, becomes one of the signs of the teacher's development in a research community. This professionalism, however, cannot be defined or characterized merely as the underlying knowledge of a profession or by the professional's ability to identify and solve problems in a situation of uncertainty. It must also be seen from the perspective of the ethical-political principles and values cultivated by the professionals in a community (Fiorentini, 2009).

This raises questions about the power relationships that exist between the school community and the academic community and, especially, public policy.

> In an inquiry community, we are not satisfied with the normal (desirable) state, but we approach our practice with a questioning attitude, not to change everything overnight, but to start to explore what else is possible; to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to provide answers to them (Wells, 1999). In this activity, if our questioning is systematic and we set out purposefully to inquire into our practices, we become researchers (Jaworski, 2008, pp. 313-314).

Neither research professionalism nor research communities are born ready-made. They are built up mainly through questioning, problematizing and denaturalizing what is taught and learnt at school which apparently seems to be normal, and, later by systematically searching for an answer or a better understanding of the questioning. We understand that to change radically a school practice, it is necessary to unravel its continuity. This in not achieved by its overlapping with something new, but by problematizing or contrasting it with the traditional and current cultures of the classroom. It is during the process of problematizing current practices that the educators' input gains importance and relevance, especially at the beginning when a community is hoping to assume a research dimension. In time, all teachers who develop a research stance begin to question practices.

The student who begins a post-graduate program becomes part of academia, which is generally composed of small research communities or research groups. Here the students produce and negotiate the meaning of what they are learning and researching. They share their reflections and knowledge; they learn to produce scientific work; they are committed to carrying out research; and, to attain their aims, they use the resources and observe the requirements of the academic community. Although debates and oral communication are widely used in these situations, written language plays a prominent role as an instrument of learning and communication. In this sense, one can view the text of the master's or PhD thesis as the main *reification* of the teacher-researcher in the academic community.

PROFESSIONAL DEVELOPMENT AND LEARNING

In this study we view teacher development as a continuous process which continues throughout the person's professional life, and begins before the graduate obtains her degree. This development «happens in the multiple areas and moments of each of our lives, involving personal, family, institutional and socio-cultural aspects» (Rocha & Fiorentini, 2006, p. 146). It is, therefore, a complex process which involves the teacher as a total human imbued with feelings, desires, utopias, knowledge, values and social and political conditioning (Fiorentini & Castro, 2003).

Within this concept of professional development, the teacher is seen, according to Ponte (1998), as the principal protagonist of her own education and professional culture. She acts from the «inside out» in search of knowledge and improvement in her teaching practice.

Day (1999), referring to certain signs of teacher development, highlights that it is a process through which

> the teachers review, renew and extend their commitment as change agents to the moral purposes of teaching. It is also the means by which they acquire and develop critically the knowledge, skills and emotional intelligence essential to good professional thinking, planning and practice with children, young people and colleagues through each phase of their teaching lives (pp. 20-21).

Day (1999) understands professional development as a process involving multiple «spontaneous learning experiences,» that are indicators or markers in the teacher's development. However, the way in which teachers learn in communities has so far been the subject of little research into professional development. The indicators and markers have been noted thanks to the perceptions of the teachers themselves in interviews or oral and written narratives. They have also been taken from other studies, without being researched in detail in practice situations or in shared community analyses.

It is our belief that the circumstances and context in which a teacher learns play an important role in understanding the process of becoming a teacher and in the construction of the teacher's professionalism. The learning context may be a workshop, the classroom itself, and/or a homogeneous or heterogeneous collaborative group that discusses and analyses teaching and learning practices.

According to Cochran-Smith and Lytle (2009), teachers learn and develop professionally, when they reproduce their local, practical knowledge by participating in research communities, theorizing and linking their work to a wider social, cultural and political context. In this sense, professional development requires that the researcher come closer and draw away from that which is circumstantial or isolated in the learning process, as the process of becoming a teacher can only be perceived and understood by the researcher in a diachronic movement, that is, over the years. Oral history and written reifications by the teacher herself may help the researcher gain access to the feelings and meanings that each teacher attributes to her professional development.

Therefore, the researcher who is interested in understanding how teachers learn and develop needs to focus on isolated moments in the teacher's learning process as well as on the diachronic movement of the development process over the years. She must consider the context, practice and interactions that may have contributed to the process of becoming a teacher. Next, we will explain our research procedures in more detail.

METHODOLOGICAL ASPECTS OF THIS STUDY

The analysis of case studies involving teachers who participate in research communities is useful in identifying, describing and understanding the learning opportunities that can arise from the participation of the teacher-researcher in these communities. The case study also helps us track the milestones of the teacher's personal development.

According to the social theory of learning (Lave & Wenger, 1991), community-based learning can best be described and analysed by examining the participation and reification of the participants in that community.

Based on the above-mentioned aim and presupposition, we have chosen the following research question as the beacon for the present study: *What learning is evidenced by the teacher who researches his own practice and participates in research communities, and how does she develop professionally through participation and reification in these contexts?*

One way of researching learning in research communities, according to Lave and Wenger (2002, p. 168), is to analyse the «historical production, transformation and change in people» who participate in them and how they evolve through time and constitute their identities.

One way of understanding and describing this process is through *narrative analysis*, which, according to Bolívar, Domingo and Fernández (2001), consists of narrating an event or a person's development process by means of attributing sense and meaning, highlighting the common and unusual elements which make up the history of each subject over time. The researcher's task in this type of analysis «is to configure the data elements in a history which unite and give significance to the data, with the objective of expressing, in an authentic way, the individual's life, without manipulating the voice of the participants» (p. 110).

With the aim of developing a narrative analysis with a certain depth, we opted to examine just one case. Our subject was a Brazilian math teacher (Eliane Matesco Cristovão)[3] who had a track record of participation and reification in three communities with research characteristics.

3 The teacher herself has kindly and gladly given us permission to use her real name.

Other teachers in our data bank had also taken part in research communities, but Eliane was the only one who had participated in three. Therefore, the research we are presenting here is a case study, as «it presents individual characteristics which makes it deserving of special research» (Fiorentini & Lorenzato, 2006, p. 110). We have also adopted a qualitative approach, since the work was submitted to a process of narrative analysis, which requires an appreciable degree of interpretation.

In the narrative analysis and interpretation of Eliane's learning process and professional development, we have made use of her participation and reification in three research communities, as we mentioned at the beginning of this article. The reifications include the elaboration and discussion of teaching tasks; recordings or classroom episodes narrated or documented by the teacher; narrative analyses of lessons; published texts such as chapters from books, journal articles, and conference proceedings; master's theses; and the minutes from or recordings of group meetings, etc. We shall also use statements made by the teacher herself, and her reflections and perceptions of her community learning experiences.

In recounting Eliane's professional development, some analytical and interpretive parameters were established: her historical-cultural background and the motives that led her to participate in each research community; her problematizing and negotiation of meanings within each community; the knowledge she mobilized in teaching and learning mathematics; the questions that informed her research; the shared analysis of the teaching and learning tasks and activities; identification of the main things she learned from her participation in the community, with an emphasis on her conceptual, didactic-pedagogic and curricular knowledge; and the identification and description of the perceived changes and the professional development she underwent.

Next, in narrative form, we shall analyse and interpret Eliane's path as a learner and a professional. This will be gleaned from the multiple oral and written reifications she produced in the three study and research communities after having obtained her degree in mathematics. Although the written narrative accompanies the teacher's professional life chronologically, the reifications selected are not in chronological order. For example, for convenience sake, we used a recent reification of the teacher's to narratively analyse her early career. The last year we used in describing Eliane's career path is 2013 when she was 42 years of age and had taught for 21 years, 20 of which were spent as a middle school teacher.

NARRATIVE ANALYSIS OF ELIANE'S LEARNING AND PROFESSIONAL DEVELOPMENT

While participating in the PraPeM Research Group, which resulted in her master's thesis (Cristovão, 2007), Eliane reflected on her schooling process and career options. She stated that she had always studied at state-run schools and, from grade 9, attended night school to be able to work during the day. Influenced by her mathematics teacher, she began a night program in mathematics at Unicamp. She remembers having had great difficulty with calculus, which made her aware of her «lack of cultural and scientific baggage». She says she did not drop out of the course because she wanted to show her father that she could overcome her difficulties, as he did not place any value on studying (Cristovão, 2007, p. 7).

She started teaching in the third year of her degree program. She remembers that at the beginning she tried to emulate the best teachers she had in school and do the opposite of what the worst teachers did. This early start in teaching enabled her to do a specialization course at FE/Unicamp after she graduated. The course had a profound effect on her, and was influential in her professional development. It allowed her to carry out, as a final assignment, research into her own teaching, which was developed with the support of a collaborative group made up of four other course colleagues and two teacher educators who were the supervisors of the five teacher-students.

This group was her first research community. As she herself says, it was two years of meetings where «each of us could count on the collaboration of everyone to prepare and analyse teaching practice, to give a (new) meaning to the history of their own professional formation» (Cristovão, 2007, p. 9). The results of Eliane's first research experience (Cristovão, 2001) were published in a book, organized by the educators (Fiorentini & Miorim, 2001a).

With regard to the research process, Fiorentini and Miorim (2001b) observed that the group had adopted, an exploratory, problematizing approach with negotiation of meanings as a teaching methodology. As this was an innovative approach for the teachers, in that the students had a voice and were asked to record their mathematical ideas, thoughts, and rationales, the teachers were frequently in a panic and called the didactic-pedagogic approach into question. Hence, the group was important in promoting analysis and furthering awareness of the students' accomplishments and difficulties.

This collaborative process is well illustrated in a small episode taken from Eliane's grade 6 teaching-research project entitled «Along the Paths of a New Experience in Geometry Teaching», which covered a period of 36 class hours. In one of her first lessons, Eliane explored notions of geometry using the tangram, having asked her pupils to compare the middle triangle with the smaller square of the tangram pieces and to find a way to prove that one of them was bigger or equal to the other. Although one of her classes produced satisfactory results by folding, cutting or overlapping the figures, to prove that the area of the two figures was the same, Eliane was disappointed with the other class where quite a few students wrote nonsensical answers such as «The square because it has 4 equal sides and 4 straight lines and the triangle only has three points and three straight lines»; «The triangle is bigger than the square, its sides are longer...»; «The triangle is bigger because the angles are bigger», etc. (Cristovão, 2001, p. 63).

When the collaborative group questioned her about how she managed the activities in the two groups, Eliane remembered that in the class where the results were satisfactory, one of the pupils, right at the beginning of the activity, had asked «bigger, how's that?» She then negotiated the meaning of bigger with the students with regard to the quantity of paper or to the area of the geometric figures. This had not occurred in the other class.

The collective analysis of this episode helped Eliane learn that as well as setting constructive tasks or *mathematical challenges*, the teacher needs to observe the Potari and Jaworski (2002) teaching triad *being sensitive to what the pupils say* in their answers and meanings, challenging the students and *managing the learning* by negotiating the meanings that are required for the development of the classroom activities.

In this and other episodes Eliane interacted with other interested parties to gain a better understanding of teaching and learning mathematics. She interacted principally with the educators and university academics who, according to Bakhtin (2003), have a broad overview, of schoolteachers, various methods of teaching and learning mathematics, and the ability to link teaching with research. This, as the teacher herself has acknowledged, was crucial to her professional development.

> It was principally in these moments of discussion with the group... that I understood the importance of having someone to share the conflicts we went through when trying to be innovative. Alone it is difficult to innovate and, even more difficult to analyse the practice (Cristovão, 2001, p. 58).

The research perspective of the project required that each of the group's teachers produce a narrative analysis of her educational experience, describing in detail: the production and negotiation of meanings; the mathematical sense-making and learning of the pupils; the personal, class or school dilemmas and tensions present in the innovative process; and the new professional knowledge each teacher produced in this process (Fiorentini & Miorim, 2001b).

Participating in this research community, Eliane started to develop *research with an attitude* of questioning and analysis with regard to her teaching. This attitude is echoed in the studies of Cochran-Smith and Lytle (1999) and Jaworski (2008). This research practice/stance can be inferred from Eliane's analyses of the relationship between her teaching practice and her research practice.

> It is difficult being a teacher and a researcher at the same time. In this educational experiment I tried to reconcile the two things. My greatest concern as a teacher: to teach and learn geometry with understanding and pleasure. As for being a researcher: to analyse and understand the process of teaching and learning when we prioritize a practice of production and negotiation of meanings (Cristovão, 2001, p. 45).

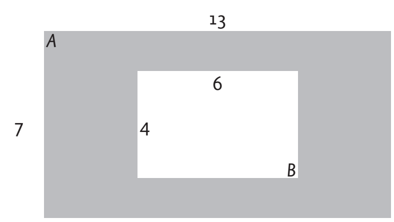
Years later, Eliane theorized about the experience in which she shared the process with colleagues and wrote about herself as a way to reflect on and research her teaching practice. When she looked back, she realized that it had contributed to her developing a critical and questioning eye with regard to her lessons. It was yet another landmark in her professional development:

> That experience began the process that formed my very research attitude. It was when I began to understand the importance of sharing our classroom experiences by writing them down, and how the process of writing and rewriting allows us to reflect on our pedagogical practice. After the book was written, my awareness in the classroom, especially of the students, became more critical and questioning (Cristovão, 2009, pp. 18-19).

Another significant experience for Eliane was her participation in the Grupo de Sábado starting in 2003. Her principal motivation in joining the group arose from her previous participation in the community that wrote *Por Trás da Porta, que Matemática Acontece?»*, which she recognized as being a period of great learning.

She immediately identified with the practices of the GdS, recognizing that in the new community, she would be able to resume the processes of reflection and research into her practice in an environment of collaboration, using written, narrative analyses of mathematics lessons. Her recognition of the benefits of being in the GdS can be seen in the episode we relate below.

The first time she participated in the GdS, the group was questioning the concept of perimeter. A colleague named Rogério gave the group a task to complete (Figure 1).



What is the perimeter of figure *A* which is hollowed out by rectangle *B* (measured in cm)?

FIGURE 1 – TASK SET BY ROGÉRIO (EZEQUIEL, 2003, P. 32).

Eliane, some of her GdS colleagues and all of Rogério's grade 8 pupils found a measurement of 40cm for the perimeter of Figure A. Others, however, obtained 60cm, including the inside perimeter.

From these results, the group began to negotiate the meanings of perimeter. Some came up with the hypothesis that the tasks and definitions used in school textbooks created an incorrect understanding of perimeter, giving rise to what Brousseau (1986) called an «obstacle of didactic origin». With his 8^{th} grade students, Rogério had also done a little research on how textbooks present tasks and definitions regarding perimeter. He confirmed that definitions such as «the perimeter is the sum total of the sides of a geometric figure» or that it is «the measurement of the contour of a geometric figure», as well as certain tasks involving figures that are not hollowed out, led to an imperfect understanding of the perimeter concept.

Meanwhile, Eliane and the academics in the group questioned how Rogério had set the task with the hollowed out figure. They argued that the way the task had been set was configured like a «trick», in which figure B could be considered overlapping figure A. This did not promote problematizing and recognition of the inside perimeter. Eliane felt motivated to set a task that could explore or problematize the meaning of perimeter, putting a dent in the meaning that had been accepted by both teachers and students.

With this aim, she brought presented the group with a task with involving various geometric figures, some of them which were non-conventional (Figure 2). Her hypothesis was that, by negotiating meanings in the group, the pupils would manage, by means of negotiating meanings in the group, to reach a better solution and meaning definition for perimeter, remaking, by doing so, the concept they had of perimeter, thus reformulating the whole concept. The figures, in the first elaboration version of the task, contained the measurements of the sides. On being questioned by the academics about the need or relevance of this information, Elaine opted not to give this information specifically. This lesson is discussed in a narrative analysis of that experiment, in which she writes about how she justified making that option. She states that it, which is «part of the training process of the pupil, who learns to obtain the data of a problem-situation, therefore thus breaking with the *facilitation pedagogy*, that is, the pedagogy that gives hands everything over to the students already chewed up and digested over to the pupils» (Cristovão, 2003, p. 36).

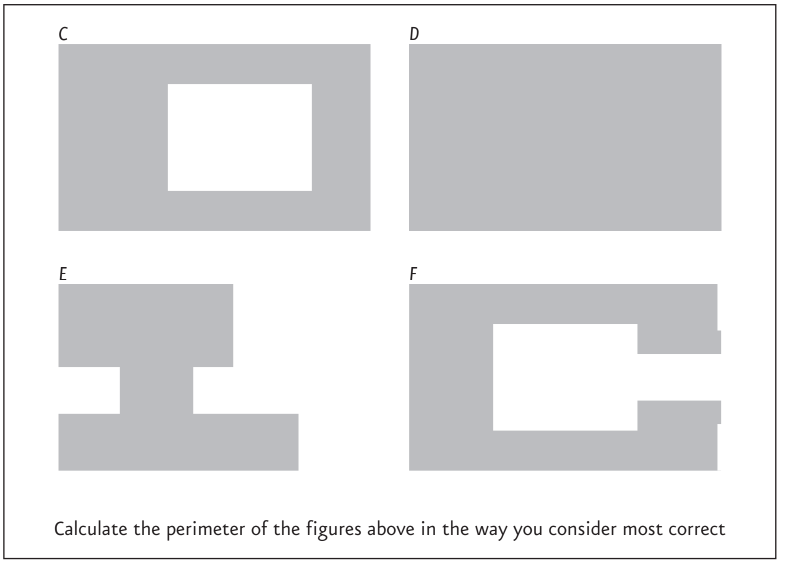


FIGURE 2 – TASK SET BY ELIANE (CRISTÓVÃO, 2003, P. 35).

We should point out that the expression *facilitation pedagogy* was introduced and orally reified in the group in 2000 and was incorporated into the GdS's discursive repertoire. It was frequently used as a way of questioning and denaturalizing the practice of mechanically using procedures or tips and was aimed at facilitating student performance.

Although the expression was already being used in the group, Jiménez Espinosa (2002), just two years after its emergence, was the first group participant to reify it in writing in his doctoral thesis, where he analysed one of the group's discussions on the meaning of facilitation pedagogy. In short, Jiménez pointed out that schoolteachers saw it as a «pedagogical culture that can create in pupils a mechanical attitude with little reflection in the light of knowledge», while the academics viewed it as obstacles of didactic origin which, according to Brousseau (1986), are simplifications used by teachers to help their students memorize a fact or computation procedure or solve a problem (Jiménez, 2002, p. 144).

Meanwhile, in the episode being analysed, Eliane had access to the expression from Rogério's narrative, as she had been a participant of the GdS since the year 2000. Eliane shows that not only did she appropriate this group's reification, but she also went on to use it in her own narrative analysis, having, however, produced her own reification: «facilitation pedagogy (...) is that which hands everything over to the students already chewed up and digested».

Looking back at Lave and Wenger (1991), we can say that, on authenticating her reification and her option not to give the measurements of the sides of the geometric figures (Figure 2), the GdS, from the very first meeting, recognized Eliane as a legitimate member of the research community, someone who identified with the practices of the group, and a person who stood to contribute to the community's development.

Applying the task that had been authenticated by the group (Figure 2) to her students, Eliane opted to cut out the figures in cardboard, so that her students would not construe the hollowed out part of figure C as one figure overlapping the other.

The results were positive and multi-faceted, and here we highlight geometric figure C. Some groups added the external and internal perimeters, giving a correct, single result for the perimeter. Others gave two separate results: the inside and the outside perimeter. Some groups only added up the outer sides, and one group presented the difference between the external and the internal perimeter as a final result.

To systematize the didactic experiment, and after discussing and negotiating the results and meanings with the whole class, Eliane used a questionnaire in which the students were to answer what a perimeter was and how it is calculated when the figure is hollowed out or cut out. She presented the results to the GdS in the form of a written narrative analysis. After collecting reflections and analyses produced with the help of the group, a second GdS book (Cristovão, 2003) was published.

In the GdS, Eliane co-authored four books for the group. She published five narrative analyses of math lessons and three essays on systematization dealing with writing in mathematics, research classes. She also took part in the development of a research stance for math teachers. Her output demonstrates how rich and fraught with reifications her participation in GdS was.

Her activities in the two research communities described here and the path of her student and professional life helped Eliane to become constructively critical in relation to school practices, and sensitive and committed to children with learning difficulties, who are in danger of dropping out or failure. She believed that these children were capable of learning and that, with better public support policy and greater appreciation for the teachers' work, both the teacher and the school would be able to design alternative pedagogies to help these pupils become protagonists and subjects of learning. It was this conviction that motivated her to work toward her master's degree.

When she started the master's in 2005, she took this issue up with the PraPeM academic group, where she studied and dealt with the issue on a theoretical-pedagogical level. Based on the works of João Pedro da Ponte and Ole Skovsmose and on studies produced by the GdS and PraPeM, she became convinced that it was necessary to break with the exercise paradigm and promote open tasks and activities of an exploratory-research nature.

Other authors were also important to her understanding of the problem. Charlot (2000), for example, helped her to understand that «school failure does not exist, what exists is failed pupils, situations of failure, school stories that ended badly» (Cristovão, 2007, p. 43). Luiz Carlos de Freitas helped her to understand «the causes for exclusion experienced not just by her pupils, but also by a good number of those attending state schools» (Cristovão, 2007, p. 43) without, however, pointing at pedagogical alternatives in order to remedy the problem.

Michel de Certeau (2007) enabled her to understand that there are always possibilities of intervening and bringing about change in the daily practices of a school. The awareness of these possibilities merged with the knowledge she had gained in the two practice communities which in turn led her to question certain suppositions and academic stances, as witnessed by the following statement: «I couldn't sit with my arms folded and wait for what Freitas called the «historical transforming project of the organizational foundations of the school and of society» (Cristovão, 2007, pp. 40-41).

The study of the pedagogical possibilities that promote the inclusion of students with math learning difficulties became the focus of her master's research. As there were Course Recovery classes for Cycle II[4], she partnered with two other teachers in designing an intervention project for her classes.

With her partners and other interested teachers she built up a local collaborative group – Grupo Colaborativo de Estudos em Educação Matemática [Collaborative Study Group in Mathematical Education] (GCEEM). However, for her master's research, she interacted with three collaborative groups (PraPeM, GdS and GCEEM). With the GCEEM she planned a series of exploratory-research tasks and lessons for the two classes. She worked together with the teachers in the classroom and acted simultaneously as a teacher and researcher in this experiment.

As a researcher, with the support of PraPeM, she looked for answers to the following question: What possibilities and contributions can exploratory-research practice involving the collaboration of a group of teachers, bring to the teaching processes and the math education of students in RCII classes that shows evidence of their school inclusion?» (Cristovão, 2007, p. 24).

She called this practice of teaching and research in the two classes **research-action of the 1st order**, as she shared and analysed the educational experiment with her partners and with GCEEM, also including time for discussion and analysis with the GdS and PraPeM.

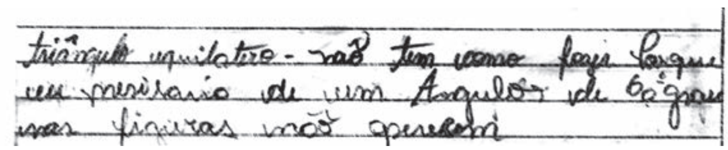
The *meta-analysis* of this research carried out later just by Eliane, with the support of PraPeM, was dubbed **research-action of the 2nd order**. In this meta-study, she obtained evidence of the social inclusion of pupils through exploratory-research activities. Due to the quantity of material collected, she only analysed the experiment with the class of Teacher RE. The material for analysis was obtained from audio and video recordings, the pupils' portfolios, questionnaires, narratives of the teacher partners and the researcher's field diary.

She chose five approaches with which she developed analyses and interpretations of the output and relationship the students established with mathematical knowledge, with themselves, with others and with the teaching and learning process. These were: the mathematical output of the pupils; mobili-

4 In Brazil, Recovery classes in Cycle II are made up of pupils at the end of their Fundamental Schooling (currently grade 9), whose performance was not satisfactory enough to move on to middle school.

zation and (re)significance of their knowledge; their changes of attitude and stance; the role and active participation of the pupils; and their resistance and *negatricities*.

She carried out various exploratory-research activities in a grade 9 class in partnership with teacher RE, who was held back to recover the cycle. The following are two episodes dealing with geometry. The task aimed to review and explore the different types of triangles and the possibility of constructing them from an isosceles triangle (non-equilateral) on which they could only draw one section of straight line. There were several answers, but Eliane highlighted the justification given by students Gi and Ta on the impossibility of constructing an equilateral triangle from a non-equilateral isosceles triangle[5] with only one line.


triângulo equilátero - não tem como fazer porque
eu necessário de um ângulo de 60° graus
nas figuras não aparecem

Another episode demonstrates the students' creativity and Eliane's ability to negotiate the authentication of the solution. It involved interpreting what two pupils (Da and Em) did with a task that Eliane set, inspired by an idea suggested by Ponte, Brocardo and Oliveira (2003) called «folds and cuts.» The pair understood that it was impossible to obtain a scalene triangle with two cuts on the folded sheet. So they initially made a cut in the lower right hand corner of the folded sheet and unfolded the cut part, obtaining an isosceles triangle. On this they made the second cut, obtaining a scalene triangle.

Seeing that Da and Em had managed to obtain a scalene triangle, Eliane and RE went to check how they had done it:

Eliane: How did you manage to make the shape?

Da: Two cuts, yeah? Now the one with the different sides, look…a cut, beautiful? [She folds the paper in half, makes a cut in the corner, as **Em** had done before, and begins to open the paper]

Eliane and RE: But it's all with it folded!

5 Transcript of the manual record: «Equilateral triangle – can't be done (built), because I'd need (at least) an angle of 60° (to then work with the other two, with a line, but) none appear in the figures» (Cristovão, 2007, p. 99) (my brackets).

Da: No, no! Here you're not saying it's just folded, the... [Pointing to item 2 of the task]

Eliane: [She reads the task sheet and agrees]...You're right...it should have said here: a sheet of paper

Eliane took these episodes to the GdS and PraPeM groups where other interpretations were added. In line with Bakhtin (2003), those with an overview were mobilized to understand the process of teaching and learning in children with significant learning difficulties. In PraPeM, for example, one of the educators found the concept of *negatricity* of the multi-referencing theory. He referred to the «incredible capacity» of the pupil to «shake things up, responding in an unpredictable and different way from the objectives outlined in our training action» (Borba, as cited in Cristovão, 2007, p. 10).

From this study the researcher concluded, among other things, that:

> Pupils of the RE are not mass consumers, who accept everything they are told. To work with them requires a change of attitude on the part of the teachers and the managers who need to see in them not just rebels who have no output, but consumers who are critical of the knowledge that is offered to them (Cristóvão, 2007, p. 15).

While Eliane was appropriating knowledge from the academic community to produce other meanings for teaching practice, she was also questioning the academic literature for not opening the possibility of other meanings and for not recognizing the complexity and richness of school practices and the students' own knowledge. Since she had experienced several episodes in which the students had surprised the teachers with their answers and creative, out-of-the-box solutions, Eliane concludes by posing a number of issues:

> Is it that students in a school failure situation do not produce knowledge or is it that they merely do not accept a rigid and closed school system where everything has to be done as prescribed and within a given time? Could it be that if we gave them more freedom to show their creativity, they would surprise us? Can school allow this type of work? In the RCII, wouldn't this be a path toward repairing these pupils' self-esteem and making them believe they are capable of learning mathematics? On interpreting EM's and DA's attitudes as the ability to argue against a set rule, we can appreciate even more

> the use of an approach that would allow them to be the subjects of learning (Cristovão, 2007, p. 104).

Eliane's sensitivity to her students' learning styles brings us back to Lave (1996) when he affirms that learning is not a problem for pupils who engage and participate in educational activities. They always learn something in the process. And that something is complex and very often hidden, since it might not coincide with the aim of the instruction.

Years later, taking stock of her professional learning in collaborative groups and research communities, Eliane commented that in these communities: we research and share classroom experiences; we count on various views to better understand those experiences, their richness and their limitations; we find support to face our problems and challenges, searching for reading material and theoretical bases which meet our needs; we analyse and write about our practice; by writing it down, we reflect and provoke collective reflections, impacting other teachers; we become critical by not merely reproducing external suggestions and recommendations (academia and public policies); we become capable of constructing our own paths, of being authors of our own practice and our own ideas; we look for the personal and professional development that we want and believe in, bearing in mind our commitment to the quality of teaching that we consider to be the most suitable for our students (Cristovão, 2009).

The research carried out by Eliane, first in professional communities and later in an academic community, made her, in Day's (1999) words, an «agent of change», engaged in reviewing, renewing and broadening her commitment to the emancipatory suggestions of children and young people. It also gave her the authority and skill to question the knowledge of others outside the local context, an aspect Cochran-Smith and Lytle (2009) broached in their work. Eliane's professional development became even more evident when she and another colleague from the GdS led a movement of protest and resistance to the Education Secretary's curricular policies in 2008 and 2009. The new policy forwarded a proposal for the entire São Paulo public network, whereby bonuses would be given to teachers whose students did the best on standardized tests. This policy was implemented without consulting the teachers and without factoring in different realities and local necessities.

Led by Eliane and her colleague, the GdS community, mindful of the results of the studies carried out by the group, opposed this *homogenizing* policy (that featured hand-outs containing ready-made lessons for the teachers to apply), and called for conditions for teachers to organize into groups and communities, so that they could design and implement projects to improve teaching, based on an assessment of local needs. The GdS claimed that, in addition to supporting these groups, the State, should make it viable for universities to take part by mobilizing educators and future teachers to act in partnership with practicing teachers.

Starting in 2008, Eliane also began working in higher education, motivated by the desire to share what she knew with future teachers. In 2011, when she began her PhD program, she stopped teaching school because her study grant was more than her salary as a school teacher. In the first semester of 2013, seeking work stability and a schedule that would be compatible with an academic career where she could also do research, she passed the test and became a math education teacher at the Universidade Federal de Itajubá.

CONCLUSIONS AND FINAL CONSIDERATIONS

The narrative analysis of learning and professional development shown in this study illustrates that the process of becoming a teacher-researcher in research communities is unique, unusual and complex. So much depends on the practices promoted by these communities, the conditions, and the inclination of each teacher to participate and throw herself into the educational experience of working, studying and researching.

Eliane's participation and reification is evidence that her initial participation in two research communities with a more professional orientation motivated her to enter a community that was more academically oriented. Here she could see the opportunity to further her understanding of issues relating to her school teaching.

After finding an academic community that was open to this type of problematizing, Eliane was able to discern other opportunities for pupils who were considered weak in math. One angle was to engage them in exploratory-research activities, in which the students, in small groups, took mathematical hypotheses and conjectures which they authenticated, tested and later reified in short written reports that were presented to and authenticated by the whole class.

Supported by authors such as Bernard Charlot, Michel de Certeau, João Pedro da Ponte and Cochran-Smith and Lytle, and by collaborating with school-based research partners and crucial partners in the PraPeM and GdS communities, Eliane carried out a positive, incisive study on the progress of pupils who were considered to have failed or were seen as having difficulties in mathematics. This analytical, interpretative study also featured important concepts such as: exploratory-research activities; *negatricity*; school failure; school inclusion/exclusion; negotiation of meanings, which, given the evolution and intellectual development of pupils with learning difficulties, became a way for the school to promote the inclusion of young people with different mind-sets.

As far as the research process was concerned, the *narrative analysis* proved to be an important methodological tool in describing learning situations and providing signs of the teacher's professional development. It melded interpretations and meanings for the researcher and the teacher being researched concerning events that impacted her classroom teaching and professional education over the years. In fact, the narrative analysis on Eliane's trajectory throughout the research communities proved that her participation and reification in those communities had exercised a fundamental role in the understanding and transformation of her teaching practice and professional development. This was especially true when it came to the construction of her way of being and working in the profession, all of which highlights her professionalism, which was born of reflection and research.

Eliane's attitude toward research, an expression of professionalism that she developed within the research communities, highlights and clarifies what Jaworski (2008) and Cochran-Smith and Lytle (1999, 2009) said about the concept. In short, research as an ingrained attitude toward one's own teaching professionalism can be witnessed in the way one permanently questions, problematizes, documents, analyses and gives fresh meaning to one's own pedagogical practice and that of others in a professional or academic research community, thus valuing the overview of critical partners even when one is not intentionally doing research.

It was not just Elaine who developed professionally. The community itself developed and continues to develop, as it produces and presents its research, and interacts with other communities, thus forming a broader learning network. This allows local communities to acquire power and recognition from the wider educational community and subsequently gain the clout to negotiate the course of education with society and the State. This is precisely what occurred with Eliane and the GdS community when they led the protest movement against the curricular policies that had been imposed by the Secretary of State for São Paulo.

The study demonstrates advances and evidence that it is important for the teacher to research her practice and participate in research communities. It provides a rich context for learning and professional development and a way to improve teaching practices, academic achievement, school culture and public policy in Brazil.

Yet the powers that be have still not come to value this type of professional. That was one of the reasons Eliane opted to work exclusively in higher education. With the exception of a few federal schools – where the teachers have a salary and work, study and research conditions equivalent to those of university teachers – the great majority of Brazilian schools continue not valuing the teacher who wants and likes to examine both the finer points and brass tacks of her own teaching. Unfortunately the normal school setting in Brazil has become a no man's land for teachers wishing to enhance their professionalism through research.

REFERENCES

Bakhtin, M. M. (2003). *Estética da criação verbal*. São Paulo: Martins Fontes.

Beillerot, J. (2001). A pesquisa: esboço de uma análise. In M. André (Ed.), *O papel da pésquisa na formação e na prática dos professores* (pp. 71-90). Campinas: Papirus.

Bolívar, A., Domingo, J., & Fernández, M. (2001). *La investigación biográfico-narrativa en educación: enfoque y metodologia*. Madrid: La Muralla.

Brousseau, G. (1986). Fondements et méthodes de la didactique des mathématiques. *Recherches en Didactique des Mathématiques, 7*(2), 33-115.

Carvalho, D. L., & Fiorentini, D. (2013). Refletir e investigar a própria prática de ensinar aprender Matemática na escola. In D. L. Carvalho *et al.* (Eds.), *Análises Narrativas de Aulas de Matemática* (pp. 11-23). São Carlos: Pedro & João Editores.

Charlot, B. (2000). *Da Relação com o saber: Elementos para uma teoria*. Porto Alegre: Artmed Editora.

Cochran-Smith, M., & Lytle, S. L. (1999). Relationships of knowledge and practice: teacher learning in communities. *Review of Research in Education, 24*, 249-305.

Cochran-Smith, M., & Lytle, S. L. (2009). *Inquiry as stance: practitioner research for the next generation*. New York: Teacher College Press.

Cristovão, E. M. (2001). Pelos caminhos de uma nova experiência no ensino de geometria. In D. Fiorentini & M. A. Miorim (Eds.), *Por trás da Porta, que matemática acontece?* (pp. 45-82). Campinas: Editora Gráfica FE/Unicamp.

Cristovão, E. M. (2003). E o perímetro de pegou! In D. Fiorentini & A. Jiménez (Eds.), *Histórias de Aulas de Matemática* (pp. 35-39). Campinas: Editora Gráfica FE/Unicamp.

Cristovão, E. M. (2007). *Investigações matemáticas na recuperação de ciclo II e o desafio da inclusão escolar* (Master thesis). FE/Unicamp, Campinas.

Cristovão, E. M. (2009). O papel da colaboração na construção de uma postura investigativa do professor de matemática. In D. L. Carvalho & K. C. Conti (Eds.), *Histórias de colaboração e investigação na prática pedagógica em matemática: ultrapassando os limites da sala de aula* (pp. 131-148). Campinas: Alínea.

Day, C. (1999). *Developing teachers: the challenges of lifelong learning*. London: Falmer Press.

De Certeau, M. (2007). *A invenção do cotidiano*. Petrópolis: Vozes.

Doig, B., & Groves, S. (2011). Japanese Lesson Study: Teacher Professional Development through Communities of Inquiry. *Mathematics Teacher Education and Development, 13*(1), 77-93.

Ezequiel, R. S. (2003). Perímetro interno ou externo? In D. Fiorentini & A. Jiménez (Eds.), *Histórias de Aulas de Matemática* (pp. 31-34). Campinas: Editora Gráfica FE/Unicamp.

Fiorentini, D. (2009). Quando acadêmicos da universidade e professores da escola básica constituem uma CoP reflexiva e investigativa. In D. Fiorentini, R. C. Grando & R. G. S. Miskulin (Eds.), *Práticas de formação e de pesquisa de professores que ensinam matemática* (pp. 233-255). Campinas: Mercado de Letras.

Fiorentini, D., & Castro, F. (2003). Tornando-se professor de Matemática: O caso de Allan em Prática de Ensino e Estágio Supervisionado. In D. Fiorentini (Ed.), *Formação de professores de Matemática: Explorando novos caminhos com outros olhares* (pp. 121-156). Campinas: Mercado de Letras.

Fiorentini, D., & Lorenzato, S. (2006). *Investigação em Educação Matemática: percursos teóricos e metodológicos*. Campinas: Autores Associados.

Fiorentini, D., & Miorim, M. A. (Eds.) (2001a). *Por trás da porta, que Matemática acontece?* Campinas: Editora Gráfica FE/Unicamp.

Fiorentini, D., & Miorim, M. A. (2001b). Pesquisar & escreve também é preciso: a trajetória de um grupo de professores de matemática. In D.

Fiorentini & M. A. Miorim (Eds.), *Por trás da porta, que Matemática acontece?* (pp. 12-37). Campinas: Editora Gráfica FE/Unicamp.

Fiorentini, D., Miskulin, R., Megid, M. A., Brum, E. D., Gama, R., Melo, M., ... & Passos, C. (2005). Learning through collaboration from professionals with different knowledges. In *XV ICMI Study* (Vol.1, pp. 1-6). Águas de Lindóia, SP, Brazil.

Foucault, M. (1977). *A Vontade do Saber*. Rio de Janeiro: Graal.

Hargreaves, A., & Fink, D. (2007). *Liderança sustentável: desenvolvimento de gestores da aprendizagem*. Porto Alegre: Artmed.

Jaworski, B. (2008). Building and sustaining inquiry communities in mathematics teaching development: Teachers and didacticians in collaboration. In T. Wood & K. Krainer (Eds.), *International handbook of mathematics teacher education: Participants in mathematics teacher education: individuals, teams, communities, and networks* (Vol. 3, pp. 309-330). Rotterdam: Sense Publishers.

Jiménez, A. (2002). *Quando professores de Matemática da escola e da universidade se encontram: ressignificação e reciprocidade de saberes* (PhD dissertation). FE/Unicamp, Campinas.

Lave, J. (1996). The practice of learning. In S. Chaiklin & J. Lave (Eds.), *Understanding practice: Perspectives on activity and context* (pp. 3-32). New York: Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: legitimate peripheral participation*. New York: Cambridge University Press.

Lave, J., & Wenger, E. (2002). Prática, pessoa, mundo social. In H. Daniels (Ed.), *Uma Introdução a Vygotsky* (pp. 165-173). São Paulo: Edições Loyola.

Ponte, J. P. (1998). Da formação ao desenvolvimento profissional. In *Actas do Profmat98* (pp. 27-44). Lisboa: APM.

Ponte, J. P., Brocardo, J., & Oliveira, H. (2003). *Investigações Matemática na Sala de Aula*. Belo Horizonte: Autêntica Editora.

Ponte, J. P., & Serrazina, L. (2005). Understanding and transforming practice: A Portuguese experience. In *XV ICMI Study* (Vol.1, pp. 1-5). Águas de Lindóia, SP, Brazil.

Ponte, J. P., Zaslavsky, O., Silver, E., Borba, M. C., van den Heuvel-Panhuizen, M., Gal, H., Fiorentini, D., & Chapman, O. (2009). Tools and settings supporting mathematics teachers' learning in and from practice. In R. Even & D. L. Ball (Eds.), *The Professional Education and Development of Teachers of Mathematics*: The 15th ICMI Study (pp. 185-210). New York, NY: Springer.

Potari, D., & Jaworski, B. (2002). Tackling complexity in mathematics teaching development: Using the teaching triad as a tool for reflection and analysis. *Journal of Mathematics Teacher Education, 5*, 351-380.

Rocha, L. P., & Fiorentini, D. (2006). Desenvolvimento profissional do professor de Matemática em início de carreira no Brasil. *Quadrante, 15*(1-2), 145-168.

Van den Heuvel-Panhuizen, M., & De Goeij, E. (2005). Offering primary school teachers a multi-approach experience-based learning setting to become a mathematics coordinator in their school. In *XV ICMI Study* (Vol.1, pp. 1-6). Águas de Lindóia, SP, Brazil.

Wenger, E. (2001). *Comunidades de práctica: aprendizaje, significado e identidad* (Original in English, 1998). Barcelona: Paidós.

ANALYSIS OF TEACHING AND LEARNING SITUATIONS IN ALGEBRA IN PROSPECTIVE TEACHER EDUCATION

Neusa Branco
neusa.branco@ese.ipsantarem.pt | Instituto Politécnico de Santarém & Universidade de Lisboa, Portugal

João Pedro da Ponte
jpponte@ie.ulisboa.pt | Universidade de Lisboa, Portugal

ABSTRACT

This paper presents a teacher education experiment that was conducted in an algebra course based on an exploratory approach and articulating content and pedagogy. We investigate the contribution of analysing teaching and learning situations, namely student answers and episodes of classroom work, in developing the mathematical and teaching knowledge of prospective primary school teachers. We use a design research methodology to probe the prospective teachers' development after having participated in an experiment in their third year of a primary education degree program. The results show that the prospective teachers' understanding of algebra and grasp of how to use different representations and strategies grew considerably. The results also show that their didactical knowledge regarding tasks, classroom organization, attention to students' reasoning, and teacher's questions grew as well. The variety of tasks proposed to the prospective teachers during the course was of vital importance to this outcome, as was the opportunity to reflect, work with elements of real practice, and participate in whole class discussions.

KEY WORDS

Teacher education; Algebra; Algebraic thinking;
Mathematics; Teaching experiment.

SISYPHUS
JOURNAL OF EDUCATION
VOLUME 1, ISSUE 3,
2013, PP. 182-213

Analysis of Teaching and Learning Situations in Algebra in Prospective Teacher Education[1]

Neusa Branco | João Pedro da Ponte

INTRODUCTION

The preparation of basic education (1st to 6th grade) prospective teachers must take into account that, when they will start teaching, they will face challenges and demands with regard to algebraic thinking that most of them did not experience as students. Kaput and Blanton (2001) acknowledge that these teachers went to school during a shift in the process of algebra learning and teaching, having had few experiences with generalization and formalization activities that, in their perspective, must form the basis of students' work in school. In this changing context, teachers face many challenges that must be addressed in the education of prospective teachers. Canavarro (2007), for example, talks about the need for teachers to value students' reasoning, to know how to select tasks, and to promote a classroom dynamic that is conducive to the development of their students' algebraic thinking.

In order to be a pre-primary or primary school teacher nowadays in Portugal teacher candidates must have a bachelor's degree in primary education and a master's degree that qualifies them to teach. This research focuses on a teacher education experiment in algebra targeting third year students in

1 This work is supported by national funds from the FCT – Foundation for Science and Technology, through the project *Professional Practices of Mathematics Teachers* (contract PTDC/CPE-CED/0989311/2008).

the bachelor's program in primary education. The quality and effectiveness of teacher education largely determines how future teachers will perform later on. This is why teacher education must include experiences that develop the knowledge and skills required to spark their students' learning.

In the education of a prospective mathematics teacher it is important to articulate mathematical and didactical knowledge (NCTM, 2000; Ponte & Chapman, 2008). This paper discusses a teacher education experiment in algebra that promotes a strong link between content and pedagogy. The goal of our experiment was to analyse and discuss – then come to conclusions – regarding how the presentation of teaching and learning situations affects the mathematical and didactic knowledge of prospective teachers. The experiment followed an exploratory approach, with especial emphasis on analysing teaching in the primary school classroom and the students' responses.

ALGEBRA IN PRIMARY EDUCATION

Over the last decade, several researchers (e.g., Blanton & Kaput, 2005; Carpenter, Franke & Levi, 2003; Carraher & Schliemann, 2007; Kieran, 2004; Lins & Kaput, 2004) and curriculum documents (ME, 2007; NCTM, 2000) have advocated encouraging algebraic thinking starting from the early school years. They claim that doing so, using both mathematical and non-mathematical contexts provides learning experiences in early schooling that will carry over to learning algebra with understanding in the students' future years. Thus, the emphasis is on meanings and understandings (Canavarro, 2007; Kaput, 1999). Kieran (2007) argues that algebra should be regarded «as a way of thinking and reasoning about mathematical situations» (p. 5), and that it cannot be seen as merely a collection of techniques. Approaching algebra in this manner thus leads to a deeper understanding of mathematics and promotes the articulation among different mathematical topics.

In order to promote algebraic thinking, attention must be given to the objects and to the relationships among them with the teacher and her students representing these relationships and engaging in wide-ranging reasoning about them (Ponte, 2006). Generalization involves analysing similarities among given situations and/or analysing regularities, procedures, structures and relationships between situations that form new objects (Kaput, 1999). It assumes a pivotal role in the development of algebraic thinking (Blanton & Kaput, 2011; Kieran, 2007). Situations that broach algebraic thinking as «an activity of generalization of mathematical ideas, using literal symbolic representations, and representing functional relationships» (Blanton & Kaput, 2011, p. 6) have now been included in the primary school classroom.

The way algebra has been addressed over the years has undergone widespread changes. The approach that prevailed for several years gave students initial contact with algebra in school when they were 12 or 13 years old (Lins & Kaput, 2004). More recent perspectives on mathematics education suggest that the work aimed at developing the students' algebraic thinking must start in primary school (Carpenter & Levi, 2000; Carraher & Schliemann, 2007; Lins & Kaput, 2004).

This «early algebra» perspective, advocates the development of algebraic thinking, rather than the teaching and learning of specific concepts of algebra, or the domain of algebraic procedures. Lins and Kaput (2004) highlight the two main features of primary school algebra: (i) generalizations and the expression of such generalizations, and (ii) reasoning with generalizations, including syntactically and semantically guided actions. This perspective does not seek to bump up formal algebra studies from secondary to primary school, but rather to promote the students' mathematics reasoning development by connecting algebra to other primary school mathematics topics. The teacher education experiment that we present in this paper approaches algebra teaching and learning as a leading thread that should run through mathematics teaching in general (NCTM, 2000), using a rationale that articulates mathematical and didactical knowledge.

PROSPECTIVE TEACHER EDUCATION

According to Ponte and Chapman (2008), prospective mathematics teacher education involves three elements: (i) knowledge of mathematics (ii) knowledge of how to teach mathematics, and (iii) a professional identity that supports both the knowledge and the teaching of mathematics. For these authors, the articulation of these aspects will develop the prospective teachers' ability to integrate their knowledge of concepts, representations, and mathematical procedures with their knowledge of students, in line with their level of education and the curricular guidelines.

It should be noted that no matter how much knowledge the teacher (or the prospective teacher) has of mathematics, it «does not ensure that one can teach it in ways that enable students to develop the mathematical power and deep conceptual understanding envisioned in current reform documents» (Mewborn, 2001, pp. 28-29). Sánchez, Llinares, García and Escudero (2000) argue that, in order to teach mathematics, one must know about mathematics, know how one learns mathematical concepts, and know about the process of teaching. Thus, teacher education needs to provide future teachers with opportunities to develop their knowledge and abilities in each mathematical topic and the understanding of the connections between both. It must also provide future teachers with the knowledge of how to effectively convey concepts in the classroom. In other words, prospective teachers «need to know the mathematics they are teaching, as well as how to teach it» (Sullivan, 2011, p. 172). Therefore, it is essential to pinpoint the mathematical contents to be taught, taking into account what teachers need to know to teach it and how the prospective teachers themselves learn (Sánchez *et al.*, 2000).

Primary school teachers must understand the concepts, representations and algebraic procedures; understand how pupils learn; and be able to use teaching strategies that foster the development of their students' algebraic thinking (Capraro, Rangel-Chavez & Capraro, 2008). It is equally important to articulate content and pedagogy in prospective teacher education courses (Askew, 2008; Davis & Simmt, 2006; Ponte & Chapman, 2008). Davis and Simmt (2006) argue against the separation of content and pedagogy that tends to prevail, emphasizing that mathematics for teaching involves mathematical objects, curricular structures, and an understanding of how the classroom works. This articulation between content and pedagogy aims to develop prospective teachers' knowledge of the students, their learning processes, and how the teacher encourages such learning, as well as how to stimulate their understanding of concepts, representations, procedures, and connections by analysing teaching situations, students' tasks and their strategies and difficulties. Prospective teachers may use this knowledge to foresee what they will face when teaching, and thus be able to identify and integrate suitable resources and set appropriate tasks to develop specific learning goals.

THE TEACHER EDUCATION EXPERIMENT

Teacher education experiments take an exploratory approach toward classroom work. They aim to develop the participants' algebraic thinking, while developing the mathematical knowledge to be taught. They also promote the development of knowledge of algebra-related mathematics teaching and its connection to other mathematics topics. Therefore, teacher education experiments combine content and pedagogy (Ponte & Chapman, 2008) and provide prospective teachers with varied and meaningful learning experiences by putting them into teaching situations that promote sharing, debating and negotiating meanings, i.e., the knowledge and skills that will be essential for their future practice.

ARTICULATION OF CONTENT AND PEDAGOGY

Our teacher education experiment included mathematical and didactic work with regard to teaching and learning of mathematics in primary education, especially in algebra, and its articulation with other mathematical topics. The integration of content and pedagogy aimed to provide prospective teachers with an understanding of mathematics teaching that differs from perceptions they held prior to their experiences in teacher education (Ponte & Chapman, 2008). Thus, teacher education should promote in-depth study of mathematics for prospective teachers, as suggested by the Conference Board of the Mathematical Sciences (CBMS) (2011) but, as this document indicates, it is not sufficient for prospective teachers to study more mathematics than the mathematics they are going to teach. What they must have are educational experiences designed to develop a deep understanding of the mathematics they will teach.

Integrating content and didactic knowledge can produce learning experiences that enhance the prospective educator's teaching of mathematics (Albuquerque *et al.*, 2006; Ponte & Chapman, 2008). In this paper, we present two experiences featuring two tasks: Task 2 (Problems with unknown quantities) and Task 4 (Pictorial sequences). These situations give the participants a chance to: (i) analyse strategies used by students, (ii) observe, explore and connect different representations, (iii) analyse the knowledge shown by students, (iv) identify potential difficulties for students, and (v) discuss working hypotheses with the students that may foster the development of their algebraic thinking.

THE EXPLORATORY APPROACH

Open tasks that allow for different solution strategies play an important part in the exploratory approach. However, the variety of tasks is also important, since students should be exposed to a wide range of learning experiences (Ponte, 2005). The prospective teachers we dealt with analysed teaching and learning situations in which students worked on a variety of tasks. During their analysis of the tasks, the student teachers identified key points regarding the use of the task in the classroom as well as the possible contribution of such tasks to student learning. The task usually requires an interpretation of the situation that may involve clarification or reformulation of questions and the appropriate use of representations (Ponte, Quaresma & Branco, 2012). Well-designed tasks also allow students to explore mathematical concepts and ideas, so that «more than a context to apply on already learned concepts, these tasks are useful mainly to promote the development of new concepts and to learn new procedures and mathematical representations» (Ponte *et al.*, 2012, p. 10).

During the teaching experiment, classroom activity involves autonomous work that is supervised and guided by the teacher, and whole class discussions, which involve submitting solutions, debating them, and systematizing the relevant concepts, establishing connections between mathematical ideas and real life situations. The participants' exploration plays a fundamental role and the teacher's activity focuses on promoting and supporting this exploration and managing the different moments of the lesson. The teacher «gathers and analyses information about the strategies and theories being employed by students» (Ruthven, 1989, p. 451), which is essential in promoting a whole class discussion that aims to present and debate ideas and strategies amongst prospective teachers (Ponte, 2005). Thus, the moments of reflection and debate are based on the prospective teachers' activity (Ruthven, 1989), and they assume a pivotal role by reflecting on their own work. On the one hand, this approach aims to promote prospective teachers' learning. On the other, it aims to provide them with the classroom dynamics experiences that they can use in the future, to promote their students' learning, while focusing on the development of their skills and mathematical knowledge.

THE TASKS

This paper presents two of the seven tasks in our teacher education experiment, in which the articulation of content and pedagogy is effected by analysing the students' solutions, describing teachers' practices, and/or discussing classroom situations. Task 2 shows the written answers of 6th grade students and the description of the learning trajectory provided by the teacher (adapted from Reeves, 2000). Analysis of the students' work and understanding of their mathematical thinking is an important part of the educator's practice (NCTM, 1991; Nickerson & Masarik, 2010). Analysing students' answers gives future teachers the opportunity to contextualized learning, gain a better understanding of how students think and hone their ability to make suitable decisions in the classroom (Crespo, 2000; Nickerson & Masarik, 2010). The activity also enabled student teachers to analyse the reasoning of the pupils they observed and the representations they used. The prospective teachers also benefited from discussing the students' written answers and the teacher's approach among themselves.

In Task 4, prospective teachers saw a video of excerpts from a 2nd grade class (described in Silvestre *et al.*, 2010) in which the students were working on a pictorial sequence. The use of video yields significant teacher education opportunities. Llinares and Valls (2009) say that video recordings of classes foster the prospective teachers' ability to analyse and identify key aspects of teaching. The authors also emphasize that classroom videos may reveal and underscore mathematics teaching practices that contrast or collide with the future teachers' own. In this type of experiment, participants can analyse the teacher's practice and line it up with the requirements of the mathematics curriculum (ME, 2007), thus using the observation as a learning opportunity.

Different phases of the class are shown in the video: the introduction of the task, students' autonomous work, and the discussion of the work that includes the formulation of further questions aimed at generalization. The analysis of real life situations offers a productive model of mathematics teaching (Ponte, 2011), and helps prospective teachers to see that it is indeed possible to implement curriculum guidelines in the classroom.

METHODOLOGY

We used a design research methodology to assess how the participants developed from the context we provided, and analysed the teacher education experiment with regard to what worked and how it worked. This method allows one to test and/or improving models of teaching guided by theoretical principles, allowing the verification of how they work, adjust them, improve upon them and re-test them (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003; Collins, Joseph & Bielaczyc, 2004).

The possibility mentioned above was included in this study, which targeted algebra education for future primary school teachers. It involved a planned intervention, implemented by the first author, which took place over a significant period of time and was based on a sequence of teaching episodes, making it possible to analyse the participants' activity (Steffe & Thompson, 2000).

The participants were 20 student teachers attending the 3rd year of the degree program in primary education (referred to in the study as Fx). The study focused on the work and learning achieved by the group as a whole and on the learning experiences of three future teachers with specific educational profiles and different future goals. This enabled us to see to what extent the experiment affected different participants with different characteristics.

PROSPECTIVE TEACHER	MATHEMATICS BACKGROUND	SEEKING THE FOLLOWING MASTER'S DEGREE
Alice	9th grade	Pre-school Education
Beatriz	10th grade	Pre-school and Primary Education
Diana	12th grade	Primary and Middle School Education

TABLE I – CHARACTERIZATION OF THE PROSPECTIVE TEACHERS: MATHEMATICS BACKGROUND AND MASTER'S DEGREE SOUGHT

Data was gathered by various means, for a detailed understanding of the situations experienced by the participants (Bogdan & Biklen, 1982) throughout the teacher education experiment. In this paper, we present evidence from three audio and video recorded interviews with the participants (named E1, E2 and E3), documents produced by the participants in the experiment (written solutions of tasks 2 and 4 and portfolios), and participant observation in the classroom, complemented by audio and video recordings. This data collection took place at different times (Figure 1).

T1, T2, T3	Internship period	T4, T5	School break	T6, T7
E1		E2		E3

FIGURE 1 — PERIODS OF DATA COLLECTION

Data analysis sought to identify the development of knowledge in mathematics and teaching offered to the three participants within the teacher education experiment. We present and discuss the work carried out in class on both tasks and the data concerning the development of Alice's, Beatriz's and Diana's knowledge in this context. In its essence, the analysis takes on an interpretative nature and tries to highlight the contribution made by analysing teaching and learning situations. The interpretative analysis documents the group's learning and highlights the development that classroom work afforded prospective teachers (Cobb, Zhao & Dean, 2009). We organized the data in order to discover regularities involving the actions and meanings the participants attributed to mathematical knowledge, the students' knowledge and their learning processes, and the knowledge of teaching practice afforded by both tasks.

ANALYSIS OF TEACHING-LEARNING SITUATIONS

In Tasks 2 and 4, the participants had the chance to analyse teaching and learning situations. In Task 2, they examined the written answers of 6th grade students, as well as the description of the work and the teacher's thoughts. In Task 4, question 3 deals with the observation and analysis of a classroom segment containing a task that features a growing pictorial sequence for 2nd graders.

TASK 2

Task 2 offers the analysis of the answers of 6th grade students to the «chicken problem» (Figure 2). On the next pagewe show the answers produced by Matt (Figure 3) and Joanna (Figure 4) (taken from Reeves, 2000, p. 399).

Problem solving by the prospective teachers. Before examining the students' answers, the participants solved the problem themselves, by working in pairs. Many participants acknowledged three unknown values in it, to which three conditions are given (F7, F8 and F19). Then, they formulated a system of three first-degree equations and solved it using the substitution method.

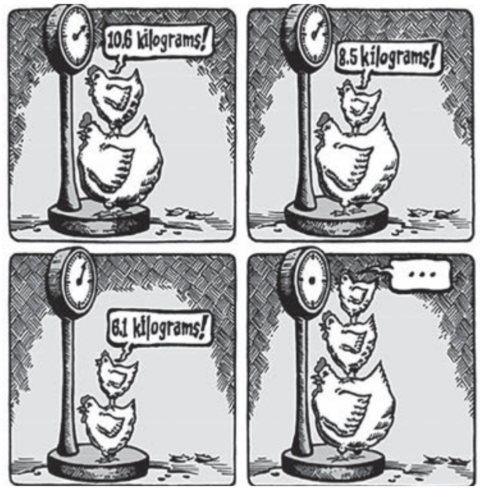


FIGURE 2 – THE CHICKEN PROBLEM,
TASK 2 (REEVES, 2000, P. 399)

Name: Matt
The mass of the big chicken is 6.5 kgs
The mass of the middle-sized chicken is 4.1 kgs
The mass of the little chicken is 2 kgs

Here's how I figured out. I put the number 1, 2, 3 and 4 around the boxes. Then I added box 2 [6.1] and box 1 [10.6]. I got the sum of 16.7. Then I subtracted box 3 [8.5] from 16.7. I got the sum of 8.2. Then I divided 8.2 by 2. I got 4.1 for the weight of the medium chicken. Then I subtracted 4.1 from box 1 which had one big and one medium chicken. I got 6.5 for the big chicken. Then I subtracted 4.1 from box 2 and got 2 for the small chicken.

FIGURE 3 – MATT'S ANSWER

Name: Joanna
The mass of the big chicken is 6.5
The mass of the middle-sized chicken is 4.1
The mass of the little chicken is 2.0

Here's how I figured it out:

$G + M = 10.6$ kg
$G + P = 8.5$ kg

$M - P = 2.1$ $\quad$ $10.6 - 8.5 = 2.1.$

$M + P = 6.1$
$M - P = 2.1$
$2M = 8.2$
$M = 4.1$ $\quad$ $G = 6.5$
$G + M = 10.6$
$G + 4.1 = 10.6$ $\quad$ $10.6 - 4.1 = 6.5$

$G = 6.5$
$G + P = 8.5$
$8.5 - 6.5 = 2.0$
$P = 2.0$

[G presents the mass of the big chicken, M presents the mass of the middle-size chicken, P presents the mass of the little chicken]

FIGURE 4 – JOANNA'S ANSWER

Most of the prospective teachers used the addition method informally, carrying out basic operations to get the value of an unknown. Some participants such as Diana (Figure 5) and Beatriz subtracted pairs of given values in order to establish relationships between the mass of two chickens, and find out which one weighs more than the other.

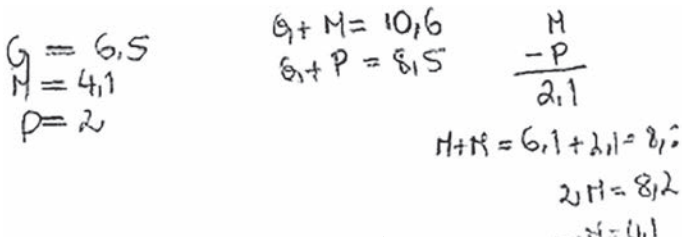


FIGURE 5 – DIANA'S ANSWER

Diana also formulated a system of three first-degree equations but, because it is a system with three equations with three unknown values, she had trouble solving it.

Most of the participants, such as Alice (Figure 6), merged two given values and got twice the weight of a chicken and the weight of the other two chickens. Based on the other equation, she subtracted the weight of the other two chickens to her result. Several participants used letters to represent unknown quantities and write algebraic expressions for the quantities they were looking for.

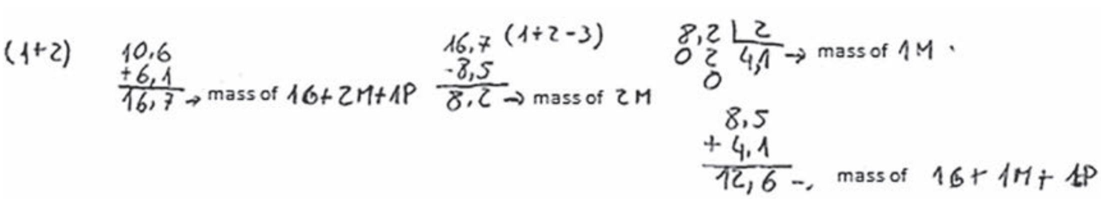


FIGURE 6 – ALICE'S ANSWER

After this autonomous work, the prospective teachers shared and discussed their strategies, presenting solutions based on doing basic operations and establishing relationships that use the algebraic language. This enabled them to analyse systems of equations and the substitution method for solving the problem. This activity was very important to the participants, as twelve of them included it in their portfolios.

Analysis of the students' answers. By its very nature, the analysis of students' answers is a challenge for the participants. Matt's answer (Figure 3) is descriptive, using verbal language. In contrast, Joanna's answer (Figure 4) is based solely on symbolic representations; but it does not explain the computations carried out. In order to analyse Matt's answer, the participants realized that they needed to specify his computations, while Joanna's answer needed to be complemented with verbal representation.

To make sense of Matt's answer, the participants used symbolic representation, showing the basic operations carried out by the student. Some participants used verbal language to indicate what those operations referred to, while others used algebraic language, as in the case of F15 (Figure 7).

His solution is descriptive, ie, Matt wrote his entire reasoning down

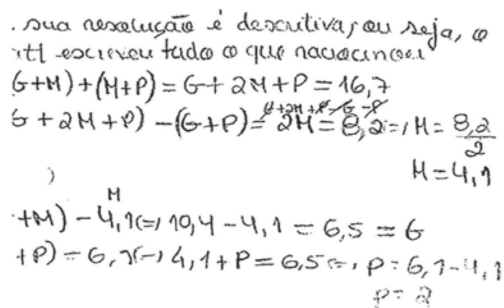

. sua resolução é descritiva, ou seja, o
tt escreveu tudo o que raciocinou
(G+M)+(M+P) = G+2M+P = 16,7
G+2M+P) − (G+P) = 2M = 8,2 => M = 8,2/2
M = 4,1
+M) − 4,1 <=> 10,4 − 4,1 = 6,5 = G
+P) = 6,7 <=> 4,1 + P = 6,5 <=> P = 6,1 − 4,1
P = 2

FIGURE 7 – ANSWER OF F15

In the case of Joanna's answer, the participants felt the need to identify the order in which the student carried out the operations. To shed light on her strategy, they used verbal language, just like F1 shows the class:

F1. – Joanna realized that the difference between the medium-sized chicken and the small chicken is 2.1, i.e., she subtracted the total weight of the big chicken with the medium-sized chicken from the total weight of the medium-sized chicken with the small chicken. She discovers that the sum of the medium-sized chicken and the small chicken equals 6.1. So, Joanna understood that if she added 6.1 and 2.1 she would get the weight of two medium-sized chickens. Right? Here it is, two medium-sized chickens [points to Joanna's answer], 8.2. So, if she divides this value by two, she'll get the weight of a medium-sized chicken. It's 4,1.

F4. – She did the compensation.

F1. – Yes. So, if she has a medium-sized chicken and has these expressions [G+M = 0.6 and G+P = 8.5] all she has to do is replace them, and she'll have all the other values.

Researcher – She knows the medium…

F1. – Exactly. After she got the value for the big chicken [6.5], she replaced it and got the value of the small chicken.

Researcher – Where did they differ? Matt was descriptive…

F1. – And Joanna did the computations.

Researcher – And she uses something else. She uses symbols to represent [unknown] quantities.

F4. – It's a way to explain all the computations but with symbols (Class, T2).

In addition to Matt and Joanna's answers, the participants also analysed Leo's answer (Figure 8) to the problem that the teacher presents with different numbers. This student numbered the different boxes from left to right (above, the scale of box 1 is 11 and box 2 is 9.8; below, the scale of box 3 is 6.4 and box 4 has all the chickens with unknown values). He added all the equations, obtaining twice the weight of the three chickens. In this new equation, he subtracted each of the initial equations. Thus, through the addition method, he got each of the unknown values. Given the conditions of the problem, this strategy proved to be very efficient. Some participants indicated that, if they eventually encounter a scenario similar to the chicken problem, they will use this solution strategy, as indicated by F6: «For me it was the simplest, quickest, most effective way to solve it, so much so, that I would use this same strategy for solving further exercises» (Portfolio).

FIGURE 8 – LEO'S ANSWER (ADAPTED FROM REEVES, 2000)

For Alice, the analysis of the students' answers promoted an understanding of their reasoning and the representations they use, and contributed to their learning of different solution strategies. As she said, «After seeing this sheet and how one of the students [Leo] solved one of the problems, in the next sheet (...), I know that I solved a problem taking into account the student's reasoning» (E2, Alice).

For Beatriz, this task contributed to deepening her mathematical knowledge. She mentioned that at first «I could only do it through experimentation» (E2) and she was surprised by the students' answers, having learned different strategies by analysing those answers. For this participant, Leo's strategy was also important:

> They considered the value of the boxes, one of them added [the value of] all the boxes, and found that if he added [the value of] all the boxes, he would obtain two times the chickens' weight (...); and if he divided that value by two, he would obtain the weight of the three chickens (E2, Beatriz).

The opportunity to analyse students' written answers was also meaningful for Diana. She recognized that the teacher has his/her own solution strategies, sometimes formal, and thought that this is not enough when you are a teacher. One must also be able to interpret the students' answers, given that they may have different strategies. Therefore, in her training phase, this prospective teacher was already valuing the knowledge and learning processes that allow students to answer the mathematical problems, and not just the development of conceptual and procedural knowledge.

> I thought it was quite amusing and right for us to analyse how children solved the exercise, because it's good for us to know how to solve the exercises, and is also good for us to understand how children solve them. Sometimes they think in ways we have not thought of, which can be equally correct. And I think it is good for us to not just practice how to do the exercises but also to understand how they did them. Because we will need it in our future teaching. We will need to understand it. We must not just know how to do it in our own way, but we also need to try to decode what they did, because many times it may be right or… Something may be right (E2, Diana).

In other situations featuring problems, the participants continued to explore less formal strategies to solve problems, closer to the primary students' work, and more formal strategies, using symbols and algebraic procedures.

Many participants stressed the importance of analysing the students' answers and checking the possibility of different strategies to solve a given problem, as F3, F8 and F19 suggest. The last two participants were particularly attentive to establishing relationships that are essential in the development of algebraic thinking in basic education. F8 recognized the possibility of solving these problems in different ways, not just t by means of the formal solution that she was acquainted with, i.e., the system of equations:

> I thought it was very interesting (...) also the fact that we analyse different answers and understand the difference between the strategies used by the students (Portfolio, F3).

> This [task] developed my ability [to] interpret and compare the students' answers. Besides that, I understood that the aim of these exercises is not to develop solution techniques but to lead the students' to understand the relationships (Portfolio, F19).

> Through this task, I realized the variety of solutions that may exist for the same exercise, because for me, until now, the only way to solve this kind of problem was through the system of equations (Portfolio, F8).

Alice, especially, was surprised by the different ways 6th graders dealt with the situation and found the answer to the problem, as well as their reasoning ability. She relates this to the difficulties she had in solving the task herself:

> I thought this was very difficult. I thought it was funny that 6th graders had a thousand ways to solve this. I was thinking... They really have ability to solve the problem and explain it in different ways, which I also found amusing (E2, Alice).

For this participant, it was very important to look at the different strategies used to solve the same problem. It enabled this future teacher to gain a deeper understanding of the diversity of representations and rationales that students may use, and especially how the relationships may be expressed in a formal or informal way. Beatriz also underlined the value seeing that students can use different strategies and that there may be «several solutions to the same problem» (E2).

Analysis of the teaching approach. This task presents a description of the work done to promote the development of students' reasoning and their ability to use organized solution strategies. Part of that description is presented here (Figure 9, on the next page).

The teacher presented the problem of the chickens in Figure 1, so that the students would solve it using intuitive approaches, before initiating the formal study of algebra.
The problem was solved at home and the answers were given in the classroom. (...)
The students [like Matt and Joanna] presented their solutions to the rest of the class and explained their reasoning. (...)

Reflections

Reeves makes the following observations:

§ Some students can learn how to solve problems like this one from listening to one another and to their parents. Not all solutions will be efficient. The problem itself, as a precursor to systems of equation, is within the reach of sixth graders.
§ If students are to teach problem-solving strategies to others, they should be asked to emphasize strategies rather than computations. They should be encouraged to explain their approaches without using numbers.
§ Students will not automatically learn to use variables even after hearing their classmates use them as shortcuts. The use of variables will have to be encouraged by the teacher if the outcomes is a goal of an algebraic-thinking strand. (2000, p. 401)

FIGURE 9 – PARTIAL DESCRIPTION OF THE WORK DONE IN CLASS, TASK 2

With the description of the teacher's strategy and reflection, the participants found that this type of work in the context of problem solving and sharing strategies is meaningful to the students. They realized the importance of students presenting their answers orally in class, as indicated by F19, since it «requires them to explain their reasoning, so that their classmates may adopt an easier way to solve this kind of problems» (T2-1.3).

They also noted that in choosing the chicken problem, the teacher provided the students with an experience that involved the use of letters to represent unknown quantities. The nature of the problem facilitated the use of algebraic symbolism, as F1 says: «Variables appear naturally with a task like this» (Class, T2). Debating students' answers allows those who use algebraic symbols to share this kind of representation with their classmates, as F1 points out: «Since they explain their reasoning, they're the ones who'll explain the existence of that variable to a fellow student» (Class, T2). F1 is referring to the variables in a broad perspective, focusing on established relationships and not specifying that in this case there is a set of conditions that must be met, that is, each letter is an unknown.

The participants verified the importance of the whole-class debate on different strategies for student learning, and they remarked that they valued opportunities such as these in teaching:

> It's important for the teacher to provide the class with problem-solving moments like the chicken problem, so that students may develop (informal) mathematical reasoning, and also to give them the opportunity, through presenting different solutions, to choose a strategy that is different from theirs, for example (Portfolio, F8).

> It is important that teachers encourage a «debate» about the solutions to the problems at the end of the task and that teachers stress this kind of problems, because the students, by discussing the results and solutions with their classmates, have the opportunity to know and learn new strategies (Portfolio, F15).

F6 highlighted that this task provided her with an experience similar to the ones she will have in the classroom – examining the students' answers and «understanding their computations, and how they reached that answer, and so on» (Portfolio). She considered it an important task, because it contributed to «the preparation for my future [professional] activity, because not all children think like me and follow the same path to reach the same outcome» (Portfolio, F6).

Alice highlighted the fact that the teacher promoted the exchange of strategies among students, so that the class saw different ways to solve the problem and students with more difficulties were able to understand it, which may enable the development of algebraic thinking. She discussed this idea in her portfolio:

> I also emphasize the pedagogical attitude and strategies used by this teacher, making her students compare and discuss their reasoning and strategies among themselves. Thus, students with more difficulties in their algebraic thinking may observe more elaborate strategies and reasoning, which may be helpful to the evolution of their own thinking (...) (Portfolio, Alice).

Alice highlighted the way the teacher offered hints and conducted teaching and learning situations by stressing reasoning over results. She stressed the importance of this in honing the students' ability to establish relationships between different quantities in order to determine the unknown values. She also indicated that the teacher must possess enough knowledge to answer the students' questions, and be able to «answer and explain in many ways, to see how the student understands best» (E2, Alice).

This task led Beatriz to reflect on the importance of teacher preparation for this kind of task:

> I would have to solve such an exercise, before proposing it to the class. Maybe I would have to know other ways of solving it, so that when the students presented several solutions, I would already know about them. But sometimes we are not able to know [all the solutions], and when we listen to theirs, we know whether they are right, even if we have not thought of it ourselves (E2, Beatriz).

Beatriz indicated that the teacher must try to solve the same problem in many ways to be prepared for any questions the students may have. She must understand and be acquainted with their solutions, even when she did not come up with the strategy they followed.

Diana, however, emphasized another important aspect of the teacher's practice: the need to adapt the tasks to the students, for example, by adjusting the values: «I think it's an amusing task to work with them. We can adjust the numbers to their age or education, and we can work with them» (E3, Diana). She also realised that based on the examples analysed, with this task middle school students would be working on problems with unknown quantities, i.e., with unknowns appearing in a natural way. As she commented, «They would be working with unknowns without even realizing it» (E3, Diana).

TASK 4

In question 3 of Task 4, the participants watched a video of a 2^{nd} grade class and analysed how students handled sequences (Figure 10).

Look at the sequence of blocks.

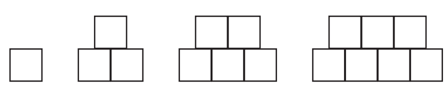


Fig. 1 Fig. 2 Fig. 3 Fig. 4

a) Continue the sequence and draw figures 5 and 6.
b) How many pieces were used to construct each of the figures? Write your answer in the following table.
c) Without using drawings, are you able to figure out how many blocks figure 20 of the sequence has? Explain how you figured it out.

FIGURE 10 – 2^{ND} GRADE TASK, T4-3

Sequence analysis by prospective teachers. Before this task, the participants completed other tasks involving sequences and patterns. Before question 3 from

Task 4, there were other questions regarding pictorial sequences. In analysing this pictorial sequence, they immediately identified relationships between parts of each term and their order, indicating an overall term for the numerical sequence, concerning the number of squares in each pictorial term.

Analysis of students' strategies. Viewing the classroom excerpts enabled prospective teachers to identify the students' real difficulties with pictorial sequences, the different representations and strategies they used, and how the teacher acted in the various situations:

> [It allowed us] to identify different approaches suggested by the students and be aware of some of their difficulties. The analysis of such strategies prepares us, prospective teachers, for the variety of answers that we can get from them (Portfolio, F3).

In the analysis and discussion of the students' work, the participants were able to identify the relationships they established, which allowed them to set an algebraic generalization using natural language:

> [Regarding the 20th term]
> Researcher – What did [the student] discover?
> F19. – That it was always an odd number.
> Researcher – It was always an odd number, correct. And what did he do?
> F11. – He did 20 twice.
> Researcher –He did 20 twice, exactly.
> F11. – Then, he realized that it had to be an odd number: either 39 or 41.
> Researcher – Exactly. What's the teacher doing?
> F15. – She keeps asking.
> [searching for the generalization. In the video, the student indicates he discovered the secret: «it's twice minus 1»]
> Researcher – What has the [student] just done?
> Beatriz: He discovered the general term.
> F15. – He discovered the secret. (Class, T4)

The prospective teachers highlighted the fact that students made a generalization using natural language, even if they did not know that it was the general term of the numerical sequence. Alice's group wrote the generalization down in symbolic language, while some students expressed it in natural language (Figure 11).

• $2n - 1$ → • $n + n - 1$	Some students gathered the row of squares on top with the one below, but since the upper row is always one square short, they always take one square from the sum total.

Figure 11 – Answer from Alice's group, T4-3.2

For Alice, it was important to observe the classroom, the students' work, and the strategies they used to determine distant terms and how they managed to express a generalization. She was surprised that 2nd graders could generalize. She recalled a strategy of a student who used his knowledge of numbers to find a general rule in order to find a term of the numerical sequence. The analysis of this teaching and learning situation gave Alice a better understanding of what students are capable of and their ability to establish generalizations from these contexts. She said that students «can relate concepts, as was the case with Bruno, whose answer was twice minus 1» (E2, Alice).

Beatriz pointed out several features of the students' work on pictorial sequences. In finding close terms and in using the table, she stressed a recursive analysis a child had performed and how another had established a direct relationship between the total number of blocks in a term and its order:

> After looking at the table filled with the figure number and the number of matching blocks [up to 6th term], a child found that each time we changed pictures, the pieces increased 2 by 2; another child added the number of the requested figure with the previous figure [based on table data] (Portfolio, Beatriz).

In finding distant terms, she emphasizes two other things as well. One has to do with what was already identified in the analysis of the table, in which the students relate the composition of the pictorial term to the order, and to the order of the previous term. This is a contextualized generalization, because the students always present an example to express it:

> He replaced the blocks with numbers. In figure two, the 1 is on top and the 2 is below. In figure three, the 2 that was below in the previous figure went on top of the next figure, and the 3 went below (...) As to figure 20, the child thought that the 19 that was below in the previous figure shifts up in the next figure and the 20 is now below this 19. If we add the 19 to 20, we obtain the number of blocks of figure 20, which equals 39 (Portfolio, Beatriz).

The other situation that she identified dealt with the formulation of a general term of the sequence in natural language by a student, which he calls «the secret». The student presents a direct rule to find any term of the sequence, relating the number of blocks to the order.

> This child suggested that the secret «is not double, but twice minus 1». The teacher asked the child to explain the «secret». The child used the order in figure 12 and said: «12 + 12 = 24, but it can't be, because 24 is even, so, it's 23, because it's always twice minus 1» (Portfolio, quotation marks in original, Beatriz).

Based on her mathematical knowledge, Beatriz interpreted the students' answer and gave it an algebraic meaning. The teacher education experience and the reflection upon this work allowed her to develop knowledge of how students work with sequences, particularly their strategies, and how they express generalizations.

The student presented a direct rule to find any term of the sequence, establishing an algebraic generalization. However, a second grader does not use symbolic algebraic language to express that rule, he uses natural language instead, as Beatriz acknowledged. Thus, the prospective teacher understood that the work with pictorial sequences acts as a precursor to the students' development of algebraic thinking:

> I found it very important and interesting that we learned from a classroom excerpt with second grade children. We were able to observe how children think, and although they have not yet been taught algebra, they have already unconsciously internalized it. That is why one of the children talked about the «secret» [refers to a general term] (Portfolio, Beatriz).

This situation also helped Diana envision the work she would be doing in primary school. She suggested that, for primary school students, «it's easier to work with pictorial sequences than with numerical sequences» (E2). She stated that at this level, students do not use symbolic algebraic language; nevertheless, it makes sense to work on these issues to find close and distant terms. She recalls the strategy of some students, to which she associates a general term:

> Diana – With or without algebraic language, I think that they may get it, for example, if we ask them the term, the closest term, I think that they get

it. That was the case yesterday [referring to the video]. (…) They knew that there was always 1 less on top, so when it was 50 they took 1 and put 49, then they put 50 below. Then, they can do it. They are doing it without realizing it, by doing $n - 1 + n$.

Researcher – $n - 1 + n$, yes it was one of the situations.

Diana – They are doing it without realizing it, but they didn't write it. Nobody from primary school or middle school wrote it, I think. (E2)

The generalization that the students establish is algebraic, as Diana saw. She says that it is important for students to analyse close and distant terms to «see the relationship between the order and the terms. I think that's it… They compare the relationship between the order and the terms and, if they understand the first cases, they will also understand more [distant ones]» (E2, Diana). According to her, the search for terms of a more distant order establishes this relationship more than an indication of terms of a very near order, because «for example, here in the fourth, they would probably draw it and say, without understanding the relationship, what was more and what was less» (E2, Diana). She realised that, for near orders, students may choose a strategy of representing and counting and still not establish the generalization. The first strategy is ineffective in finding distant terms and, therefore, the generalization becomes easier.

Analysis of the teacher's practice. Based on the video of the lesson, the participants identified significant aspects regarding the dynamics and organization of the class when it came to their work on pictorial sequences. Some participants stated that the experience was important to their teacher training, because it enabled them «to know the classroom environment and to analyse the methodologies used by the teacher (…)» (Portfolio, F3).

The experience led the participants to recognize how important the teacher's role is in managing and leading the classroom in ways that provide opportunities for their students to develop their algebraic thinking. The participants saw that the teacher asked students to find distant terms so that they could use both strategies discussed in the classroom, to add to the order, the order of the previous term (based on the pictorial representation) used by most students, and the subtraction of double the order plus one, followed by one student. At the end of the class, the grade 2 teacher suggests that they all use the latter strategy in finding some distant terms.

Alice's group highlighted various important aspects of the teacher's practice, namely, task presentation, the organization of the students' work and the use of resource materials (Figure 12).

- Reading the statement and explaining the tasks
- Student questions
- Teacher's orientation of the work, ie, in pairs and in case there are different opinions or explanations register them
- Providing materials for the students to solve the work sheet (checkered sheet and pieces – squares)

FIGURE 12 – THE TEACHER'S PRACTICE (ALICE'S GROUP)

Moreover, the group underlined the teacher's role in the classroom dynamics, particularly during the presentation and discussion of student strategies (Figure 13).

To explore the solutions and explanations that students had, the teacher was always asking the students questions about their solutions.

FIGURE 13 – THE TEACHER'S ROLE (ALICE'S GROUP)

Alice examined what the teacher in the video said to promote the student involvement: «The teacher asked many questions and never gave the answer, and she let the students figure out the answer themselves» (E2). Moreover, she indicated that teachers must know the students so that they can «adapt the knowledge to the students inside the classroom. You can try one way or another, and use several strategies in order to meet the students' needs» (E2). For this participant, the teacher should promote a classroom dynamic aimed at the students' learning progress, without directly providing the answer.

Analysis of the classroom enabled Beatriz to see that students must have the opportunity to explore the assignments and that the teacher must be prepared to assist them if they have any questions or difficulties as the teacher in the video did. Thus, with regard to working with sequences, she suggested:

> First, either (...) say nothing and let them explore and succeed, but if they fail, warn them and (...) tell them, for example, to try and look at the order

> and the number of elements of various orders, see what each of them has in common, how often it increases, for instance (E2, Beatriz).

Diana noted in the teacher's practice that she offered blocks to students to represent the pictorial terms. She discusses the possibility of using manipulative materials in primary school to make the situations more concrete:

> *For* example, to divide the class into groups and to give them materials for them to work with. Making bead strings with different colours, making a sequence. Then, for example, one thing that the program now encompasses, is to present and explain to the class how they made the sequence. What could be made after that… The teacher may ask questions to the class to see if the kids understood what the regularity was. Which is the tenth piece... Or the tenth bead, for example. I think they must do practical work that they can manipulate (E3, Diana).

Diana suggested that, for the situation she presented, group work is more appropriate, followed by a whole class discussion. This is what happened in the taped excerpt, in which the students shared what they did and the teacher asked the children listening if they understood the regularity that was presented. For primary school students, she suggests completion of a practical assignment that engages them, where they can manipulate objects and discuss ideas with their classmates, justifying their answers.

DISCUSSION

The prospective teachers' mathematical knowledge grew considerably, particularly their ability to establish relationships and use different representations and strategies to solve tasks. The experience also enabled them to learn more formal strategies, geared toward establishing relationships and abandoning trial and error strategies, as Beatriz did. It also taught them about strategies that were closer to the students' knowledge, so that they could solve situations other than by using only systems of equations, as F8 did. The study shows that the analysing students' answers to the task 2 problem with unknown quantities, helped the participants to understand different representations and solution strategies and may have acted as the precursor to the use of letters to designate the unknown.

The experience also made them aware of how they themselves learned and how they could use different representations and strategies to suit the students' grade level. In task 2, Diana highlighted the different solutions that the problem evoked, and Beatriz became cognizant of how important the teacher's knowledge was in dealing with the different solutions. Beatriz also emphasized the importance of knowing different strategies and the skill required by the teacher to assess the students' answers and orient classwork so as to encourage discussion and sharing of ideas. Task 4 contributed to the participants' understanding of the various ways in which students can analyse a pictorial sequence, and how they express generalization. During the teacher education experiment, the participants became more aware of the teacher's role in promoting student participation and encouraging them to share their reasoning.

The prospective teachers homed in on specifics of the teacher's practice, such as the selection of tasks, how they used different strategies to complete the tasks, and the importance of understanding the students and their learning processes. Analysing teaching and learning situations involving grade 6 students gave the participants a glimpse of the material and methods they could design for their future students, as well as the representations and reasoning that they could use. Diana stressed the importance of fostering moments of autonomous work and group discussions, and Alice noted that the time for sharing and discussing strategies contributes to students' learning, especially in the case of students who are struggling with math. As Crespo (2000), Capraro *et al.* (2008) and Nickerson and Masarik (2010) suggested, the participants in this study also showed development in their understanding of students' reasoning and the ability to see the teaching of algebra in terms of the tasks and ways of working in the classroom.

The results of the participants' work in these two tasks show that the analysis of students' solutions and of teacher's practice is a meaningful addition to the education of prospective teachers. In line with the theoretical principles that guided the teacher education experiment and the results in each task, the analysis of classroom situations was conducted throughout the teacher education experiment in other tasks on these and other topics, such as the study of functions.

The study also bears out the importance of using video excerpts for mathematics teaching development. The viewing of the grade 2 class with pictorial sequences enabled the participants to jointly analyse the students' and the teacher's work, the students' strategies, and how they communicated them. Without the video it would have been hard to broach the context during the prospective teachers' course.

Beatriz valued this learning opportunity for the chance it gave her to analyse the different ways students look at a pictorial sequence. All the participants were surprised that the students succeeded in generalising the pictorial sequence, thereby finding distant terms. They acknowledged that the students do not use symbolic algebraic language, but rather express such generalizations in natural language. The study shows, as Llinares and Valls (2009) mention, that video analysis identifies key aspects of teaching and learning, particularly with regard to the students' ability to generalize; it also sheds light on the way they express such generalizations. Since analysing what goes on in the classroom is eminently served by videos, it would be logical to use them even more in prospective teacher education.

CONCLUSION

The experiment gave future teachers a general view of how algebraic thinking could be promoted in primary school. It also led the participants to appreciate the teacher's role as a promoter of learning through suitably designed tasks, one who fosters communication, and as a classroom manager and supervisor. Experiments such as these also promote the analysis of the students' role and work, their degree of understanding, their strategies, and the difficulties they demonstrate, the representations they use and the connections they establish.

By analysing teaching and learning situations in primary school, future teachers are also learning mathematics themselves. They see representations and strategies that are different from their own, and this enhances their own knowledge. The experience also enabled Alice, Beatriz and Diana to understand the teaching of algebra, and allowed them to envision situations that were conducive to algebraic thinking, which is an important aspect of the teacher's role (Canavarro, 2007). This aspect is of particular significance, because not all participants had experiences with generalization and formalization when they were in primary school (Kaput & Blanton, 2001).

The discussion of the work done in the classroom with sequences and relationships gave the prospective teachers a vehicle for reflecting on key aspects of teaching practice, classroom dynamics, and how to follow the students' train of thought and get them to adjust their reasoning. Analysing teaching and learning through video put a spotlight on teaching practice itself, in particular the way the task is introduced, what materials are provided, and how student communication is promoted by asking questions.

Our study indeed suggests that by analysing and reflecting on teaching and learning situations help future teachers learn how to teach algebra, as Doerr (2004) suggests. More specifically, it helps them learn to engage their future students in situations that promote their algebraic thinking, taking into account aspects of the teacher's practice close to those analysed during the teacher education experiment and if it is in line with the exploratory approach.

The participants were aware of the importance of a classroom environment that is focused on generalization, sharing, and discussing students' reasoning, features that Kaput and Blanton (2001) and Blanton and Kaput (2011) recommend for teaching practice. Overall, the prospective teachers were surprised by the students' ability to generalize and by the diversity of representations and strategies they used.

Thus, being able to analyse a wide range of teaching and learning situations and the class-wide discussion of the experience help to develop the prospective teachers' algebraic thinking, in particular their ability to use and interpret different representations and strategies. It also hones the participants' skill at algebra teaching, making them aware of the challenges they face in selecting tasks, conducting the class, and focusing on reasoning and its representations. In all, this study showed how teacher education based upon a variety of tasks, aimed at exploring concepts, generalizing and analysing student output and teacher practice, combined with moments of reflection and group discussion, enhances the prospective teachers' understanding of algebra and of how they can foster it in their students.

REFERENCES

Albuquerque, C., Veloso, E., Rocha, I., Santos, L., Serrazina, L., & Nápoles, S. (2006). *A Matemática na formação inicial de professores.* Lisboa: APM & SPCE.

Askew, M. (2008). Mathematical discipline knowledge requirements for prospective primary mathematics teachers, and the structure and teaching approaches of programs designed to develop that knowledge. In P. Sullivan & T. Wood (Eds.), *International handbook of mathematics teacher education* (Vol. 1., pp. 13-35). Rotterdam: Sense.

Blanton, M., & Kaput, J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education, 36*(5), 412-446.

Blanton, M., & Kaput, J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 5-23). Berlin: Springer.

Bogdan, R., & Biklen, S. K. (1982). *Qualitative research for education: An introduction to theory and methods.* Boston, MA: Allyn & Bacon.

Canavarro, A. P. (2007). O pensamento algébrico na aprendizagem da Matemática nos primeiros anos. *Quadrante, 16*(2), 81-118.

Capraro, M. M., Rangel-Chavez, A., & Capraro, R. M. (2008). Effective preparation for teaching of algebra at the primary level. Paper presented at *11th International Conference on Mathematics Education (ICME-11),* Monterrey, México. Retrieved from http://tsg.icme11.org/document/get/382.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann.

Carpenter, T., & Levi, L. (2000). *Developing conceptions of algebraic reasoning in the primary grades. Research report.* Madison, WI: Wisconsin Center for Education Research, University of Wisconsin, School of Education. Retrieved from http://www.eric.ed.gov/ERICDocs/data/ericdocs2sql/content_storage_01/0000019b/80/1a/8d/19.pdf.

Carraher, D., & Schliemann, A. (2007). Early algebra and algebraic reasoning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 669-705). Charlotte, NC: NCTM & IAP.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Cobb, P., Zhao, Q., & Dean, C. (2009). Conducting design experiment to support teachers' learning: A reflection from the field. *Journal of the Learning Sciences, 18*(2), 165-199.

Collins, A., Joseph, D., & Bielaczyc, K. (2004). Design research: Theoretical and methodological issues. *The Journal of the Learning Sciences, 13*(1). 15-42.

CBMS – Conference Board of the Mathematical Sciences. (2011). *The mathematical education of teachers II*. Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Crespo, S. (2000). Seeing more than right and wrong answers: prospective teachers' interpretation of students' mathematical work. *Journal of Mathematics Teacher Education, 3*, 155-181.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61*, 293-319.

Doerr, H. M. (2004). Teacher's knowledge and teaching of algebra. K. Stacey, H. Chick & M. Kendal (Eds.) *The future of the teaching and learning of algebra: The 12th ICMI Study* (pp. 267-289) New York, NY: Kluwer.

Kaput, J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema & T. Romberg (Orgs.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Erlbaum.

Kaput, J., & Blanton, M. (2001). Algebrafying the elementary mathematics experience, Part I: Transforming Task Structures. In H. Chick, K. Stacey, J. Vincent & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The future of the teaching and learning of algebra* (Vol. 1, pp. 344-351). Melbourne, Australia: The University of Melbourne.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator, 8*(1), 139-151.

Kieran, C. (2007). Developing algebraic reasoning: The role of sequenced tasks and teacher questions from the primary to the early secondary school levels. *Quadrante, 16*(1), 5-26.

Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick & M. Kendal (Eds.), *The future of the teaching and learning of algebra The 12th ICMI Study* (pp. 47-70). Dordrecht: Kluwer.

Llinares, S., & Valls, J. (2009). The building of pre-service primary teachers' knowledge of mathematics teaching: Interaction and online video case studies. *Instructional Science, 37*, 247-271.

Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Education Research Journal, 3*, 28-36.

ME – Ministério da Educação. (2007). *Programa de matemática do ensino básico*. Lisboa: ME.

NCTM – National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: NCTM.

NCTM – National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Nickerson, S., & Masarik, D. (2010). Assessing teachers' developing interpretive power: Analysing students thinking. *Mathematics Teacher Education and Development, 12*(1), 19-29.

Ponte, J. P. (2005). Gestão curricular em Matemática. In GTI (Org.), *O professor e o desenvolvimento curricular* (pp. 11-34). Lisboa: APM.

Ponte, J. P. (2006). Números e álgebra no currículo escolar. In I. Vale, T. Pimentel, A. Barbosa, L. Fonseca, L. Santos & P. Canavarro (Orgs.), *Números e Álgebra na aprendizagem da Matemática e na formação de professores* (pp. 5-27). Lisboa: SEM-SPCE.

Ponte, J. P. (2011). Using video episodes to reflect on the role of the teacher in mathematical discussions. In O. Zaslavsky & P. Sullivan (Eds.), *Constructing knowledge for teaching secondary mathematics: Tasks to enhance prospective and practicing teacher learning* (pp. 249-261). New York, NY: Springer.

Ponte, J. P., & Chapman, O. (2008). Preservice mathematics teachers' knowledge and development. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 225-263). New York, NY: Routledge.

Ponte, J. P., Quaresma, M., & Branco, N. (2012). Práticas profissionais dos professores de Matemática. *Avances en Investigación en Educación Matemática, 1*, 65-86.

Reeves, C. A. (2000). The chicken problem. *Mathematics Teaching in the Middle School, 5*(6), 398-402.

Ruthven, K. (1989). An exploratory approach to advanced mathematics. *Educational Studies in Mathematics, 20*, 449-467.

Sánchez, V., Llinares, S., García, M., & Escudero, I. (2000). La formación de profesores de primaria desde la didáctica de las matemáticas. *Números: Revista de Didáctica de las Matemáticas, 43-44*, 143-146.

Silvestre, A., Faria, A., Sousa, H., Cristo, I., Santos, I., Molarinho, M. J., & Veladas, M. (2010). Sequências pictóricas: estratégias de generalização dos alunos de 2.º, 3.º e 5.º anos. In GTI (Org.), *O professor e o programa de matemática* (pp. 89-119). Lisboa: APM.

Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Sullivan, P. (2011). Identifying and describing the knowledge needed by teachers of mathematics. *Journal of Mathematics Teacher Education*, *14*, 171-173.

DEVELOPING KNOWLEDGE OF INQUIRY-BASED TEACHING BY ANALYSING A MULTIMEDIA CASE: ONE STUDY WITH PROSPECTIVE MATHEMATICS TEACHERS

Hélia Oliveira
hmoliveira@ie.ulisboa.pt | Universidade de Lisboa, Portugal

Márcia Cyrino
marciacyrino@uel.br | Universidade Estadual de Londrina, Brazil

ABSTRACT

This study aims to understand the knowledge of inquiry-based teaching that prospective mathematics teachers reveal when analysing a multimedia case featuring one mathematics teacher´s practice in one 7th grade lesson. Data analysis focused on individual reflections that prospective teachers had written, covering four major points: the nature of the task, articulation and purpose of the lesson phases, the teacher's role in inquiry-based teaching, and anticipated challenges. The results show that the use of the multimedia case enabled prospective teachers to develop an understanding of various dimensions of inquiry teaching practice and its complexity, while they simultaneously began to predict specific difficulties they would face in their future professional practice. Significant elements of a dialogical perspective of learning also emerged from the study, as the prospective teachers recognize the central roles of language in knowledge construction, and of the interaction between teacher and student and among the students, in different moments of the lesson.

KEY WORDS

Pre-service teacher education; Inquiry-based teaching; Multimedia cases; Dialogical perspective of learning; Mathematics teaching.

SISYPHUS
JOURNAL OF EDUCATION
VOLUME 1, ISSUE 3,
2013, PP. 214-245

Developing Knowledge of Inquiry-Based Teaching by Analysing a Multimedia Case: One Study with Prospective Mathematics Teachers

Hélia Oliveira[1] | *Márcia Cyrino*[2]

INTRODUCTION

Teacher education at a national and international level, coupled with innovative curriculum guidelines, has contributed to the promotion and discussion of new perspectives in mathematics teaching. In Portugal, the basic education mathematics curriculum (ME, 2007) and research into Mathematics education with projects that have focused on mathematical tasks for students, have introduced the concept of «exploratory learning» in mathematics teaching practice (Ponte, 2005). The type of teaching required is demanding, and given the limited or non-existent contact prospective teachers have had with this new reality, the complexity has become compounded. It is therefore important to provide them with observation experiences in which they can analyse the various aspects of inquiry-based teaching.

During the P3M project, multimedia cases were developed that featured inquiry-based teaching. The cases have been used in pre-service and in-service mathematics teacher education (Canavarro, Oliveira & Menezes, 2012).

1 This work is partially supported by a national grant through the FCT – *Fundação para a Ciência e a Tecnologia*, Portugal, as part of Project P3M - *Professional Practices of Mathematics Teachers* (contract PTDC/CPE-CED/098931/2008).

2 This study is part of the author's Productivity Grant for Research and a post-doctoral grant for study abroad, both from CNPq – *Conselho Nacional de Desenvolvimento Científico e Tecnológico*, Brazil.

Widely used in recent years, classroom videos have been the object of a growing body of research on the implications of using them during the initial education of mathematics teachers (Climent, Romero-Cortés, Carrillo, Muñoz-Catalán & Contreras, 2013; Llinares & Valls, 2010; Santagata & Guarino, 2011). Yet none of these studies has specifically focused on inquiry-based teaching.

The authors assume that mathematics teaching practice as presented in the multimedia case takes a dialogical perspective of knowledge construction (Wells, 2004), ultimately aiming to augment the prospective teachers' understanding of this practice. Thus, with this study featuring the teaching practice of a 7^{th} grade teacher, we intend to examine the kinds of knowledge prospective teachers reveal of the complex, and challenging practice known as inquiry-based teaching, after they have explored one multimedia case. The study was carried out in an initial teacher education course of the master's degree program in mathematics teaching at the University of Lisbon.

THEORETICAL FRAMEWORK

PERSPECTIVES OF INQUIRY-BASED TEACHING

The word *inquiry*, in the context of education, has been used for decades with various meanings that were in line with a number of theoretical approaches to learning. Many authors who propose the *inquiry-based approach* as an alternative to expository teaching (Chapman & Heater, 2010; Towers, 2010; Wells, 2004) base their theories on Dewey's perspectives of learning. This type of practice is also linked to *big ideas* such as «learner-focused, question driven, investigation/research, communication, reflection, and collaboration» (Chapman & Heater, 2010, p. 448), which played a major role in the reformist ideas of curriculum recommended by the NCTM (2000). In Portugal, some research projects into math education propounded that investigative activity be brought into the classroom, using open-ended tasks in which the students ask questions and get involved in formulating, testing and proving conjectures (Ponte, Brocardo & Oliveira, 2006).

To focus on the inquiry-based teaching approach, we took the theoretical perspective of Wells (2004) who believes that knowledge «is constructed and reconstructed between participants in specific situations, using the cultural resources at their disposal, as they work toward the collaborative achievement of goals that emerge in the course of their activity» (p. 105). It is a perspective that views knowledge as being situated and constructed in cooperation with others, and reflecting on what has been learned.

Wells (2004) derives several implications for teaching, of which we highlight the following: i) knowledge must be constructed from problems and issues that are meaningful to the students, encouraging their understanding; (ii) it is important to develop individual autonomy and capacity for action, while stimulating interdependence and the value of collaboration; (iii) this knowledge can only be constructed from previous experiences by dealing with problems that arise in the course of specific, practical activities ; and (iv) language has a central role as it mediates knowledge in a process of meaning assignment, which is at the core of teaching and learning activities.

Wells (2004) asserts that the teaching and learning process should be seen as a process of *inquiry* that is built jointly by the teacher and the students through dialogue. He advocates a teaching practice that emphasizes the students and the conditions that favour participation in inquiry activities that are both collaborative and individual. With this perspective, what we learn is that we do and what we do depends on the practices that are available in the communities in which we participate. All of this underscores the fact that students' mathematical learning (or learning in any other area) is strongly influenced by the teaching that is going on in a given context.

In Portugal, the expressions «exploratory teaching», which describes a certain teacher practice (Canavarro, 2011) and «exploratory teaching and learning» (Ponte, 2005) have been used to describe an approach that differs starkly from directive teaching which is posited on knowledge transmission effected by the teacher who explains the content after which the students practice by applying the concepts and procedures that were taught. Exploratory teaching focuses on students' activity, through challenging tasks that allow multiple entry points while stimulating students' mathematical thinking. This approach gives students a greater degree of autonomy. In this study, we adopt the expression «inquiry-based teaching» in line with the international literature.

One of the aspects highlighted by inquiry-based teaching is the selection of tasks that engage the students in significant mathematical activity. These activities are aimed at honing their reasoning and understanding of mathematical concepts and processes. The NCTM (1994) refers to mathematically valid tasks that promote the development of mathematical understanding and skills, and encourage the establishment of connections, formulation and solution of problems, mathematical reasoning and communication, and the elevation of mathematics learning as a permanent human activity.

Within an inquiry-based teaching framework, the mathematical tasks proposed are of particular importance. It is from them that the student's mathematical activity unfolds. Thus, the tasks assigned should allow students «to think mathematically about important ideas and assign meaning to mathematical knowledge that emerges from the collective discussion on these tasks» (Canavarro, Oliveira & Menezes, 2012, p. 256). These tasks may be problems, investigations or explorations (Ponte, 2005) but should exhibit several common traits. They should be challenging and be based on a concrete situation; allow the students to rely on their experience when solving them and, therefore, make use of various strategies with different levels of mathematical sophistication. They should be anchored in the curriculum and be aimed at a deeper understanding of mathematical concepts that have a strong connection with the knowledge students build during the lessons.

In the context of the P3M project, a framework for the practice of inquiry-based teaching (Annex 1) was designed that was based on the literature and the analysis of Portuguese teachers' practices (Canavarro, Oliveira & Menezes, 2012). This framework has a four-phase classroom structure (introduction of the task, students' autonomous work on the task, collective discussion and systematization), in which the authors identify specific actions of the teacher with two distinct but interrelated main objectives: (i) to promote the students' mathematical learning; and (ii) to manage the students, the class and the functioning of the classroom. This appears to be a fairly complete picture of what may be the teacher's deliberate actions in inquiry-based teaching, yet we do not find all these aspects in a single lesson with these characteristics. Summing up some of the most important aspects of the teacher's role during the course of this type of lesson (Canavarro, Oliveira & Menezes, 2012), we then intend to broach the various phases of the lesson.

In the *introduction of the task* phase, the teacher has to ensure that the students «own» the task. He/she must see that they are willing to take part, so that their mathematical activity can develop. He/she also has to organize the class and provide resources for the task to get done.

In the next phase, the *students work on the task* in small groups or pairs. The teacher has to ensure that the task runs smoothly, without compromising student's autonomy or lessening the task's cognitive challenge (Stein & Smith, 1998). The teacher also needs to pay particular attention to the quality of the interaction between the students, while ensuring that they come up with materials that are suitable for presentation to the whole class. By this stage, teacher should have a good grasp of the work being done by the different groups in order to choose and sequence the solutions to be submitted to the larger group. This will be done in accordance with the criteria the teacher has defined in advance.

Then there is the *discussion of the task* as a whole group, in which previously selected solutions are presented by the students. The literature recognises this as a particularly demanding moment for the teacher in management terms, especially if the students have been working on challenging tasks (Cengiz, Kline & Grant, 2011; Scherrer & Stein, 2013). During this phase, the teacher must create and maintain an appropriate environment for presentation and discussion, by both promoting and managing student's participation. The teacher should ensure that the students listen and intervene appropriately and productively so that meaningful mathematical discourse will develop. Promoting the mathematical quality of the students' presentations is fundamental to achieving lesson's goals and furthers the students' mathematical understanding.

Finally, there is the phase in which the teacher *systematizes the key learning points* highlighted by the task, particularly points that have emerged during the discussion of the students' problem-solving strategies. Here there is more focus on the teacher. New concepts may arise or be synthesized and reviewed and procedures with which the students are already acquainted may be linked to other topics or concepts, and to transverse mathematical processes (Canavarro, Oliveira & Menezes, 2012). This is a crucial stage in which the knowledge building activity gives way to an understanding of mathematical ideas in the sense described by Wells (2004).

Within this framework of inquiry-based teaching, the teacher assumes a demanding and important role in promoting learning: from her/his choice and selection of tasks, to the structuring of the lesson and the support she/he gives to the students' mathematical activity. Inquiry-based teaching, as we understand it in this study, does not imply that the students participate in designing the curriculum by creating their own issues. Rather, it is based on a dialogical learning perspective in which knowledge is actively and collaboratively constructed by students, in environments that have been purposely created and sustained by the teacher.

Research has shown that the new perspectives on teaching underlying many teacher education programmes and curriculum reforms that have flourished in recent decades are very complex for prospective teachers (Lampert & Ball, 1998). These new perspectives are sometimes considered «ambitious practice» (Kazemi, Franke & Lampert, 2009), raising important questions about how to support prospective teachers to develop a professional knowledge that takes that practice as a goal. Although many prospective teachers are receptive to innovative ideas they sometimes misconstrue these ideas, and believe the myths regarding the impact such reforms have on the classroom (Oliveira & Hannula, 2008). These misconceptions are fuelled by the prospective teachers' inexperience and the lack of past exposure to such ideas and approaches (Towers, 2010).

Inquiry-based teaching has its own specificity that involves unknown dimensions for prospective teacher, who neither experienced it as students nor had many opportunities to observe it in classroom settings. Therefore, we have to create new contexts that further their grasp of this type of teaching.

Assuming a dialogical perspective regarding the knowledge for teaching (Wells, 2004), according to which the knowledge is constructed through personal experiences in progressive cycles that lead to understanding, the prospective teacher may gradually develop an understanding of inquiry-based teaching. The prospective teacher's first contact with this perspective of teaching and learning should be seen as an encounter with «information» (Wells, 2004). The theoretical ideas that are conveyed by the teacher educator or are present in the literature have to be processed and articulated through personal experience (Wells, 2002). This process of knowledge construction is essentially interactive and social, to the extent that «the individual is engaged in meaning-making with others in an attempt to extend and transform their collective understanding with respect to some aspect of a jointly undertaken activity» (Wells, 2004, p. 84); and it involves «constructing, using and progressively improving representational artefacts of various kinds» (Llinares & Valls, 2009, p. 249). Finally, understanding is seen as the culmination of this process of knowledge construction, which is action-oriented, and as a continuous process of enrichment of the personal framework from which the new experiences will be interpreted (Wells, 2004).

This model of knowledge which assumes the shape of a spiral that takes in all four quadrants (experience, information, knowledge building and understanding, according to Wells, 2002), enables us to sustain that the prospective teacher's understanding of inquiry-based teaching will develop at different moments, arise from multiple experiences, and be enriched by interaction with others.

In a study conducted with prospective teachers, Llinares and Valls (2009) maintain that the prospective teachers built up knowledge of mathematics teaching as a result of the arguments they put forward while interacting with their colleagues. These authors witnessed the emergence of reifications that became the subjects of discussion regarding the lessons they had observed that contributed to expand their knowledge about teaching.

This topic is also featured in a study by Towers (2010), who presents the case of a teacher, starting his career, who had difficulty in describing his vision of inquiry-based teaching to his peers. Although he put various aspects of this type of teaching into practice – for example, by promoting student's participation in large and small groups, demonstrating interest in student's alternative strategies and fostering their mathematical thinking –, he was ultimately unable to articulate his ideas, which made it difficult to collaborate with other teachers who did not share his vision of teaching.

MULTIMEDIA CASES IN INITIAL TEACHER EDUCATION

The use of videos and other multimedia resources in the classroom has become increasingly more frequent in initial teacher education. On the one hand, it compensates for the prospective teachers' lack of contact with in-class practice, and enables them to deepen their knowledge of teaching. On the other hand, it encourages the development of analytical skills targeting classroom practice, which is considered relevant to the growth of a professional teaching perspective (Koc, Peker & Osmanoglu, 2009; Stürmer, Königs & Seidel, 2013).

The video is a powerful resource that conveys a realistic image of the classroom. It contains real images of the students and teachers, and captures the voices, body language and environment of the classroom (McGraw, Lynch, Koc, Budak & Brown, 2007). By analysing a teaching and/or learning situation, prospective teachers may focus on a number of target points such as the students' thinking, the teachers' role or classroom discourse (Alsawaie & Alghazo, 2010; Koc, Peker & Osmanoglu, 2009; McGraw *et al.*, 2007). However, the focus can also be directed toward a particular curriculum topic which enhances mathematical knowledge for teaching, or toward principles associated with pedagogical knowledge (Climent *et al.*, 2013; Seidel, Blomberg & Renkl, 2013).

Many teacher education courses integrate other resources, such as theoretical elements, interviews with teachers, resources used in lessons, students' productions, etc., which can be accessed electronically at any moment (Llinares & Valls, 2010; McGraw *et al.*, 2007). With this diversity of elements, one can design multimedia classroom cases that encompass the complexity and the different strands of teacher practice, include information of different types, and in some studies, be taught in tandem with virtual discussion settings (Koc, Peker & Osmanoglu, 2009; Llinares & Valls, 2010). These contexts have proved to be instrumental in building bridges between theoretical and practical knowledge. They allow the prospective teachers to develop a reference framework for analysing observed practice and a «metalanguage» to discuss it with their peers (Llinares & Valls, 2010).

The multimedia case in this study illustrates the inquiry-based teaching practice of a 7th grade teacher that unfolds from a mathematical task entitled «Election of the Class Representative» (Annex 2) (Oliveira, Canavarro & Menezes, 2012a). The case exposition has a narrative structure, as it contains a sequential analysis of the lesson and its preparation, starting with the task selected and the lesson plan designed by the teacher. It also discusses the teacher's intentions with regard to each phase of the lesson (introduction, students' autonomous work, discussion and systematization).

For each phase of the lesson, video segments are presented, accompanied by a transcription of teacher and student's interventions. Questions are asked to help prospective teachers examine particular aspects of the teacher's actions that seek to promote learning and the classroom management. The student's solutions, and the teacher's analyses and explanations about those are also available.

Finally, the prospective teachers are asked to focus on the teacher's reflections within the framework, «Intentional Actions of the Teacher in an Inquiry-Based Classroom» (Annex 1). Prospective teachers also use this framework to retrospectively examine the analysis of the teacher's practice that they did throughout the case, particularly to identify points that were included or omitted.

Along the multimedia case, small text excerpts, called «Synthesizing» are presented, enabling prospective teachers to systematise the main ideas the authors wish convey with regard to the teacher's role during each phase of the lesson. Additionally, the website on which the case has been posted contains suggestions of readings on inquiry-based teaching.

We chose this particular teacher for this multimedia case because she consistently espouses inquiry-based teaching. The case also includes interviews with the teacher about the lesson, in which she explains the reasons behind her choices and the doubts and difficulties she faces. By including these details, we hope to provide prospective teachers with further insights into authentic teaching practice (Oliveira, Menezes & Canavarro, 2012b). This multimedia case is, therefore, a contextualized narrative that aims to be an instance of more general classes of ideas about inquiry-based teaching hence allowing multiple readings and interpretations (McGraw *et al.*, 2007).

The lesson featured takes place during the last part of the topic «First Degree Equations». According to a perspective of algebraic thinking development, the teacher hopes that the students, through the solution and discussion of the task, can apply and systematize the knowledge they have acquired and establish connections between topics in algebra (Oliveira, Canavarro & Menezes, 2012a). Since this was the first year the students worked with algebraic language and solved equations, this task was challenging for the students both in interpreting the situation and in the solving process.

The work on the multimedia case formed one of the modules in the course on Mathematics Teaching Methods, which targets prospective mathematics teachers. It is given during the 3rd semester of the master's in Mathematics Teaching for middle and secondary school. The module was designed by the two authors of this study. One is the teacher of the master's course, while the other is exclusively a classroom researcher. However, during the whole process, there was close collaboration between both authors, and significant time spent on discussing how the work was progressing. Student evaluation was done by the course instructor.

The multimedia case was analysed in the classroom in four sessions of 2.5 hours apiece over the course of two weeks. The prospective teachers worked in pairs or groups of three, sharing a computer. Assuming the dialogical perspective of knowledge construction (Wells, 2004), the group was encouraged to read and interpret the material as autonomously as possible and to dispel their doubts by asking each other for clarification. However, they were free to seek the teacher's support on details regarding the case content.

In general, each exploration session began with a brief reference, by the teacher, to the written work done by the different groups in the previous session. The intention was to provide feedback on how this matched established objectives. Using a dialogical perspective of knowledge construction, although no collective discussion had taken place, prospective teachers, in small groups, discussed the ideas and negotiated what they would present as a written product of their work in each session. Analysis of the multimedia case assumed inquiry-based characteristics as well, since the questions posed were predominantly of an open-ended nature.

As far as working methodology was concerned, we must also emphasize the prominent role writing played, both as a product of small group work, and the written expression of individual reflections after these working sessions. This writing option is consistent with Wells' (2002) dialogic perspective, since according to this author, the construction of knowledge also occurs by means of writing. It is a dual dialogue: between the one who writes and the audience to whom it is addressed, and between the person himself and the text that emerges.

METHODOLOGY

The study took place within the framework of a broader Design Research project. Design Research involves a family of methodological approaches in which the research and development are mutually dependent (Cobb, Zhao & Dean, 2009). The module orienting the present study comprises experimentation with multimedia cases in teacher education that were built on research into inquiry-based teaching. The research reported in this paper focuses particularly on one of the aspects of learning targeted by this teacher education setting: the development of discourse on inquiry-based teaching.

Concerning data collection methods, we opted to focus on the learning that the prospective teachers displayed in the reflections they wrote on inquiry-based teaching. We did so because, as Wells, we believe that writing «encourages one to interrogate one's interpretations of others' utterances as well as of one's own personal experiences and beliefs in order to add to the ongoing dialogue» (2004, p. 129).

The class consisted of ten prospective mathematics teachers, the great majority of whom had no classroom teaching experience. Two had taught other subjects, while one had taught in higher education. Some had experience in individual remedial work or in small groups, giving private lessons.

For ethical reasons, one of the prospective teachers has not been included in this study, since the first author was also supervising his teaching practice.

For the data analysis, four major dimensions were defined: the nature of the task, articulation and purpose of each lesson phase, the teacher's role in accordance with the Intentional Actions' Table, and anticipated difficulties.

The first issue, the nature of the task, was not explicitly discussed with the prospective teachers at the multimedia case, since it is a subject that is addressed throughout the course and in others that preceded it. The issue also arises in some of the texts listed as additional readings for the multimedia case (for example, Ponte, 2005), as well as in the Synthetizing section dealing with introduction of the task, on which the prospective teachers based themselves when they were completing their individual assessment work. Knowing the importance of the task within the framework of this type of teaching, through this dimension of analysis, we aim to understand to what extent the prospective teachers master this topic, namely by identifying specific characteristics of the proposed task in the case.

The phases in an inquiry-based lesson are one of the constructs highlighted and the multimedia case examination was clearly structured by the lesson phases. As a result of their work the prospective teachers were able to clearly distinguish the four phases and what they were composed of. However, it is also crucial that prospective teachers realize how the phases are articulated, how they contribute to the mathematical goals of the class (Canavarro, 2011), and that they do not develop a compartmentalized view of this type of practice.

The examination of the multimedia case in the course focuses strongly on the analysis of the teacher's practice, particularly in her role in the promotion of learning and in the lesson's management. Therefore, the teacher's role was also an object of analysis. The Framework for the Teacher's Intentional Actions (Annex 1) is an organizing tool specifically designed for prospective teachers to be able to analyse the teacher's practice in the final phase of exploration of the case. Therefore, we wanted to understand to what extent it becomes a reference to prospective teachers when they make a more global reflection on their experience in analysing an inquiry-based lesson.

Finally, we propose ourselves to analyse the challenges posed by inquiry-based practice, particularly those prospective teachers might face when they start teaching. This is important since research has demonstrated that both knowledge and disposition or belief about teaching should be taken into account in initial teacher education (Beswick, Callingham & Watson, 2012; Oliveira & Hannula, 2008).

For each of the dimensions we set up categories from the data analysis that we summarised in tables, naming the prospective teachers that they are associated with. We present illustrative examples of these categories, using excerpts from the prospective teachers' written reflections.

RESULTS

The results of the study are organized into four sections according to the dimensions mentioned. Throughout, we sought to characterize the knowledge of inquiry-based teaching the prospective teachers demonstrated.

THE NATURE OF THE TASK

The nature of the task that is presented in the multimedia case was not explicitly discussed in our sessions. However, the prospective teachers did discuss the topic in their written reflections, highlighting its importance in this type of teaching, and identifying specific characteristics of the proposed task.

> This is, therefore, an example of a lesson in which the idea of inquiry-based teaching of Mathematics was put into practice, suggesting students can work on interesting tasks, creating their own strategies and constructing knowledge in a way that highlights the need or benefit of a particular idea, concept or mathematical procedure (Simone).

> I think we should present several tasks, some of a more closed nature and others that lead students to explore, conjecture and discover for themselves, always with the teacher's support. The tasks should also enable students to progressively use symbolic notation and appropriate forms of representation. This multimedia case was a good example of a task with the features that I listed earlier (Vânia).

The prospective teachers highlighted that the task proposed in the multimedia case enabled the students: to submit an original answer, by resorting to previous knowledge and experiences; create their own strategies for solution; build knowledge that reveals the need or the advantage of an idea, concept or procedure; develop autonomy to explore, conjecture and discover; use progressively appropriate symbolic representations and notations; and develop their mathematical thinking.

The task worked in the multimedia case was included in the didactics unit that the students were working on. While the teacher worked through the Algebra theme, she displayed a marked intention to promote the students' algebraic thinking (ME, 2007). The prospective teachers recognized the task's potential to enhance the students' mathematics skills.

> (...) the task requires the students to be able to interpret and represent a contextualised situation, using algebraic language and procedures. It also requires them to solve problems, communicate, reason and shape the situation by resorting to mathematical concepts (equations, sequences ...) (Matilde).

The prospective teachers acknowledged that the task allowed students to interpret, represent, reason and solve a problematic situation using algebraic language and procedures. They realized that the work involving regularities, which had been dealt with before, allowed the students to establish a connection with equations. The fact that the students resorted to trial and error strategies, sequences and regularities and the construction of tables, enabled the teacher to establish connections between the different strategies adopted, and to promote the development of the students' algebraic thinking, thus deepening the study of algebraic relations and their symbolization, which is essential for the development of the notion of variables and the understanding of algebraic language (ME, 2007).

In Table 1 we present a summary of the characteristics of the task, as highlighted by the prospective teachers:

THE TASK	PROSPECTIVE TEACHERS
It is not a simple exercise in which the students have to apply previously acquired knowledge; it allows for different solution strategies.	Antónia, Lourenço, Margarida, Simone, Vânia, Bárbara
It has the potential to develop skills in students: problem-solving, mathematical thinking and mathematical communication.	Antónia, Lourenço, Matilde, Vânia, Sílvio, Simone
It enables the development of the students' algebraic language and thinking.	Antónia, Matilde, Vânia, Sílvio, Simone
It promotes the understanding of the need or benefit of a particular idea, concept or mathematical procedure.	Lourenço, Margarida, Matilde, Simone

TABLE 1 — TASK CHARACTERISTICS

Most of the prospective teachers refer to at least two characteristics of the task dealt with in the lesson, although some of them also reported on the mathematical tasks within a framework of inquiry-based teaching in a more general way. Only one prospective teacher (Sandra) does not mention this matter in her reflection and another (Bárbara) refers to only one of these aspects.

THE LESSON PHASES

One of the constructs that was highlighted the most was the definition of the four phases associated with the task because, as we have mentioned, the structuring of the case was clearly oriented by the different phases. However, in addition to recognizing and distinguishing the phases, prospective teachers were able to see how they are articulated and contribute to the mathematical goals of the lesson, and thus the development of algebraic thinking.

The prospective teachers said that in the introduction phase the students had time to take ownership of the task, understand its goals and organize themselves to work. According to them, this allowed the students to interpret the task correctly, clarify doubts about language, and thus be able to distinguish what information they should retain and use for the solution.

In the prospective teachers' point of view, the presentation phase is also crucial to ensuring that the students are engaged, so that they feel challenged and confident enough to go on to the subsequent activity.

> It is important to challenge them, so that the students take ownership of the task with enthusiasm and curiosity, committing themselves to solving it. It is also the moment when the class organizes their work, defines timing, and manages resources and classroom working methods (Lourenço).

With regard to the second phase of inquiry-based teaching, the prospective teachers acknowledged that the main objective was for the students to work autonomously to complete the task. They could see the videotaped students had been given the opportunity to develop their reasoning, discuss their ideas, explain and defend their ways of thinking, question the solutions presented by their classmates, use different records, explore their findings and acknowledge their mistakes and difficulties. They also recognized how important this moment was in building the students' skill at mathematical communication.

> (...) this moment proved to be important to the extent that the students were able to make different representations, discuss their ideas, explain their reasoning aloud to classmates, highlight different solution strategies, understand the reasoning of others, explore their discoveries, deal with their mistakes and difficulties and write their conclusions (Bárbara).

Referring specifically to the linkage between the different phases of the lesson, one of the prospective teachers mentioned that discussion of the task depends to a large degree on the activity that occurred in the previous phase.

> (...) the four phases are related, each one depends strongly on what they achieved, during the discussion held in the previous phase. For example, in my view, there is a strong relationship between the phase of students' autonomous work and the segments devoted to discussion and systematization, because, while during the phase of autonomous work students explore different strategies, and the students' solutions are selected and sequenced by the teacher, in the discussion, those same ideas are discussed collectively (Bárbara).

According to one of the prospective teachers, interaction among the students while they are working autonomously allows them to feel safe enough to share their solutions, voice their doubts and question their classmates' strategies during the discussion.

> Another thing that stood out in this case was the interaction with the dyads when they were completing the task. I consider the simple act of interacting as a way of helping the student believe in himself and feel safe. (...) During this time of collective discussion, the students had the opportunity to present their strategies, share their doubts and question the strategies presented by their classmates (Antónia).

The prospective teachers saw that in the task discussion phase the students were able to understand the differences between the various strategies presented, in particular, their mathematical effectiveness. They became involved in processes of explanation and justification and developed their mathematical communication skills. The following comment also shows that this prospective teacher understood how discussion of strategies can contribute to the systematization of learning:

> In this exploratory work, by trying out different solutions during their discussions, the students were also able to see that some ways of tackling the solution are more effective than others (Margarida).

They also believed that discussion of strategies had contributed to the systematization of mathematical learning, because students had been able to give meaning to the mathematical concepts being introduced or reviewed, establish connections and formalize and generalize the concepts in question.

> (...) the students [could] cement, clarify their ideas and establish connections between the various strategies (...) Thus, I believe that this phase has helped the students to have a global view of what they have learned and the different strategies that they could have chosen to solve the problem (Bárbara).

From the previous quotations, we can see that the prospective teachers were able to recognize specific student actions and reactions that occurred during each one of the phases (Table 2). The majority of prospective teachers highlighted eight or more student actions during the different phases. Three of the prospective teachers (Bárbara, Matilde, and Vânia) identified more than ten actions. Just one prospective teacher (Sandra) only recorded two actions. However, as we have seen, they also acknowledged the existence of different relationships between the lesson phases.

THE TEACHER'S ROLE IN AN INQUIRY-BASED LESSON

The prospective teachers linked some of the videotaped teacher's actions and intentions to each phase of this inquiry based teaching lesson, and interpreted the role of her actions in promoting lesson management and student learning.

The prospective teachers picked up on the fact that, during the introductory phase, the teacher had sought to ensure that the pupils understood the task. She read the task out to the students, discussed the meaning of expressions and concepts that could have caused confusion, established connections with previous experiences and made sure that all the students were on board with the task. They also grasped that the teacher had ensured that the pupils understood what was expected of them, informed them of how long they had, and offered resources that could help them complete their task.

PHASES OF LESSON	THE STUDENTS HAD THE OPPORTUNITY TO:	PROSPECTIVE TEACHERS
Introduction of the task	- take ownership of the task, and understand its goals	Bárbara, Lourenço, Margarida, Matilde, Vânia
	- organize the work	Lourenço, Matilde, Simone
	- correctly interpret the wording of the task and distinguish important information to retain and use	Bárbara, Margarida, Matilde, Simone, Sílvio
	- clarify doubts regarding language	Bárbara, Margarida, Sílvio, Vânia
	- be challenged	Lourenço, Sílvio
Students' autonomous work	- be autonomous	Antónia, Matilde, Sandra, Sílvio, Simone, Vânia
	- discuss ideas	Bárbara, Matilde, Vânia
	- explain and defend their ways of thinking, exploring their findings	Antónia, Bárbara, Matilde
	- question the solutions presented by classmates	Antónia, Bárbara, Matilde
	- use different records	Bárbara, Matilde, Vânia, Margarida
	- highlight different solution strategies	Antónia, Bárbara, Vânia
	- understand the reasoning of others	Bárbara, Margarida
	- deal with their mistakes and difficulties	Antónia, Bárbara, Matilde
	- formalize their conclusions	Bárbara, Vânia
Discussion of the task	- understand the differences between the various solution strategies and records	Bárbara, Lourenço, Margarida, Matilde
	- present solutions and justifications	Bárbara, Matilde, Vânia, Simone, Sílvio
	- share their doubts	Antónia, Simone
	- analyse the effectiveness of strategies and records	Bárbara, Lourenço, Margarida, Simone
	- understand concepts, processes and procedures	Bárbara, Lourenço, Margarida, Simone, Sílvio
Systematization	- give meaning to mathematical concepts presented or reviewed during the completion of the task	Bárbara, Sílvio, Vânia
	- establish a connection between the different strategies and concepts	Bárbara, Matilde, Simone, Vânia
	- record conclusions, definitions and properties	Sandra, Simone
	- accomplish learning	Bárbara, Matilde, Sílvio

TABLE 2 – STUDENTS ACTIONS IN THE LESSON PHASES IDENTIFIED BY PROSPECTIVE TEACHERS

> One of the teacher's concerns was to ensure that there were no language issues, in particular regarding the meaning of blank and null votes. Finally, she defined the working methodology needed to carry out the task, taking into account the characteristics of the room and the students' behaviour. She reminded the students that they would have to present their rationales. She also provided material (Vânia).

The prospective teachers highlighted the ways in which the videotaped teacher made sure that the task was completed. She gave students the space to develop their own solution strategies; monitored them and gave them feedback, mindful not to change the level of the task's cognitive demand; helped students who were having more difficulty or were more uninterested, without giving them the answer; challenged the students who were more advanced; encouraged the use of mathematical communication in the classroom; furthered the students' skill at argumentation; tried to interpret and understand how the students had solved the tasks; identified unexpected responses and errors; identified the potential of the strategies used by the students and supported them in the strategies they had chosen; and matched the students' written work with the expectations she had lined out when planning the task, so that she could organise the sequence in which the students presented their solutions during the discussion (third phase).

> Claudia, the teacher, always supported students in the strategy that they had chosen, not influencing them to take a certain path. This teacher was also very careful never to validate the students' mathematical output, in order not to dash their expectations regarding the work they had already done or were about to do. However, it is crucial for the teacher to understand the mathematics of her students, who sometimes, despite being correct, go in a different direction than the teacher expected/planned (Antónia).

The prospective teachers stressed that during the discussion phase, the teacher had been committed to making sure that the pairs shared and compared several solution strategies. She also made it clear that there is no one single strategy that is ideal and infallible. The future teachers also noted that she had analysed the potential of their strategies and discussed their weaknesses, so that the students could learn – not only from the activity itself – but also from the reflections about it. In addition, as they noted, she articulated the students' ideas of what they were expected to learn; discussed the mathematical learning involved in the task; and established connections between the various mathematical topics involved.

> The teachers' main concern should be to promote mathematical communication, through reflection and instruction because this is how enriching learning experiences emerge. It highlights how positive questioning can be since it focuses the students' attention on the task and helps them to make their reasoning explicit. Encouraging communication, the explanation of ideas and questioning among classmates, and not just between the teacher and the class (Sandra).

This phase contributed to the accomplishment of some of the lesson's mathematical goals. The presentation of different equations for the same problem allowed the students to discuss whether or not the equations were equivalent. The students also had the opportunity to establish connections between different types of mathematical records and to see the different strategies they could use to solve the problem.

> One particularity was that there were groups with different equations for the problem. The teacher explored the idea that these equations were not all the same because the variable represented different things in different cases (Simone).

According to the prospective teachers, during the systematization phase, the teacher assumed the role of protagonist, formalizing concepts, ideas and procedures related to the topic and focusing on the transverse skills that had been highlighted during the discussion phase. The teacher revisited concepts learned during the teaching unit, established connections with previous learning, summarized the conclusions reached during the task and guaranteed that the main ideas were written down.

Table 3 shows the videotaped teacher's actions, during the lesson's four phases, that the prospective teachers noted. Seven prospective teachers refer to at least eight actions that typify the teacher's role within a framework of inquiry-based teaching. However, there is one prospective teacher (Matilde) who only mentions two aspects.

PHASES OF LESSON	THE TEACHER SOUGHT:	PROSPECTIVE TEACHERS
Introduction of the task	- to ensure the understanding of the task and that all students joined in	Bárbara, Lourenço, Simone
	- to discuss the significance of the task's expressions and concepts	Bárbara, Lourenço, Vânia
	- to establish connections with students' previous experiences	Simone
	- to guide the class organization	Bárbara, Lourenço, Simone, Vânia
	- to offer instruments/records that assist the students in solving the task	Lourenço, Margarida, Vânia
Students' autonomous work	- to promote autonomy	Lourenço, Margarida, Sandra, Simone, Vânia
	- to monitor and give feedback to students during the interactions	Bárbara, Sandra, Simone, Vânia
	- to assist students with difficulties	Margarida
	- to challenge the students	Bárbara, Margarida, Simone, Vânia
	- to promote communication in the classroom	Bárbara, Lourenço, Simone, Vânia
	- to interpret and understand how students solved the task	Antónia, Vânia, Sandra
	- to identify the students' unexpected responses and errors	Antónia, Margarida
	- to identify the potential of the students' strategies and to support them in developing them	Antónia, Bárbara
	- to compare the students' written work with the expectations that she had when the task was planned	Antónia, Margarida
Discussion of the task	- to provide the sharing of several strategies and mutual respect with regard to their classmates' explanations	Antónia, Lourenço, Sandra, Sílvio
	- to define the order in which the solutions would be presented	Lourenço, Margarida, Sílvio, Simone
	- to design questions that promote student reflection and to assist them in clarifying their ideas, without validating or discrediting their work	Lourenço, Margarida, Sandra, Sílvio, Simone
	- to compare solutions, analyse their potential and discuss their weaknesses	Lourenço, Margarida, Sandra, Sílvio, Simone
	- to approximate and articulate the students' ideas	Antónia, Sandra
Systematization	- to formalize concepts, ideas and procedures	Antónia, Matilde, Simone, Vânia
	- to foster transverse skills	Simone
	- to establish connections between the types of learning targeted	Antónia, Bárbara
	- to ensure that the main ideas are written down	Lourenço, Matilde, Simone

TABLE 3 – TEACHER'S ACTIONS IN THE COURSE OF THE FOUR PHASES OF THE LESSON, AS NOTED BY THE PROSPECTIVE TEACHERS

ANTICIPATING CHALLENGES

The multimedia case led prospective teachers to reflect on the challenges that arise in inquiry-based teaching, particularly those that they may experience – or have already experienced – in their own teaching. They stressed that, due to their lack of experience, they would find it hard to design interesting and productive tasks; to anticipate different solutions the students can come up with and how to link those solutions to the mathematical goals that had been set; to establish connections among mathematical ideas; to select student strategies and sequence and manage the discussion among students.

Other classroom management issues the prospective identified as potentially challenging were: how to manage time, student interaction, certain events that occur simultaneously in the classroom and discussions that involve decision-making and affect the students' learning.

> Regarding difficulties (...) I identified three that seem relevant:
> – Being able to reach all students and pay attention to various aspects that occur simultaneously in the classroom;
> – Making decisions that can affect [students'] learning opportunities. For example, in one phase of the lesson, the teacher decided to go beyond the time she had planned and justified her decision by the richness and diversity of strategies that emerged;
> – «Orchestrating» the discussion in large groups and promoting student reflection. This is one of the roles that I find is frequently referred to as being one of the most difficult for the teacher (Lourenço).

To encourage students to reflect, and to maintain the level of cognitive demand posed by the task, to explore their thoughts, poll and give meaning to their ideas, to design questions that enable students to progress without giving answers were the points prospective teachers deemed as potentially challenging in this type of teaching.

> Other difficulties that may arise are managing the interactions among students; getting all the groups to work and seeing that within each group all the students work; ensuring that the students do not digress but keep focused during the solution; and being able to resist giving direct answers to the students (Margarida).

The following table lists the challenges of inquiry-based teaching, as noted by prospective teachers. Most of them describe four or more aspects. Two prospective teachers made no reference to this topic in their reflections.

ANTICIPATED CHALLENGES:	PROSPECTIVE TEACHERS
– finding interesting , enriching tasks to create learning opportunities	Margarida, Sílvio, Vânia
– anticipating different student strategies	Sílvio
– relating the solutions to the mathematical purpose of the lesson	Sílvio, Vânia
– establishing connections between mathematical ideas	Bárbara, Matilde
– selecting the strategies and sequencing them	Matilde
– managing the discussion so that everyone participates	Lourenço, Margarida, Matilde, Sílvio, Simone
– managing interactions among the students	Lourenço, Margarida, Simone
– preparing questions for the students to think about, so that they can accomplish their tasks	Bárbara, Margarida, Matilde, Vânia
– managing time	Lourenço, Vânia

TABLE 4 – CHALLENGES OF INQUIRY-BASED TEACHING ANTICIPATED BY PROSPECTIVE TEACHERS

DISCUSSION AND CONCLUSIONS

Throughout this study, we have sought to understand the knowledge prospective teachers display regarding inquiry-based teaching. We have based our work on their analyses of a multimedia case featuring a teacher teaching a 7th grade mathematics lesson. We identified four important dimensions that enabled us to examine how this knowledge was expressed in the prospective teachers' written reflections.

Realizing how important the first dimension – the nature of the task – is to this type of teaching, we tried to judge to what extent the prospective teachers highlight this topic and identify the specific characteristics of the task in the case. Although at the outset, this task could have been construed merely as an attempt to solve an equation, the prospective teachers were able to discern its contribution to achieving the mathematical goals of the lesson. They were also able to relate it to inquiry-based teaching practice. They acknowledged that the task had given the students the freedom to create their own strategies and assess the need for or the advantage of a particular mathematical idea, concept or procedure. They recognised that it had also helped students develop their mathematical skills, in particular algebraic thinking, mathematical reasoning and mathematical communication.

Understanding and reflecting on the task's role enables prospective teachers to choose tasks that suit their teaching goals and prioritize them according to their potential to trigger complex ways of thinking that help students establish connections to meanings or to ideas and mathematical concepts (Cyrino & Jesus, to appear). However, since not all prospective teachers elaborated on this topic sufficiently in their reflections, we did not obtain a complete picture of the knowledge that the whole group had acquired on this aspect of an inquiry-based teaching.

Most of the prospective teachers concluded that the mathematical task does not guarantee a significant mathematical activity by itself. They have overcome a naive vision that in order to transform the mathematics teaching, it is enough to propose good tasks. They very obviously came to realise that learning indeed occurs through enriching mathematical activities, but that is necessary that students reflect on these activities, an idea that this was well illustrated in the way the videotaped teacher organised the lesson and in her oral observations about it.

The multimedia case we used was clearly driven by the different *phases of an inquiry-based lesson* (introduction, implementation, discussion and systematization). The prospective teachers were able to recognize and distinguish between each of the phases, realizing that they articulate and complement each other in developing the students' algebraic thinking, problem-solving skills, mathematical reasoning and communication. This recognition is fundamental if prospective teachers are to understand the advantages of this type of lesson structure, and see that it is not rigid.

Thus, it is significant that the prospective teachers were able to link specific actions and/or features in the students' learning to each phase of the lesson, noticing that the students' role is front and centre in this type of teaching, although this is not the analytical perspective adopted in the multimedia case which focuses on the teacher's intentional actions (Annex 1). We see, with their comments, the elements of a dialogical learning perspective (Wells, 2004) in which language is assumed to play a central role in the knowledge processes, in the interaction between teacher and students, and among the students, throughout the various phases of the lesson.

Regarding the teacher's role in inquiry-based teaching, prospective teachers pinpointed several teacher actions belonging to each lesson phase of the videotaped class. They emphasized aspects of teaching practice such as questioning in order to understand the students' thinking and promoting communication among students, which showed that they were picking up on the dialogical perspective of learning.

In the last section of the multimedia case, we presented the framework entitled «Intentional Actions of the Teacher in Inquiry-Based Teaching» (Annex 1). This framework has become an important tool for prospective teachers as far as the two major dimensions of teaching practice are concerned – promoting learning and classroom management. It has also served to clarify the various approaches that the phases encapsulate. It seems quite significant that, as prospective teachers, they chose to talk about the teacher's role in inquiry-based teaching in a way that was not restricted to the ideas and vocabulary that were presented in the framework and in a way that cogently covers the various phases of the lesson.

Finally, when it came to identifying the challenges of inquiry-based teaching practice, it appears that the majority of prospective teachers did anticipate many of the difficulties that may arise. They refer to aspects that show that they recognise the characteristics of this type of teaching and the actions required of the teacher. Among the challenges they cite are: choice of appropriate tasks; classroom management (including communication); dealing with the student ideas and output through appropriate questioning, while not limiting their mathematical activity; and establishing connections among mathematical ideas. It should be noted that most of the prospective teachers went beyond the videotaped lesson being analysed and were able to anticipate the challenges on a more general level. Even so, there were two students who did not mention any type of challenge. This lack of foresight is critical because, without the ability to predict, the future teacher will not be able to problematize. By anticipating challenges, beginning teachers are projecting themselves into the teaching role while developing a more realistic view of this type of teaching, which is indeed complex.

By watching the lesson featured, the prospective teachers were given access, to a type of inquiry-based teaching they were not familiar with because they had not experienced it as students, nor had they been exposed to it before in their teacher education programme. Knowledge is constructed from personal experiences in progressive cycles that lead to understanding (Wells, 2004). Hence, exploration of the multimedia-based lesson proved to be very promising because firstly, it gave the student teachers access to a body of information, by means of successive approximations. It enabled them to witness the classroom experiences of both a teacher and her students in a way that embodied the theory. The construction of knowledge was intensified, since the prospective teachers became involved in the production of meanings; constructing and reconstructing collective concepts about the nature of the task by working among themselves and in conjunction with the teacher educator; understanding and articulating the purpose behind the lesson phases; grasping the nature of the teacher's role within the framework of inquiry-based teaching; and fathoming the complexity associated with this type of practice. This process of construction of knowledge was essentially social and interactive; and this enabled them to engage in a type of discourse about inquiry-based teaching, an approach that is considered to be complex and challenging for prospective teachers.

By viewing real videotaped classroom material, prospective teachers improve their ability to interpret the overarching topics and nuances of teaching, and get to see how much the teacher's actions impact classroom dynamics and the students' learning (Alsawaie & Alghazo, 2010). The opportunity to witness relevant teaching practice plays an important role in the development of their knowledge about teaching; the same has been true in other studies where classroom video analysis has been used (Koc, Peker & Osmanoglu, 2009; Stürmer, Königs & Seidel, 2013).

In their written reflections, the prospective teachers often refer to particular elements in the multimedia case that illustrate or substantiate statements about inquiry-based teaching. This reinforces the idea that the knowledge that they are developing about this type of teaching is situated, although it may represent more extensive classes of ideas (McGraw *et al.*, 2007). In this multimedia case the theory was not presented in the beginning. It was, rather, built from the analyses that were done. In the final phase, when the students already had the reference framework, Intentional Actions of the Teacher in Inquiry-Based Teaching, in their possession, they were able to review the practice having observed the theory in action, discuss it, and attribute meaning. The overall comprehension of inquiry-based teaching that they demonstrated in the reflective essay they wrote afterwards showed that they had taken in the core features of inquiry-based teaching. However, the role of theory in analysing or reflecting on a situation or observed practice is not linear (Llinares & Valls, 2010), which is why we need to further explore the issue in future research on the topic.

In conclusion, our study shows that, in general, the prospective teachers who participated indeed gained knowledge of inquiry-based teaching, recognizing its main characteristics and potential, and the challenges it may pose. The study highlights the potential such a teacher education setting has for the development of a dialogical perspective of learning by prospective teachers. In the study, they acknowledged the central role language plays in the knowledge processes, in the interaction between teachers and students, and among the students themselves. They also highlighted the role of teacher questioning in understanding the students' thinking, and in promoting communication.

Using this study as a departure point, it would be interesting to embark on a more individualized analysis, using other data sources, so that we might develop an even deeper understanding of the knowledge each prospective teacher gains with regard to inquiry-based teaching. This analysis and further research will enable us to determine what areas of this teacher education setting should be invested in and which should be reformulated, so that prospective teachers come to realise that this complex teaching approach is not only the bailiwick of a few expert teachers (Kazemi, Franke & Lampert, 2009; Santagata & Guarino, 2011). Future research should attempt to unveil how prospective teachers integrate these ideas into the next phase of teacher education – their supervised teaching practice – which occurs in different contexts, some of which are more conducive to inquiry-based teaching than others.

REFERENCES

Alsawaie, O., & Alghazo, I. (2010). The effect of video-based approach on prospective teachers' ability to analyze mathematics teaching. *Journal of Mathematics Teacher Education, 13*(3), 223-241.

Beswick, K., Callingham, R., & Watson, J. (2012). The nature and development of middle school mathematics teachers' knowledge. *Journal of Mathematics Teacher Education, 15*(2), 131-157.

Canavarro, A. P. (2011). Ensino exploratório da Matemática: Práticas e desafios. *Educação e Matemática, 115*, 11-17.

Canavarro, A. P., Oliveira, H., & Menezes, L. (2012). Práticas de ensino exploratório da Matemática: O caso de Célia. In L. Santos, A. P. Canavarro, A. M. Boavida, H. Oliveira, L. Menezes & S. Carreira (Eds.), *Investigação em* *educação matemática 2012: Práticas de ensino da matemática* (pp. 255-266). Portalegre: SPIEM.

Cengiz, N., Kline, K., & Grant, T. J. (2011). Extending students' mathematical thinking during whole-group discussions. *Journal of Mathematics Teacher Education*, *14*(5), 355-374.

Chapman, O., & Heater, B. (2010). Understanding change through a high school mathematics teacher's journey to inquiry-based teaching. *Journal of Mathematics Teacher Education*, *13*(6), 445-458.

Climent, N., Romero-Cortés, J., Carrillo, J., Muñoz-Catalán, N. P., & Contreras, L. (2013). Qué conocimientos y concepciones movilizan futuros maestros analizando un vídeo de aula? *Relime*, *16*(1), 13-26.

Cobb, P., Zhao, Q., & Dean, C. (2009). Conducting design experiments to support teachers' learning: A reflection from the field. *Journal of the Learning Sciences*, *18*(2), 165-199.

Cyrino, M., & Jesus, C. (to appear). Análise de tarefas matemáticas em uma proposta de formação continuada de professoras que ensinam Matemática. *Ciência & Educação*.

Kazemi, E., Franke, M., & Lampert, M. (2009, July). Developing pedagogies in teacher education to support novice teacher's ability to enact ambitious instruction. *Keynote Address to the 32nd Annual Conference of the Mathematics Education Research Group of Australasia, New Zealand*. Retrieved from http://www.merga.net.au/documents/Kazemi.pdf.

Koc, Y., Peker, D., & Osmanoglu, A. (2009). Supporting teacher professional development through online video case study discussions: An assemblage of preservice and inservice teachers and the case teacher. *Teacher and Teacher Education*, *25*(8), 1158-1168.

Lampert, M., & Ball, D. L. (1998). *Teaching multimedia, and mathematics: Investigations of real practice*. New York, NY: Teachers College Press.

Llinares, S., & Valls, J. (2009). The building of pre-service primary teachers' knowledge of mathematics teaching: Interaction and online video case studies. *Instructional Science: An International Journal of the Learning Sciences*, *37*(3), 247-271.

Llinares, S., & Valls, J. (2010). Prospective primary mathematics teachers' learning from on-line discussions in a virtual video-based environment. *Journal of Mathematics Teacher Education*, *13*(2), 177-196.

McGraw, R., Lynch, K., Koc, Y., Budak, A., & Brown, C. (2007). The multimedia case as a tool for professional development: an analysis of online and face-to-face interaction among mathematics pre-service teachers, in-service teachers, mathematicians, and mathematics teacher educators. *Journal of Mathematics Teacher Education, 10*(2), 95-121.

ME (2007). *Programa de Matemática do Ensino Básico*. Lisboa: DGIDC.

National Council of Teachers of Mathematics. (1994). *Normas profissionais para o ensino da Matemática*. Lisboa: APM.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Oliveira, H., Canavarro, A. P., & Menezes, L. (2012a). *Eleição para o delegado de turma – caso multimédia*. Retrieved from http://p3m.ie.ul.pt/caso-3-eleicao-para-o-delegado-de-turma-3-ciclo.

Oliveira, H., & Hannula, M. (2008). Individual Prospective Mathematics Teachers: Studies on their Professional Growth. In T. Wood & K. Krainer (Eds.), *International Handbook of Mathematics Teacher Education* (Vol. 3, pp. 13-34). Rotterdam: Sense Publishers.

Oliveira, H., Menezes, L., & Canavarro, P. (2012b). The use of classroom videos as a context for research on teachers' practice and teacher education. In *Proceedings of ICME 12* (pp. 4280-4289). Seoul: ICME.

Ponte, J. P. (2005). Gestão curricular em Matemática. In GTI (Ed.), *O professor e o desenvolvimento curricular* (pp. 11-34). Lisboa: APM.

Ponte, J. P., Brocardo, J., & Oliveira, H. (2006). *Investigações matemáticas na sala de aula* (2ª edição). Belo Horizonte: Autêntica.

Santagata, R., & Guarino, J. (2011). Using video to teach future teachers to learn from teaching. *ZDM Mathematics Education, 43*(1), 133-145.

Scherrer, J., & Stein, M. K. (2013). Effects of a coding intervention on what teachers learn to notice during whole-group discussion. *Journal of Mathematics Teacher Education, 16*(2), 105-124.

Seidel, T., Blomberg, G., & Renkl, A. (2013). Instructional strategies for using video in teacher education. *Teaching and Teacher Education, 34*, 56-65.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*, 268-275.

Stürmer, K., Königs, K., & Seidel, T. (2013). Declarative knowledge and professional vision in teacher education: Effect of courses in teaching and learning. *British Journal of Educational Psychology, 83*(3), 467-483. doi: 10.1111/j.2044-8279.2012.02075.x.

Towers, (2010). Learning to teach mathematics through inquiry: a focus on the relationship between describing and enacting inquiry-oriented teaching. *Journal of Mathematics Teacher Education*, *13*(3), 243-263.

Wells, G. (2002). Learning and teaching for understanding: The key role of collaborative knowledge building. *Social Constructivist Teaching*, 9, 1-41.

Wells, G. (2004). *Dialogic inquiry: Towards a sociocultural practice and theory of education*. Cambridge: Cambridge University Press.

	PROMOTION OF MATHEMATICS LEARNING	CLASS MANAGEMENT
Introduction of the task	*Guarantee the appropriation of the task by the students by:* – clarifying unfamiliar vocabulary – mobilizing and verifying prior knowledge – setting goals *Promote students' commitment to the task by:* – challenging them to work – establishing connections to students' prior experiences	*Organize the students' work by:* – establishing the time for each phase of the lesson – establishing the organizational structure of the work (individual, pairs, small groups, whole-class) – organizing classroom materials
Students' work on the task	Ensure the students carry the task through by: – posing questions and giving clues – suggesting representations – focusing on productive ideas – requesting clarification and justifications *Keep the cognitive challenge by:* – promoting student's reasoning – trying not to validate the mathematical correctness of students' answers	*Promote the work of students/groups by:* – setting up interaction among students – providing materials *Guarantee the production of materials for students presentations by:* – requesting their records – providing appropriate materials – setting aside time to plan the presentation *Consider the selection and sequencing of students presentations by:* – identifying solutions that are more or less comprehensive, complete and/or formal – identifying solutions with common errors – sequencing the solutions selected
Discussion of the task	*Promote the mathematical quality of the presentations by:* – asking for clear explanations with mathematical evidence – asking for justifications of the outcomes and representation used – discussing the difference and the efficacy of the solutions presented. *Promote interaction among students in the discussion of mathematical ideas by:* – encouraging questioning for the clarification of ideas – encouraging analysis, debate and comparison of ideas – identifying and making available to discuss questions or mistakes in the presentations	*Create a favorable environment for presentation and discussion by:* – putting an end to students' autonomous work – providing the reorganization of places to focus on a common resource (whiteboard, overhead…) – promoting an attitude of respect and attentiveness while presentations are being given *Effectively manage relationships among students by:* – establishing the order of presentations – justifying reasons some students do not present (to avoid repetition, ...) and changing the order of groups for the next task – promoting and managing student participation in the discussion
Systematizing mathematical learning	*Institutionalize concepts or procedures on mathematical topics by:* – identifying key mathematical concept(s) from the task, clarifying their definition and exploring their multiple representations – identifying key mathematical procedure(s) from the task, clarifying the conditions of their implementation and reviewing how they are used *Institutionalize ideas or procedures concerning the development of transversal skills by:* – identifying and connecting them – enhancing the key factors involved in developing them *Establish connections with prior learning by:* – highlighting links with mathematical concepts, procedures and transversal skills previously explored	*Create an appropriate environment for systematization by:* – focusing students attention on collective systematization – emphasizing the importance of this phase of the lesson for student learning *Guarantee that the systematization ideas are written down by:* – recording them on the computer or other physical devices (boards, interactive boards, transparencies, posters ...), which may be done by the students or the teacher – asking students to write down their work

ANNEX I – INTENTIONAL ACTIONS OF THE TEACHER IN INQUIRY-BASED TEACHING (OLIVEIRA, CANAVARRO & MENEZES, 2012A)

projeto
P3M

TASK «ELECTION OF FOR THE CLASS REPRESENTATIVE»

The teacher coordinating the election of the class representative reported that:

- All students in the class voted (30 students) and that there were no null nor blank votes
- Only three students received votes: Francisca, Lucas and Sandra
- Lucas received two votes fewer than Francisca
- Sandra received twice as many votes as Lucas.

Who won the election? With how many votes?

Do not forget to present and explain how you found the solution.

ANNEX 2 – THE MATHEMATICAL TASK (OLIVEIRA, CANAVARRO & MENEZES, 2012A)

MATHEMATICS PROFESSIONAL DEVELOPMENT RESEARCHERS AS STAKEHOLDERS

Paola Sztajn
psztajn@ncsu.edu | North Carolina State University, United States of America

ABSTRACT

In this paper I argue that since the publication of *The Reflective Practitioner* (Schön, 1983), mathematics professional development researchers have focused on bringing teachers' knowledge to the foreground, leaving behind the value of their own research community's knowledge. I revisit Schön's criticism of the technical rationality and use examples from my own practice in mathematics professional development to suggest that instead of continuing to reject technical rationality, mathematics professional development researchers should consider a revised version of it to move the field forward: one that values both teachers' and researchers' knowledge.

KEY WORDS

Mathematics education; Professional development; Technical rationality.

SISYPHUS
JOURNAL OF EDUCATION
VOLUME 1, ISSUE 3,
2013, PP. 246-269

Mathematics Professional Development Researchers as Stakeholders

Paola Sztajn

INTRODUCTION

This paper addresses a major current challenge for mathematics education researchers: to demonstrate the unique contribution of their research knowledge to discussions about K-12 mathematics. I take for shared that the ultimate goal of mathematics education research is to contribute to the improvement of K-12 mathematics for all children. However, I believe there is no shortage of situations designed to bring together stakeholders in K-12 mathematics that do *not* include mathematics education researchers. Parents, mathematics teachers, mathematics specialists, school administrators, professional development providers, mathematicians, engineers, scientists and others come together to discuss K-12 mathematics without considering that perhaps a mathematics education researcher should be included or is missing in the conversation. These stakeholders do not necessarily acknowledge the potentially unique contributions mathematics education researchers can make.

The situation just described is more complex in the case of mathematics *professional development* researchers, that is, the subset of members of the larger mathematics education research community who have teacher professional development as their research foci. As researchers in a newer field within mathematics education, mathematics professional development researchers have yet to organize and present their growing body of research-based knowledge in a coherent standardized way (Sztajn, 2011) that highlights its contributions. The discussion in the field still focuses on issues of rigour (e.g., NRC, 2002; Simon, 2004) when only attending to both rigour and relevance will position mathematics professional development researchers as stakeholders.

Further, as I will argue, mathematics professional development researchers have put their efforts in the past decades into making the case that teachers are key stakeholders in mathematics professional development research (Kieran, Krainer & Shaugnhessy, 2013). This effort has deviated attention from the goal of making mathematics professional development researchers stakeholders in K-12 mathematics.

Whereas I understand and respect colleagues who may not want to become stakeholders in K-12 mathematics because they believe the current system needs to be reconsidered from a more critical stance, I suggest the field needs to come together to establish that mathematics professional development research results are fundamental for K-12 mathematics. More important, mathematics professional development researchers make a unique contribution to discussions about the field that others cannot make.

Thus, in this paper, I position myself as a mathematics professional development researcher who is interested in establishing the value of research-based, scientific knowledge for K-12 mathematics. I contend that in discarding the technical rationality, Schön (1983) separated relevance from rigour and placed relevance with practitioners' knowledge whereas researchers' knowledge, at best, accounted for rigour. Therefore, since *The Reflective Practitioner* (Schön, 1983), mathematics professional development researchers have focused their attention on bringing teachers' knowledge to the foreground, leaving behind the value of their own research community's knowledge.

In what follows, I first revisit Schön's (1983) criticism of the technical rationality. I contend that, although *The Reflective Practitioner* was important because it made researchers attend to other professional rationalities including teachers' knowledge, researchers are inexorably connected to the scientific knowledge. I then attend to researchers' and teachers' knowledge to compare and contrast two professional development programs in which I have worked. The first took place in the early 2000s and was focused on establishing teachers as knowers (Sztajn, Hackenberg, White & Allexshat-Snider, 2007). The second took place in 2010 and focused on teachers and researchers as boundary crossers between researchers' and teachers' knowledge (Sztajn, Wilson, Edgington & Myers, in press). I use these examples to suggest that instead of rejecting the technical rationality, mathematics professional development researchers embrace a revised version of it to move the field forward.

REVISITING *THE REFLECTIVE PRACTITIONER*

The crisis of professionalism is the theme that sets the stage for Schön's (1983) discussion of *The Reflective Practitioner*. Considering the various challenges that the 1970s and early 1980s posed to the 1960s' glorification of professionals, Schön claimed «In 1982, there is no profession which would celebrate itself» (p. 11). He argued that the professional claim to knowledge monopoly was questioned when professionals could no longer make their knowledge fit the inherently unstable nature of problems of practice. He noted that:

> [Leading professionals and educators] are disturbed because they have no satisfactory way of describing or accounting for the artful competence which practitioners sometimes reveal in what they do. They find it unsettling to be unable to make sense of these processes in terms of the model of professional knowledge which they have largely taken for granted (p. 19).

For Schön, questioning professional knowledge meant questioning the technical rationality that defined professional activity as the instrumental application of scientific theories and techniques to solving problems of practice. Although professionals can adapt their knowledge to the problems at hand, the technical rationality suggested they practiced «rigourously technical problem solving based on specialized scientific knowledge» (p. 22). Therefore, professionals used a knowledge base that was not only specialized but also scientific and standardized, and carried out solutions from one problem to the next as the application of general theories and principles.

Contrary to the notion of technical rationality, Schön argued that the problems of practice escaped scientific categories and presented themselves as unique and unstable. Therefore, competent practice could not be accomplished solely through the use of techniques derived from applied research. Schön proposed:

> Let us then reconsider the question of professional knowledge, let us stand the question on its head. If the model of Technical Rationality is incomplete,

> in that it fails to account for practical competence in «divergent» situations, so much the worse for the model. Let us search instead for an epistemology of practice implicit in the artistic, intuitive processes which some practitioners do bring to situations of uncertainty, instability, uniqueness, and value conflict (p. 49).

Changing the focus of attention from professionals to practitioners, he noted that competent practitioners recognize phenomena they cannot describe, make judgments based on quality for which there are no criteria, and apply skills for which there are no prescribed procedures. Further, competent practitioners turned thoughts into action and attended to the knowing that was implicit in the action. Schön concluded:

> Once we put aside the model of Technical Rationality, which leads us to think of intelligent practice as an application of knowledge to instrumental decisions, there is nothing strange about the idea that a kind of knowing is inherent in intelligent action. Common sense admits the category of know-how, and it does not stretch common sense very much to say that the know-how is in action (p. 49).

The Reflective Practitioner presented reflection-in-action as the way to account for how practitioners are knowledgeable in practice. Schön proposed that knowing is tacit and knowledge is implicit in the action of practice. Later, Schön (1987) explained the difference between reflecting on action and reflecting-in-action. The former (on action) occurs when, after the fact, one thinks back on accomplished practice to examine how knowing-in-action contributed to the outcomes of the situation. The latter (in action) occurs when unexpected situations arise and thinking reshapes practice as the practice is being carried out. Schön proposed that reflection in action questioned assumptions about the structure of knowing-in-action. Further, reflection-in-action led to on-the-spot experiment and thinking that affected what was being done. However, Schön noted that it was the careful reflection *on* previous reflections-in-action that began what he called «a dialogue of thinking and doing» (p. 31) through which one became skilful and acquired the artistry of practice.

Thirty years after its publication, *The Reflective Practitioner* continues to influence mathematics professional development researchers (e.g., Krainer, 2011). Schön's work directed attention to tacit knowledge in the practice of teaching and questioned the assumption that expert teaching was based on research knowledge. He called for a re-examination of the relation between educational research and teaching practice and a reconsideration of the value of research-based knowledge in relation to other forms of knowledge that exist in teaching. Schön helped establish that teachers had knowledge and their practice could not be reduced to an application of scientific knowledge.

The importance of establishing the value of teachers' knowledge was Schön's fundamental contribution and that continues to be important. The debate that followed, however, was whether this knowledge should replace research knowledge in improving mathematics teaching and learning. A review of the work spearheaded by *The Reflective Practitioner* is beyond the scope of this paper. However, in what follows, I briefly discuss the work of Munby (1989) and Eraut (1995). I borrow from these authors to disagree with the interpretation that bringing teachers' knowledge to the forefront eliminated the space for scientific knowledge in teaching.

Although Munby was a follower of Schön's work and Eraut was a strong critic, both argued that Schön did *not* call for the elimination of research knowledge in professional practice and did *not* suggest that research knowledge had no value for teachers. Rather, both Munby and Eraut proposed that although different, knowledge emerging from both scientific inquiry and reflection-in-action had a place in teaching and professional development. Their interpretation of *The Reflective Practitioner* called for increased attention to the knowledge generated through reflection-in-action, but not for a new hegemony of this knowledge at the expense of scientific knowledge.

Offering Language

Munby (1989) directly responded to criticisms that *The Reflective Practitioner* separated practitioners from the products of science and isolated technical rationality from reflective practice. He explained that he did not interpret Schön to be claiming that reflection-in-action was «the sole source of professional knowledge» (p. 6). Munby characterized Schön's contributions as highlighting the exaggerated emphasis that had been placed on formal knowledge at the expense of practical knowledge. He pointed out that knowledge from practice had gone unrecognized because it was not perceived as rigourous within scientific traditions. Thus, for Munby, what Schön did was to offer the field language to recognize and attend to overlooked elements of learning from teaching.

Munby (1989) attended to the concept of reflection-in-action as opposed to knowledge-in-action, as Schön's major contribution. He explained that in Schön's work, the focus of attention was on the meaning of «in-action» and not on the meaning of «reflection.» Munby considered fundamental the concept that teachers gained knowledge as they were teaching, that is, in the practice of teaching. However, Munby also considered that Schon's concept of reflection-on-action was powerful. Through this process, practitioners could attend to research knowledge as they «undoubtedly use the knowledge of technical rationality in their work» (p. 6). Thus, it was both through reflection-in-action during practice, and afterwards, reflection-on-action that could include scientific knowledge, that teachers' developed expertise.

Knowledge Generation

Eraut (1995) criticized Schön's work for being unclear and inconsistent in its analysis of knowledge. He recognized that there had been disagreements as to whether Schön's alternative epistemology was meant to replace or complement the technical rationality. He criticized Schön's work for oscillating between a radical rejection and an accommodatory stance toward scientific knowledge.

> A generous interpreter of Schön might argue that he is not discarding research-based professional knowledge but challenging inflated views of its practical significance. In particular, he is attacking the ideological exclusivity of a paradigm in which only knowledge supported by 'rigourous' empirical research is accorded any validity (Eraut, 1995, p. 10).

More interested in the discussion of innovation within professional knowledge instead of classification of types of knowledge, Eraut (1995) proposed that reflection-in-action was a process for knowledge generation and not a new kind of knowledge. He characterized knowing-in-action as being used in routine situations, whereas reflection-in-action was triggered by recognising that the situation being faced was in some respect unusual. However, Eraut was concerned with the time period for this reflection-in-action, and noted that such reflection, for teachers, was different than for other professionals, given the speed, amount, and uniqueness of the interactions teachers face daily in their classrooms.

RECENT DEBATES

Similarly to the argument I will make in the remainder of this paper, researchers in various fields have recently returned to Schön's work to suggest that, despite the prevalence of reflective practice in the education of professionals, development of professional knowledge does not require the abandonment of technical rationality. Kinsella (2007) examined the contributions of Schön and Dewey and suggested that, in the field of nursing, the common interpretation of reflective practice as a «theory that sets up a dichotomy between technical rationality and an epistemology of practice» (p. 109) was an oversimplification. Kotzee (2012) noted that in continuing education, reflective practice had become mainstream but lacked attention to social aspects of learning and of practice. This attention to social aspects necessitated a review of reflective practice. In architectural education, Webster (2008) examined how Schön's ideas had become the dominant theory of practice and acknowledged the important contribution Schön made in establishing that professionals developed tacit knowledge through experience and reflection. However, she highlighted the role other theories of knowledge play in the development of architectural learning and suggested that those who value Schön's contributions should also recognize «the 'partial' nature» (Webster, 2008, p. 72) of his contributions. In line with these current re-examinations of the role of the *The Reflective Practitioner* in professional education, I consider the role of Schön's ideas in mathematics professional development.

MATHEMATICS PROFESSIONAL DEVELOPMENT AND *THE REFLECTIVE PRACTITIONER*

Stimulated by *The Reflective Practitioner*, the discussion about the nature of teachers' knowledge impacted mathematics professional development. Within the technical rationality, professional development was characterized by the transmission of research-based knowledge to teachers. This view of professional development, however, had to change when Schön established that there was another type of knowledge in teaching and this knowledge came from teaching practice itself. And, even though teachers' knowledge was not to replace researchers' knowledge, teachers needed new mechanisms to access their knowledge from practice, which required a different model for professional development. Thus, following Schön's work, mathematics professional development researchers turned their attention to knowledge coming from teaching and renewed discussion about what was needed to educate mathematics teachers.

One influential interpretation of the criticism of the technical rationality in teacher education that impacted mathematics professional development came from Cochran-Smith and Lytle (1999). These authors questioned the assumption that teachers who knew more taught better and claimed that «radically different views» (p. 249) existed for what it meant to know more and to teach better. These views were based on different conceptions of professional practice and teacher learning. Cochran-Smith and Lytle proposed three categories of knowledge, two of which related directly to Schön's work: knowledge-for-practice and knowledge-in-practice.

Knowledge-for-practice hinged on the idea that knowing more subject matter, educational theory, pedagogy, instructional strategies, etc., leads to more effective practice. In this case, the knowledge needed for teaching came from formal knowledge composed of theories and research findings that established a knowledge base. Skilled practitioners, therefore, had deep knowledge acquired from research that produced this knowledge outside the classroom. Knowledge-in-practice placed its emphasis on knowledge-in-action, which Cochran-Smith and Lytle (1999) explained as «what very competent teachers know as it is expressed or embedded *in* the artistry of practice, *in* teachers' reflections on practice, *in* teachers' practical inquiries, and/or *in* teachers' narrative accounts of practice» (p. 262, emphasis in the original). Skilled practitioners acquired this knowledge through experience and deliberate reflection into practice (in and on action) that made explicit the tacit knowledge that existed in the action of competent teachers.

The distinction between knowledge-for-practice and knowledge-in-practice separated professional development into settings that transmitted research knowledge and settings that engaged teachers in examining and reflecting on practice. Emerging research in the 1990s showed that teachers who worked together as colleagues to examine their teaching found themselves better prepared to teach (Little, 1990) and teachers who de-privatized their practice strengthened their pedagogical preparation (Louis, Kruse & Marks, 1996). This research heightened the call for researchers to attend to the need to build communities among teachers (Wilson & Berne, 1999). Reviewing the literature on mathematics professional development, Sowder (2007) listed the development of professional communities, establishment of professional development schools, and implementation of lesson studies as some of the approaches that emerged to promote knowledge-in-practice.

It is also important to note that this shift of mathematics professional development researchers' attention to communities and teachers' examining their teaching practice happened at the time when mathematics education at large was engaged in a «social turn» (Lerman, 2000), shifting from a more constructivist perspective that attended to the individual acquisition of knowledge to a focus on the social origins of knowledge from a socio-cultural perspective.

This shift brought attention to learning as participation in communities of practices (Lave & Wenger, 1991), which aligned with some of the ideas promoted under the search for teacher knowledge-in-practice. Together, the concepts of knowledge-in-practice and communities of practice strengthened the teacher's role and valued the teacher's knowledge. For example, examining a variety of project that reconceptualized the relationship between researchers' and teachers' knowledge in mathematics professional development research, Kieran, Krainer and Shaughnessy (2013) highlighted the importance of settings designed to harness teachers' expertise and build from collaboration among teachers.

In summary, by the end of the 1990s, mathematics professional development research was engaged in an important movement to value teachers' knowledge and examine the role this knowledge played in mathematics teaching and professional development. Teachers were placed at the centre of mathematics professional development, and mathematics professional development researchers turned their attention to examining how teachers organize in learning communities to promote teacher participation and knowledge exchange in ways that leveraged and valued teachers as knowers. At that point, as interpreted in mathematics professional development research, *The Reflective Practitioner* supported a shift that led to the predominance of teachers' knowledge over researchers' knowledge.

My claim in this paper is that in turning their attention to studying teachers' knowledge and communities, mathematics professional development researchers helped establish the importance of teachers as stakeholders. However, unfortunately, in the process of supporting teachers, while diminishing the attention given to the role of researchers as stakeholders. Therefore, renewed attention to researchers' knowledge is timely.

To support my claim, in what follows, I share two examples from my own practice in mathematics professional development research. In analysing the first project, I discuss how attention to teachers and their knowledge became fundamental in mathematics professional development research and made the role of research knowledge in professional development less clear. This first project is, in many ways similar to other mathematics professional development research projects of its time (2000s), allowed me to critically examine the field while criticizing my own work.

In examining the second project (2010s), I propose one way in which mathematics professional development researchers can continue to promote teachers' knowledge while also recognizing the importance of researchers' knowledge. The analysis of the second project highlights a venue to think about mathematics professional development in relation to teachers' communities, but in interaction with the researchers' communities, making *both* mathematics teachers and mathematics professional development researchers stakeholders in K-12 mathematics.

TWO EXAMPLES FROM MY OWN PRACTICE

In turning my attention to two projects, I share professional reflections about my work as a mathematics professional development researcher. I bring forth and discuss successes and tensions experienced in navigating between researchers' and teachers' knowledge in mathematics professional development. Although I cannot claim that others share my experiences, I expect that other mathematics professional development researchers have worked in similar situations. Highlighting similarities across the projects, I call attention to their school-based design, that is, both projects took place at the participating teachers' schools and all teachers from the partner elementary school in each project were invited to participate.

In both cases, mathematics professional development researchers met with teachers at the school and the school principals supported the professional development, making it a part of the school activities. In different ways, teachers' voices were important in both projects. Also, in both projects, members of the mathematics professional development research group had worked with some of the teachers at the school in various capacities for about one year prior to the beginning of the project. Thus, when each project started, researchers and teachers already knew each other.

The descriptions that follow are not meant to be exhaustive. They are based on the language used in each project's publications to allow for analysis of how the two research teams conceived the work of each project at the time it was carried out. The description of the first project was compiled from Sztajn, Allexsaht-Snider, White and Hackenberg (2004); White, Sztajn, Allexsaht-Snider and Hackenberg (2004); Sztajn, Hackenberg, White and Allexshat-Snider (2007); and Sztajn, White, Hackenberg and Allexsaht-Snider (2010). The description of the second project came from Sztajn, Wilson, Edgington and Confrey (2011); Sztajn, Confrey, Wilson and Edgington (2012); Sztajn, Wilson, Decuir-Gunby and Edgington (2012); Wilson, Sztajn, and Edgington (2012); and Sztajn, Wilson, Edgington and Meyers (in press).

Although both projects included multiple years of collaboration with the partner school and participating teachers, the focus of the description is on the first year in which mathematics professional development researchers and mathematics teachers at the partner schools engaged in professional development activities. The presentation of the projects is followed by a discussion about the role of researchers' and teachers' knowledge in each project, and the topic of researchers as stakeholders is revisited in the conclusion of the paper.

THE SUPPORT AND IDEAS FOR PLANNING AND SHARING PROJECT (SIPS)[1]

Project SIPS was a partnership with an urban elementary school in the South of the United States, where 90% of the children qualified for free or reduced-price lunch. In its school district, this school had the highest percentage of Hispanic children at the time (39%), although the school population was mostly African American (51%). In an initial project survey, twenty of the twenty-two teachers who participated (91%) in the project said they had not completed any professional development program or graduate courses in the previous five years in which recent research on children's learning of mathematics was discussed.

1 Supported through an Eisenhower Teacher Quality Grant.

Describing SIPS

Project SIPS was designed to help teachers improve the quality of their mathematics instruction by building a supportive mathematics education community within their school. In its first year, the main goal was to build such a community, creating a space for teachers to engage in reflections about their mathematics instruction. The mathematics professional development research team worked with the school administration to provide teachers with time to meet and discuss mathematics teaching and learning at their schools. As these conversations evolved, based on what teachers highlighted as their needs, mathematics professional development researchers provided teachers with vocabulary and ideas to think and talk about student learning. Thus, with teachers' input and recommendations, SIPS was designed to infuse the emerging mathematics education community with activities that focused on instruction; but it also increased teachers' mathematical content and pedagogical knowledge.

During the first year of SIPS, teachers participated in two types of SIPS meetings: work sessions and faculty meetings. SIPS work sessions took place at the school during school hours and teachers met within grade-level groups. Each group met for a half-day of activities every other month, and substitute teachers were hired to allow for teacher participation. Each work session addressed children's learning of those mathematics topics selected by teachers as critical to the grade-level. For example, one 2nd grade work session focused on place value and subtraction. During the work sessions, teachers explored their knowledge of and teaching strategies for the mathematical topic in focus. They discussed the work of their students, were introduced to research-based ideas for teaching those particular mathematics topics and co-planned lessons to implement in their classrooms.

The after-school mathematics faculty meetings were attended by the whole school staff and, whenever possible, by school administrators. These meetings were devoted to building and maintaining a mathematics education community within the school. During these meetings, teachers had the opportunity to share what they were doing in their mathematics classrooms with their colleagues across grade levels. They also solved some mathematics problem together and discussed their visions for mathematics teaching and learning at their school.

As a research project, SIPS aimed at understanding «the complex world of lived experience from the point of view of those who lived it» (Schwandt, 1994,

p. 118). From a qualitative research standpoint, SIPS researchers attempted to «elucidate the process of meaning construction and clarify what and how meanings are embodied in the language and actions of social actors» (p.118). Thus, in its research component, SIPS was interested in unveiling teachers' perceptions about the development of trust within the mathematics education community.

In answering the question what factors in Project SIPS supported the development of trust among mathematics professional development researchers and mathematics elementary teachers as the community was formed, the project research showed that teachers valued the mathematics professional development researchers' flexibility, respect for teachers' knowledge and awareness of school realities as important to developing trust. Teachers also appreciated the time SIPS provided for them to meet and the practical activities they developed to implement in their classrooms.

Examining SIPS

With a focus on teachers' knowledge, Project SIPS was designed to promote teachers working together and support them in examining their mathematics teaching. In line with the attention given at the time to the teachers' knowledge-in-action and knowledge-in-practice, the project highlighted the importance of teachers talking to each other and analysing their practice. Most of project SIPS time was spent in collectively planning for and sharing of mathematics instruction, with a focus on topics teachers deemed important.

Working together with teachers in a community of learners, SIPS researchers brought suggestions for classroom activities for discussion with the community, and, in the context of discussing such activities, they shared research knowledge on student mathematics learning. Thus, researchers' knowledge was not at the forefront of the SIPS community conversations – classroom practice was. Further, research knowledge only emerged as part of the conversation of the community when teachers saw a need for it.

In project writings, teachers and researchers were called «school-based educators» and «university-based educators,» respectively. These names were purposefully selected to represent the proximity of teachers and researchers in the project, indicating that in the SIPS community, all participated together as educators interested in mathematics teaching and learning. Although teachers and researchers obviously brought different contribution to the SIPS community, mathematics classroom practices was what brought them together.

Mathematics professional development researchers in Project SIPS, nonetheless, also had a goal of promoting change in mathematics instruction at the partner school through teacher learning of research on children's mathematics. This goal was not at the forefront of the project or at the centre of the SIPS community. Elsewhere (Sztajn, 2008) I discussed the dilemmas of being a researcher in such a community and trying to build trust and promote research when not all members of the community shared the goal of learning research results to transform practice.

THE LEARNING TRAJECTORY BASED INSTRUCTION PROJECT (LTBI)[2]

The LTBI project was a partnership with one elementary school in a mid-size urban area in the southeast of the United States. The school had approximately 600 students: 35% Caucasian, 29% Hispanic, 25% African American, 7% Asian, and 4% other; 54% of the children qualified for free or reduced-price lunch. Teachers at the partner school volunteered to participate in the project and all professional development meetings were conducted at the school, at hours deemed convenient by teachers, researchers and school administrators. Of the 24 teachers who started the professional development in July 2010, 22 completed the program one year later. The initial group of teachers included six kindergarten teachers, three grade 1, five grade 2, three grade 3, two grade 4, and one grade 5 teacher. Four teachers taught multiple grade levels.

Describing LTBI

The LTBI Project was designed to share research-based knowledge on student mathematics learning with teachers and, in the process, investigate how teachers came to learn about and use this knowledge in practice. In its first year, the main goal of the project was to examine teacher learning of the students' learning trajectories, with learning trajectories being defined as «a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representa-

2 Supported through a National Science Foundation REESE grant.

tion, articulation, and reflection, towards increasingly complex concepts over time» (Confrey, Maloney, Nguyen, Mojica & Myers, 2009, p. 347).

The LTBI professional development was designed for a 12-month period beginning with a 30-hour summer institute in which teachers learned about one particular learning trajectory. Following the institute, teachers and mathematics professional development researchers met regularly throughout the school year, after school hours, to continue to build teachers' knowledge of the trajectory and discuss their classroom implementations of instruction that used the trajectory.

The model of instruction emphasized in the professional development highlighted the importance of open instructional tasks to elicit students' mathematical thinking, together with a set of pedagogical practices that allowed teachers to build on this thinking to promote mathematical discourse in the classroom. Although teachers learned about the learning trajectory and the model of instruction promoted in the project throughout the duration of the project, the summer professional learning tasks were designed to support teacher learning of the learning trajectory, whereas during the rest of the year, the professional development focused more on the learning trajectory-based instructional model. The two components of the professional development totalled 60 hours of face-to-face, whole group interactions over one school year.

As a research project, LTBI used a design experiment research methodology to investigate teacher learning of students' learning trajectories. Research questions focused on teacher learning, including both questions about teachers' participation in the professional development and teachers' acquisition of the learning trajectory itself. For example, an initial conjecture in the project stated that as teachers learned about the trajectory, they gained specialized language that brought their participation closer to the centre of their professional community and strengthened their positioning and voice in the discourse of the group.

However, very early in the ongoing data analysis, teachers' discourse indicated the prevalence of language that talked about students as being «high» or «low,» for example, or not being able to complete a task because it did not align with their «experiences outside school.» This use of language to explain students' mathematical work led researchers to attend – not only to the ways in which teachers positioned *themselves* in the community - but also to the ways in which teachers positioned *students* in their discourse within this community. Further, it led to investigations of the ways in which knowledge of trajectory disrupted teachers' discourse about students.

Examining LTBI

With a focus on research-based knowledge, LTBI was designed to share recent knowledge about student mathematical learning with teachers and investigate how teachers come to learn and use this knowledge. Thus, in many ways, the project shared features of the Technical Rationality and knowledge-for-practice. However, LTBI was also designed to strongly build on teachers' interests as the partnership between teachers and researchers in the project was built with teachers' input with attention to their questions about learning trajectories as new ways to represent student learning in mathematics. Research knowledge was at the centre of the LTBI professional development community, and the project investigation focused on how teachers appropriated this knowledge.

The roles of researchers and teachers in the LTBI professional development community were clearly different. Researchers organized the professional development sessions with the goal of supporting teacher learning of the trajectory. Teachers knew researchers were interested in their use of the trajectory in their classrooms and felt they were teaching the researchers about classroom constraints and the realities of implementation.

In conceiving and designing for this relationship, LTBI mathematics professional development researchers conceptualized the LTBI community as a temporary boundary encounter (Wenger, 1998) among researchers and mathematics teachers. In such encounter, boundary practices emerged that brought the group together to work on shared goals. Further, mathematics professional development researchers designed for these boundary practices by creating professional learning tasks around boundary objects, that is, objects form the researchers' or the teachers' community that created shared knowledge across the two communities. Among such objects, for example, were sample of students' work and various representation of the learning trajectory.

LOOKING ACROSS SIPS AND LTBI

In contrasting SIPS and LTBI, both similarities and differences are important. Because LTBI followed SIPS and since the projects were about 10 years apart, changes in LTBI reflect tensions experienced in SIPS, combined with the maturity gained in conducting investigations within the emerging field of the mathematics professional development research. By understanding these similarities and differences, one can support the claim that it is important to attend to teachers' knowledge while also maintaining mathematics professional development researchers as stakeholders in K-12 mathematics.

By design, LTBI built on features that supported the successes of the community-building experiences from project SIPS. The trusting and caring relationship (Sztajn, 2008) established between researchers and teachers, which was at the centre of SIPS, continued to be important in LTBI. Teachers' knowledge-in-practice, which was central for SIPS, continued to be respected in LTBI, and teachers engaged in a variety of discussions focused on their own knowledge.

These discussions positioned teachers as expert in the professional development community. Both projects included teachers across grade levels at one partner school and allowed for conversations about mathematics teaching and learning within the school as a whole. Teachers from different grade levels got to discuss and gain a better grasp of K-5 mathematics instructional goals and teaching strategies, acquiring a better-aligned perspective of their collective work in mathematics teaching.

These fundamental similarities were an important motive of teacher engagement and satisfaction in both projects. Yet LTBI was also designed to address the tensions perceived by researchers in project SIPS. One of these was allowing researchers to have a stronger voice in the project. Whereas SIPS was built around process goals only, focusing on teachers' knowledge and the development of a teacher-based community, LTBI included both content and process goals (Simon, 2008), focusing on both the teachers' and researchers' knowledge and the development of a boundary community across teaching and research. In LTBI, research-based knowledge was more openly shared with teachers during professional development, because sharing knowledge was one of the goals of professional development. One of the LTBI's explicit aims was for teachers to learn about the learning trajectory, which was always front and centre in the project because it brought research-based knowledge into the community.

LTBI reclaimed the role of researchers as knowers and stakeholders in professional development, while allowing the teachers to maintain the voices and positions they had acquired as knowers. LTBI accepted the importance of both research-based and practice-based knowledge, and looked for ways in which both knowledge-types could interact. This approach was more in line with the notion supported by both Munby (1989) and Eraut (1995) that knowledge-in-action was to be respected without eliminating the importance of research knowledge. Thus, LTBI professional development did not have to choose a focus on knowledge-for-practice *or* knowledge-in-practice; rather, it attended to the intersection between the two.

Revisiting the Technical Rationality: Teachers and Researchers as Stakeholders

In this paper, I claimed that mathematics education researchers are facing the challenge of demonstrating the unique contribution they make to discussions about K-12 mathematics, and that this challenge is more acute for mathematics professional development researchers because they are a newer subset of the larger research community. I noted that in the recent past mathematics professional development researchers made an effort to establish that teachers are key stakeholders in mathematics professional development. This effort made significant gains in supporting teachers, but also hindered the concept of mathematics professional development researchers as stakeholders in K-12 mathematics. I traced the effort to attend to and strengthen teachers' knowledge to the concept of knowledge-in-action and the increased attention to practitioners as knowers. I related the attention to teachers as knowers to the concept of knowledge-for-practice and knowledge-in-practice used in the professional development literature.

Sharing my own experiences, I presented two research projects: one with process goals focused on establishing a teacher community in a partner school, the other maintaining the process goals but adding a clear content goal of teaching teachers about students' learning trajectories. I described and examined each project, later comparing and contrasting their similarities and differences. I noted that by working with teachers in a boundary community, researchers' knowledge could have a stronger role in the project with a content goal while also respecting teachers' knowledge.

Thus, I showed one venue for mathematics professional development researchers to continue to value teachers' knowledge and return to an increased attention to researchers' knowledge. I contend that highlighting research knowledge in mathematics professional development and clearly articulating the contributions of research for teaching is a possible route to establishing that mathematics professional development researchers are, indeed, stakeholders in K-12 mathematics.

I conclude this paper calling for mathematics professional development researchers to embrace a revised version of the technical rationality that highlights the importance of both knowledge from practice *and* knowledge from research. Without recognizing the former, there is no understanding of what teaching entails. However, without valuing the latter, there is no connection between teaching and innovations from various areas. Thus, more productive than attending to whether a project is about researchers' or teachers' knowledge, or whether it is about knowledge-for-practice versus knowledge-in-practice, is focusing on the connections between the types of knowledge and the ways in which knowledge-for-practice becomes knowledge-in-practice and vice-versa. In our work, we have addressed this connection by examining the concept of boundary encounters (Wenger, 1998).

A revised technical rationality highlights the value of teachers' knowledge – a fundamental contribution of Schön's workwhile also maintaining the importance of knowledge from research. A revised technical rationality allows mathematics professional development researchers to bring rigour and relevance – separated in *The Reflective Practitioner* –, which suggested that only practice was relevant – together once again. A revised technical rationality establishes that both mathematics education researchers and mathematics teachers are key stakeholders in K-12 mathematics.

NOTE

This report is partially based upon work supported by the National Science Foundation under grant number DRL-1008364. Any opinions, findings, and conclusions or recommendations expressed in this report are those of the authors and do not necessarily reflect the views of the National Science Foundation.

REFERENCES

Cochran-Smith, M., & Lytle, S. L. (1999). Relationship of Knowledge and Practice: Teacher Learning in Communities. *Review of research in education, 24*, 249-306.

Confrey, J., Maloney, A. P., Nguyen, K. H., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the Psychology of Mathematics Education: In Search for Theories in Mathematics Education* (Vol. 2, pp. 345-352). Thessaloniki, Greece: PME.

Eraut, M. (1995). Schön shock: A case for reframing reflection-in-action? *Teachers and Teaching, 1*(1), 9-32.

Kieran, C., Krainer, K., & Shaughnessy, M. J. (2013). Linking research to practice: Teachers as key stakeholders in mathematics education research. In M. A. Clements *et al.* (Eds.), *Third International Handbook of Mathematics Education* (pp. 361-392). New York: Springer.

Kinsella, E. A. (2007). Technical rationality in Schön's reflective practice: dichotomous or non-dualistic epistemological position. *Nursing Philosophy, 8*, 102-113.

Krainer, K. (2011). Teachers as stakeholders in mathematics education research. In B. Ubuz (Ed.), *Proceedings of 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 47-62). Ankara, Turkey: PME.

Kotzee, B. (2012). Private practice: Exploring the missing social dimension in 'reflective practice'. *Studies in Continuing Education, 34*(1), 5-16.

Lave, J., & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge, UK: Cambridge University Press.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19-44). Westport, CT: Ablex Publishing.

Little, J. W. (1990). The persistence of privacy: Autonomy and initiative in teachers' professional relations. *Teachers College Record, 91*(4), 509-536.

Louis, K. S., Kruse, S. D., & Marks, H. M. (1996). Schoolwide professional community. In F. W. Newman (Ed.), *Authentic Achievement* (pp. 179-203). San Francisco: Jossey-Bass.

Munby, H. (1989). Reflection-in-Action and Reflection-on-Action. Paper presented at the *Annual Meeting of the American Educational Research Association*. San Francisco, CA.

NRS – National Research Council. (2002). *Scientific research in education* (R. J. Shavelson & L. Towne, Eds.). Washington, DC: National Academy Press.

Schön, D. A. (1983). *The Reflective Practitioner*. New York: Basic Books.

Schön, D. A. (1987). *Educating the Reflective Practitioner*. San Francisco: Jossey-Bass.

Schwandt, T. A. (1994). Constructivist, interpretivist approaches to human inquiry. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of Qualitative Research* (pp. 118-137). Thousand Oaks, CA: Sage.

Simon, M. (2008). The challenge of mathematics teacher education in an era of mathematics education reform. In B. Jaworski & T. Woods (Eds.), *International Handbook of Mathematics Teacher Education. Vol. 4 -The mathematics teacher educator as a developing professional* (pp. 17-30). Rotterdam, Netherland: Sense Publishers.

Simon, M. A. (2004). Raising issues of quality in mathematics education research. *Journal for Research in Mathematics Education, 35*, 157-163.

Sowder, J. T. (2007). The mathematical education and development of teachers. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 157-223). Charlotte, NC: Information Age Publishers.

Sztajn, P. (2008). Caring relations in the education of practising mathematics teachers. In B. Jaworski & T. Woods (Eds.), *International Handbook of Mathematics Professional development. Vol. 4 - The mathematics teacher educator as a developing professional* (pp. 299-314). Rotterdam, Netherland: Sense Publishers.

Sztajn, P. (2011). Standards for reporting mathematics professional development in research studies. *Journal for Research in Mathematics Education, 42*(2), 220-236.

Sztajn, P., Alexsaht-Snider, M., White, D. Y., & Hackenberg, A. (2004). School-based community of teachers and outcomes for students. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceeding of 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 273-281). Bergen, Norway: Bergen University College.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: toward a theory of teaching. *Educational Researcher, 41*, 147-156.

Sztajn, P., Hackenberg, A., White, D. Y., & Allexshat-Snider, M. (2007). Mathematics professional development for elementary teachers: Building trust within a school-based mathematics education community. *Teaching and Professional Development, 23*(6), 970-984.

Sztajn, P., White, D. Y., Hackenberg, A., & Allexsaht-Snider, M. (2010). Developing trusting relations in in-service education of elementary mathematics teachers. In G. Anthony & B. Grevholm (Eds.), *Teachers of mathematics: Recruitment and retention, professional development and identity* (pp. 167-176). Sweden: Swedish Society for Research in Mathematics Education.

Sztajn, P., Wilson, P. H., Decuir-Gunby, J., & Edgington, C. (2012). Teachers' attributions for students' mathematical work. In L. R. Van Zoest, J. J. Lo & J. L. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 482-487). Kalamazoo, MI.

Sztajn, P., Wilson, P. H., Edgington, C., & Confrey, J. (2011). Learning trajectories and key instructional practices. In T. S. Lamberg & L. R. Wiest (Eds.), *Proceedings of the 33rd Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 434- 442). Reno, Nevada: PME-NA.

Sztajn, P., Wilson, P. H., Edgington, C., & Meyers, M. (in press). Mathematics professional development as design for boundary encouters. *ZDM: The International Journal on Mathematics Education.*

Webster, H. (2008). Architectural education after Schön: Cracks, blurs, boundaries, and beyond. *Journal for Education in the Built Environment, 3(2),* 63-74. Retrieved September 17, 2013 from http://journals.heacademy.ac.uk/doi/pdf/10.11120/jebe.2008.03020063.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity.* New York: Cambridge University Press.

White, D. Y., Sztajn, P., Alexsaht-Snider, M., & Hackenberg, A. (2004). Building a mathematics education community that facilitates teacher sharing in an urban elementary school. In D. E. Dougall & J. A. Ross (Eds.), *Proceedings of the 26th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 977-984). Toronto: OISE/UT.

Wilson, P. H., Sztajn, P., & Edgington, C. (2012). Designing professional learning tasks for mathematics learning trajectories. In T. Tso (Ed.), *Proceedings of the 36th Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 227-234). Taipei, Taiwan: PME.

Wilson, S., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of contemporary professional development. *Review of Research in Education*, *24*, 173-209.

NOTES ON CONTRIBUTORS

Neusa Branco is an assistant professor in the Higher School of Education at the Polytechnic Institute of Santarém (Portugal). In the last few years, she has worked on a number of research projects in mathematics education and directed master's students in education. She co-coordinates an educational program of pre-service elementary and kindergarten teachers. Currently, her main research interests are the teaching and learning of algebra and teacher education for elementary school.

Olive Chapman is professor and associate dean of undergraduate education programs at the University of Calgary in Canada. Her research interests include mathematics teaching practices and mathematics teacher thinking, learning and change, with a focus on mathematics knowledge for teaching, problem solving, algebraic thinking, and inquiry-based mathematics pedagogy. She led the development and implementation of an undergraduate teacher education program at her institution (2010-2012). She has (co)authored several book chapters, articles in international journals and papers in proceedings of refereed international conferences. She is editor-in-chief of the *Journal of Mathematics Teacher Education* and a member of the International Committee of the International Group for the Psychology of Mathematics Education.

David Clarke is a professor and director of the International Centre for Classroom Research (ICCR) in the Melbourne Graduate School of Education at the University of Melbourne. His current research interests are international comparative research, classroom research, mathematics education, science education, assessment, teacher professional learning, research synthesis and multi-theoretic research design. Video plays a significant part in many of his research projects, in particular the *Learner's Perspective Study* which has generated a book series reporting the analysis of video material from the mathematics classrooms of more than a dozen countries. The findings of other major international research projects are reported in over 150 journal articles, book chapters and conference proceedings.

Márcia Cyrino is a professor in the Mathematics Department and in the Post-Graduate Programme in Science and Mathematics Teaching at the State University of Londrina (Brazil). Her main research interests are mathematics teacher education, with a particular focus on the practicum and the teacher's professional identity, algebraic thinking, and proportional reasoning. She is a productivity researcher of CNPq and has conducted several projects supported by the Fundação Araucária and CNPq. She has authored a number of book chapters and papers on mathematics education and teacher education.

Dario Fiorentini is a professor and coordinator of the Graduate Program in Education of the Faculty of Education at Unicamp (State University of Campinas), Brazil. His research interests centre mainly on teacher education and professional development in professional and inquiry communities. He has published several papers and books on teacher education and professional development, collaborative research, and state-of-the-art and qualitative meta-analysis of research in mathematics education. Internationally, he co-edited the book entitled *International Approaches to Professional Development for Mathematics Teachers* (2011) with Nadine Bednarz and Rongjin Huang.

Radhika Gorur is a post-doctoral research fellow at The Victoria Institute, Victoria University, Melbourne. Her primary research focus is the sociology of numbers and measurement, with a focus on large-scale comparative assessments. She is also interested more broadly in the making of scientific and social knowledge, and in educational policy, education reform and teacher professional practice. She has published several articles on educational policy and large-scale assessments in such journals as *Education Philosophy* and *Theory and Discourse*.

António Guerreiro is an adjunct professor and director of the School of Education and Communication, University of the Algarve in Portugal, where he also teaches. His main research interests are mathematics teacher practices and teacher education, with a special emphasis on mathematical communication in primary school. He was a member of the coordinating team of the teacher training programme in mathematics for basic education (2005-2010). He is the author of articles and papers in refereed national and international conferences.

Hilary Hollingsworth is a senior research fellow at the Australian Council for Educational Research. Her main areas of interest are teacher professional learning and the assessment of student learning. Her current work foci include: teacher evaluation and assessment, classroom observation frameworks, the use of video for teacher professional learning, the assessment of student learning and primary mathematics education. Her doctoral studies led her to co-develop an elaborated model of teacher professional growth (2002), and she was the Australian representative on the TIMSS Video Study projects, co-authoring the International and Australian reports (2003) and numerous related publications.

Salvador Llinares is a professor in the Department of Innovation and Education of the University of Alicante (Spain). His current research interests are mathematics teaching practices and teacher education, with a special focus on teaching and learning and the students' development of mathematics reasoning. In Spain, he was a member of the assessment committee of I+D+i research activities (2006-7) and the accreditation committee for university teachers (2008-9). He collaborated on the management of I+D+i research projects (2012-13) and is the (co)author of several books and articles in national and international journals.

Maria Helena Martinho is an assistant professor at the Institute of Education of University of Minho in Portugal. Her main research interests are teachers' practices, mathematics communication, reasoning and literacy. She was a member of the curriculum development team for East Timor, where she contributed to the mathematics curriculum for basic education (2008-2010). She served on the coordination committee for the implementation of the new mathematics curriculum for basic education (2009-2012). Currently, she coordinates a research project on literacy and is the deputy director of the mathematics education research journal, *Quadrante*. She is (co)author of several articles in national and international journals.

Luís Menezes is an adjunct professor in the Viseu Higher School of Education (ESEV) at the Polytechnic Institute of Viseu, Portugal. His main research interests are mathematics teaching practices and teacher education. His interests lie on the role of communication in the mathematics classroom, the processes of construction and the transfer of knowledge. He was a member of the team that developed the new mathematics curriculum for basic education (2007). Between 2005 and 2011, at ESEV, he coordinated the In-Service Teacher Education Programme in Mathematics. He is (co)author of several articles in national and international journals and has produced curriculum support materials for mathematics teacher education.

Hélia Oliveira is an assistant professor at the Institute of Education of the University of Lisbon. Her current main interests are teacher education, in particular initial teacher education, multimedia resources, the development of students' algebraic thinking, and statistics teaching. She has co-authored chapters and books for an international audience and papers in national and international journals and in proceedings of refereed national and international conferences.

João Pedro da Ponte is a professor and director of the Institute of Education of the University of Lisbon. Currently, his main research interests are mathematics teaching practices and teacher education, with a special focus on the teaching of algebra, rational numbers and statistics, and developing students' mathematics reasoning. He coordinated a government report on pre-service teacher education (2006) and a new mathematics curriculum for basic education (2007). He is (co)author of several books, book chapters and articles in national and international journals such as *Educational Studies in Mathematics* and is associate editor of the *Journal of Mathematics Teacher Education*.

Tim Rowland is a professor of mathematics at the University of East Anglia, UK; emeritus reader in mathematics education at the University of Cambridge, UK; and professor II at Sør-Trøndelag University College in Trondheim, Norway. He is co-editor of the journal *Research in Mathematics Education* and Chair of the Joint Mathematical Council of the United Kingdom. His book *The Pragmatics of Mathematics Education* reflects research interests in mathematics, linguistics, epistemology and education. More recent books and articles have reflected a 15-year corpus of research into the role of mathematical knowledge in teaching.

Paola Sztajn is a professor of mathematics education at North Carolina State University and currently serves as head of the Department of Elementary Education. She previously served as a program officer at NSF and is the incoming chair for the Special Interest Group for Research in Mathematics Education for the American Educational Research Association. Her research focuses on elementary teachers' professional development in mathematics. She is currently the principal investigator for three grants from the National Science Foundation,all Included in Mathematics, Professional Development Models, and Learning Trajectory Based Instruction. Her work has been published in journals such as *Educational Researcher, Journal for Research in Mathematics Education*, *Journal of Mathematics Teacher Education*, *Teaching and Teacher Education*, and *Teaching Children Mathematics*.

Rosa Antónia Tomás Ferreira is an assistant professor of the Faculty of Sciences at Porto University in Porto, Portugal. Currently, her main research interests are mathematics teacher education, teacher communication practices and didactical knowledge, and inclusive mathematical problem solving competitions, especially the role of affective factors and feedback. She coordinated a national project that promoted teaching practices to address students with learning difficulties (2007-2012). She also was a member of coordination committee for the implementation of the new mathematics curriculum for basic education (2009-2012). She has presented several research studies at national and international conferences and is a member of the board of directors of the Portuguese Society for Research in Mathematics Education.

SUBMISSION GUIDELINES

Publication frequency

3 issues per year

Each number will focus on a specific theme. Along with the articles, the journal will include other research materials such as case-study reports, experiences and inquiries, conceptual and methodological discussions, on-going research papers and book reviews.

Submission

Authors are requested to submit their papers electronically by using the e-mail sisyphus@ie.ulisboa.pt or our website http://revistas.rcaap.pt/sisyphus.

Submission declaration

Sisyphus – Journal of Education only publishes original articles, explicitly under the strict condition that they have not been published already, nor are they under consideration for publication or in press elsewhere (excluding abstracts or writings extracted from conferences or theses). Articles must be sanctioned by all authors in order to be published.

Instructions for authors

Review criteria

All submitted articles must be rigorous, technically precise, and should put forward a progressive perspective in relation to the state of the art. They should also elucidate and circumscribe the significance of the subject matter, as well as the conceptual and methodological orientations; the research enquiry; the revision of the correlative and most relevant publications on the subject; and the presentation of all results and conclusions. The manuscripts must be essentially problematical; that is, they should draw research vectors that open up new theoretical paths while suggesting methods to deal with intrinsic interrogations. They must also add new perspectives to current writings.

In order to be published, the articles must focus on issues that can resonate with an international audience, which is why they should promote and be engaged in wide-ranging issues and debates that can be inscribed within a non-local agenda.

Language and use of language

All articles must be written in English (U.K.), so that they may be read by an international audience. National, colloquial terms and idiomatic expressions should be avoided. Non-discriminatory language is mandatory for all manuscripts: any words or expressions conveying social prejudices are to be avoided.

Articles

Length: Articles submitted should not exceed 10,000 words.

Abstract: An abstract of 150 words is required for all submitted papers. It must synthetize the article's main query, its results and conclusions.

Key words: Each article should have 3 to 5 key words.

Use of word-processing software

All articles should be submitted in any editable format (docx, doc), and formatting should be minimal.

References

All submitted articles must follow APA Style, 6th edition. For detailed information, please see the Publication Manual of the American Psychological Association, 6th edition (www.apastyle.org).

THE PROFESSIONAL PRACTICE AND PROFESSIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

JOÃO PEDRO DA PONTE

ISSN: 2182-8474 03

9 7 2 2 1 8 2 8 4 7 0 1 8

Instituto de Educação

FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR